REMARKS

(i) THE KEY EQUATION (24) CAN BE OBTAINED IN A MORE DIRECT WAY WITH A LITTLE TRICKERY.

BEGIN WITH $$L_0 \Phi - \chi w^3 \int w^2 \Phi \Big/ \int w^3 = \lambda \Phi$$

RECALL $L_0(w^2) = 3w^2$. FORM LAGRANGE'S IDENTITY USING DECAY AT ∞ TO GET

$$\int_{-\infty}^{\infty} \left(w^2 L_0 \Phi - \Phi \, L_0(w^2) \right) dy = 0$$

$$\int_{-\infty}^{\infty} \left[w^2 \chi w^3 \frac{\int w^2 \Phi}{\int w^3} + \lambda w^2 \Phi - 3 w^2 \Phi \right] dy = 0$$

FACTOR OUT $\int w^2 \Phi$ TO GET

$$\int w^2 \Phi \, dy \left[\chi \frac{\int w^5}{\int w^3} + \lambda - 3 \right] = 0.$$

THUS FOR EIGENFUNCTIONS FOR WHICH $\int w^2 \Phi \, dy \neq 0$ WE HAVE

$$\chi \int w^5 \Big/ \int w^3 + \lambda - 3 = 0.$$

THUS, $$\chi = \frac{\int w^3}{\int w^5} (3 - \lambda) = \frac{2}{3} (3 - \lambda)$$

SO λ MUST SATISFY $$g(\lambda) = \frac{1}{\chi(\lambda)} - \frac{3}{2(3-\lambda)} = 0$$

(ii) A RATHER CURIOUS ISSUE AGAIN OCCURS FOR OTHER POWERS WHERE THE NLEP IS EXPLICITLY SOLVABLE.

CONSIDER $$V_t = \varepsilon^2 V_{xx} - V + V^p / U^q$$

$$\tau U_t = D U_{xx} - U + \frac{1}{\varepsilon} V^m / U^s$$

THEN THE NLEP FOR ONE-SPIKE FOR THE <u>INFINITE LINE</u> PROBLEM IS

$$L_0 \Phi - \chi w^p \frac{\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} = \lambda \Phi$$

WITH $\chi(\lambda) = \dfrac{q m}{s + \sqrt{1+\tau\lambda}}$

WHERE $w'' - w + w^p = 0, \quad L_0 \Phi = \Phi'' - \Phi + p w^{p-1} \Phi.$

WE OBTAIN THAT λ IS A ROOT OF $g(\lambda) = 0$ WITH

$$g(\lambda) = \frac{1}{\chi(\lambda)} - \underbrace{\frac{\int_{-\infty}^{\infty} w^{m-1} (L_0 - \lambda)^{-1} w^p \, dy}{\int_{-\infty}^{\infty} w^m \, dy}}_{\equiv \mathcal{F}(\lambda)} \qquad (26)$$

IF $p = 5$ THEN $L_0(w^3) = 8 w^3$. THUS IF $m = 4$ THE NLEP IS EXPLICITLY SOLVABLE. WE USE THE SAME APPROACH AS BEFORE.

DEFINE
$$\mathcal{J}(\lambda) = \int_{-\infty}^{\infty} w^3 (L_0 - \lambda)^{-1} w^5 \, dy$$
$$= \frac{1}{8} \int_{-\infty}^{\infty} L_0 (w^3) (L_0 - \lambda)^{-1} w^5 \, dy$$
$$= \frac{1}{8} \int_{-\infty}^{\infty} ([(L_0 - \lambda) + \lambda] w^3) (L_0 - \lambda)^{-1} w^5 \, dy$$
$$= \frac{1}{8} \int_{-\infty}^{\infty} ((L_0 - \lambda)^{-1} [(L_0 - \lambda) + \lambda] w^3) \, w^5 \, dy \qquad \text{BY IBP}$$
$$\mathcal{J}(\lambda) = \frac{1}{8} \int w^8 \, dy + \frac{\lambda}{8} \int_{-\infty}^{\infty} [(L_0 - \lambda)^{-1} w^3] w^5 \, dy$$

NOW USE IBP AGAIN,
$$\mathcal{J}(\lambda) = \frac{1}{8} \int w^8 + \frac{\lambda}{8} \int_{-\infty}^{\infty} w^3 (L_0 - \lambda)^{-1} w^5 \, dy$$

THIS GIVES $\mathcal{J}(\lambda) = \frac{1}{8} \int w^8 \, dy + \frac{\lambda}{8} \mathcal{J}(\lambda)$ SO $\mathcal{J}(\lambda) = \dfrac{\int w^8}{8 - \lambda}$.

WE CONCLUDE THAT $\mathcal{F}(\lambda) = \dfrac{\int w^8 \, dy}{\int w^4 \, dy} \dfrac{1}{8 - \lambda} = \dfrac{2}{8 - \lambda}$ BY THE INTEGRAL EVALUATION.

AS A REMARK, WE CAN CALCULATE $\int w^8 / \int w^4 = 2$ IN A 1-LINE CALCULATION. THE ORIGINAL INTEGRALS EARLIER ON CAN'T GIVE THIS RESULT.

<u>PROOF</u> $L_0(w^3) = 8w^3$ AND $L_0(w) = (p-1)w^p$. LET $p=5$ AND
(FOR $p=5$)

CALCULATE $\int_{-\infty}^{\infty} (w^3 L_0 w - w L_0 w^3)\, dy = 0$ BY LAGRANGE.

THUS $\int_{-\infty}^{\infty} \left(w^3 [\ 4w^5] - w(8w^3)\right) dy = 0.$

THIS GIVES $4\int_{-\infty}^{\infty} w^8\, dy = 8\int w^4\, dy$ SO $\int w^8 / \int w^4 = 2.$

<u>REMARK</u> THE STUDY OF ROOTS OF (26) FOR GENERAL VALUES OF m, p AND FOR VARIOUS $\chi(\lambda)$ IS A CENTRAL TOPIC IN <u>NLEP THEORY</u>. VARIOUS RESULTS EXIST FOR DIFFERENT p AND m. THE THEORY IS INTRICATE AND STILL "INCOMPLETE". THE PROTOTYPICAL GM MODEL IS $p=2$, $m=2$, $q=1$, $s=0$.

FINALLY, WE RETURN TO OUR ORIGINAL PROBLEM AND SEE IF WE CAN CHARACTERIZE THE HOPF BIFURCATION MORE PRECISELY.

RECALL FROM (24) THAT λ IS A ROOT OF

(28) $$\left.\right\} g(\lambda) = \frac{1}{\chi(\lambda)} - \frac{3}{2(3-\lambda)} = 0 \quad \text{ON A DOMAIN } -1 < x < 1$$

WITH $\chi(\lambda) = \dfrac{6}{\sqrt{1+\tau\lambda}} \dfrac{\tanh \varphi_0}{\tanh(\varphi_\lambda)}$ $\qquad \varphi_0 = 1/\sqrt{D}$, $\quad \varphi_\lambda = \sqrt{1+\tau\lambda}\,\varphi_0$

- FOR INFINITE LINE PROBLEM FOR WHICH $1/\sqrt{D} \to \infty$ WE HAVE

$$\chi(\lambda) = \frac{6}{\sqrt{1+\tau\lambda}} \qquad (28)_1$$

- FOR THE LARGE D LIMIT, REFFERED TO AS THE SHADOW LIMIT, WHERE $1/\sqrt{D} \to 0$ WE HAVE

$$\chi(\lambda) = \frac{6}{1+\tau\lambda}. \qquad (28)_2$$

ALTHOUGH WE CANNOT SOLVE (28) FOR ARBITRARY D, WE CAN FIND A UNIQUE HOPF BIFURCATION POINT FOR THE TWO LIMITING FORMS $(28)_1$ AND $(28)_2$.

<u>SHADOW LIMIT</u> $(1/\sqrt{D} \to 0)$

FIND ROOTS OF $\left[\dfrac{6}{1+\tau\lambda}\right]^{-1} = \dfrac{3}{2(3-\lambda)}$ SO $\dfrac{1+\tau\lambda}{6} = \dfrac{3}{2(3-\lambda)}$

WE GET $(1+\tau\lambda)(3-\lambda) = 9$

$$-\tau\lambda^2 + \lambda[3\tau - 1] - 6 = 0$$

$$\lambda^2 - \lambda(3 - 1/\tau) + 6/\tau = 0$$

THUS WE HAVE

$\text{RE}\,\lambda < 0$ IFF $3 - 1/\tau < 0$ OR $0 < \tau < 1/3$.

THE HOPF BIFURCATION OCCURS AT $\tau = \tau_H = 1/3$ WITH FREQUENCY

$$\left.\begin{aligned} &\tau = \tau_H = 1/3 \\ &\lambda = \lambda_{\tau_H} = 3\sqrt{2}. \end{aligned}\right\} \quad (29)$$

<u>INFINITE LINE LIMIT</u>

WE NOW WANT TO FIND A HB POINT FOR INFINITE LINE PROBLEM

$$\frac{\sqrt{1+\tau\lambda}}{6} = \frac{3}{2(3-\lambda)}$$

OR EQUIVALENTLY $\sqrt{1+\tau\lambda} = \dfrac{9}{3-\lambda}$. WHERE WE HAVE TAKEN PRINCIPAL VALUE OF $\sqrt{1+\tau\lambda}$. AT THE EXPENSE OF CREATING SPURIOUS ROOTS WE SQUARE BOTH SIDES TO GET

$$(1+\tau\lambda)(3-\lambda)^2 = 81.$$

PUT $\lambda = i\lambda_H$, $\tau = \tau_H$ TO SEE A HB POINT.

$$(1 + i\tau_H\lambda_H)(9 - \lambda_H^2 - 6i\lambda_H) = 81.$$

TAKING REAL AND IMAGINARY PARTS OF BOTH SIDES

$$\text{REAL}(\quad) = 81 \longrightarrow 9 - \lambda_H^2 + 6\lambda_H^2\tau_H = 81$$

$$\text{IM}(\quad) = 0 \longrightarrow 9\lambda_H\tau_H - 6\lambda_H - \tau_H\lambda_H^3 = 0.$$

DIVIDING BY λ_H $\quad 9\tau_H - 6 - \tau_H\lambda_H^2 = 0$. ELIMINATING λ_H^2 IN FIRST EQUATION WE GET QUADRATIC EQUATION FOR τ_H WITH TWO ROOTS. WE <u>MUST</u> CHOOSE ROOT FOR WHICH $\text{RE}\left(9/3-\lambda\right) > 0$.

WE GET

$$\left.\begin{aligned} &\tau_H = \frac{1}{36}\left[2c^2 + 12 + 2c\sqrt{c^2+12}\right] \\ &\lambda_H = 3\sqrt{1 - 2/3\tau_H} \qquad \text{WITH} \quad c = 3\sqrt{3}/\sqrt{2}. \end{aligned}\right\} \quad (30)$$

SHADOW LIMIT $D \gg 1$

CAN WE DERIVE THE NLEP CORRESPONDING TO THE LIMIT $D \to +\infty$ OF

$$(31)\quad \begin{cases} V_t = \varepsilon^2 V_{xx} - V + V^3/U^2 & -1 < x < 1 \;; \quad U_x = V_x = 0 \text{ AT } x = \pm 1. \\ \tau U_t = D U_{xx} - U + \frac{1}{\varepsilon} V^3 \end{cases}$$

TO LEADING ORDER FOR $D \gg 1$ WE EXPECT $U = U_0(t)$ SINCE $D U_{xx} \simeq 0$ WITH $U_x = 0$ AT $x = \pm 1$ GIVES $U = U_0(t)$. USE

$$U = U_0(t) + \frac{1}{D} U_1 + \cdots$$

TO OBTAIN
$$U_{1xx} = \tau U_0' + U_0 - \frac{1}{\varepsilon} V^3$$
$$U_{1x}(\pm 1) = 0.$$

BY DIVERGENCE THEOREM $\quad (\tau U_0' + U_0)|\Omega| = \frac{1}{\varepsilon}\int_{-1}^{1} V^3\, dx$, WHERE $|\Omega| = 2 1$.

THUS IN THE SHADOW-LIMIT $D \to +\infty$, (31) REDUCES AT LEAST FORMALLY TO THE <u>PDE-ODE SYSTEM</u>

$$(32)\quad \begin{cases} V_t = \varepsilon^2 V_{xx} - V + V^3/U^2, & -1 < x < 1 \\ \tau U_0' = -U_0 + \frac{1}{|\Omega|\varepsilon}\int_{-1}^{1} V^3\, dx \end{cases}$$

THE STEADY-STATE SPIKE SOLUTION IS $U = U_e$ AND $V \sim U_e\, w(y)$ WHERE $w'' - w + w^3 = 0$. TO FIND U_e WE USE

$$U_e = \frac{1}{|\Omega|\varepsilon}\int_{-1}^{1} V^3\, dx \sim \frac{1}{|\Omega|} U_e^3 \int_{-\infty}^{\infty} w^3\, dy$$

THIS YIELDS
$$U_e = \left(\frac{|\Omega|}{\int_{-\infty}^{\infty} w^3\, dy} \right)^{1/2}.$$

NOW TO ANALYZE THE LINEAR STABILITY WE PUT

$$V = V_e + e^{\lambda t}\,\Phi, \qquad U_0 = U_e + e^{\lambda t}\,\Lambda$$

where $\Phi = \Phi(x/\varepsilon)$ AND Λ IS A CONSTANT.

WE OBTAIN FROM (32) THAT

$$(33)\quad \begin{cases} \lambda\,\Phi = \Phi'' - \Phi + 3\,V_e^2/U_e^2\,\Phi - 2\,\dfrac{V_e^3}{U_e^3}\,\Lambda & \text{ON } -\infty < y < \infty. \\[2ex] \lambda\tau\,\Lambda = -\Lambda - U_e + \dfrac{1}{|\Omega|\varepsilon}\displaystyle\int_{-1}^{1}\left(V_e^3 + 3V_e^2\,\Phi\right)dx \end{cases}$$

SINCE $V_e/U_e \sim w(y)$ IN CORE OF SPIKE WE OBTAIN THAT

$$\Phi'' - \Phi + 3w^2\,\Phi - 2w^3\,\Lambda = \lambda\,\Phi. \qquad (34)$$

FROM THE SECOND EQUATION $\quad U_e = \dfrac{1}{|\Omega|\varepsilon}\displaystyle\int_{-1}^{1} V_e^3\,dx.$

THUS,

$$\lambda\tau\,\Lambda + \Lambda = \frac{3}{|\Omega|}\cdot\int V_e^2\,\Phi\,dy = \frac{3}{|\Omega|}\,U_e^2\int w^2\,\Phi\,dy.$$

BUT $U_e^2 = |\Omega|/\int w^3$ SO THAT

$$\Lambda(1+\tau\lambda) = 3\,\frac{\int w^2\,\Phi\,dy}{\int w^3\,dy}$$

NOW SUBSTITUTING INTO (34) WE GET THE NLEP FOR THE SHADOW PROBLEM

$$\Phi'' - \Phi + 3w^2\,\Phi - \chi\, w^3\,\frac{\int w^2\,\Phi\,dy}{\int w^3\,dy} = \lambda\,\Phi$$

WITH $\quad \chi \equiv \dfrac{6}{1+\tau\lambda}.$

STABILITY OF A STRIPE

PROBLEM.
$$V_t = \varepsilon^2 \Delta V - V + V^3/U^q \qquad -\infty < x_1 < \infty, \quad 0 < x_2 < d.$$
$$\tau U_t = \Delta U - U + \frac{1}{\varepsilon} V^3/U^s$$

WITH $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$, $q > 0$, $s \geq 0$, $\frac{3q}{2} - (s+1) > 0$.

ASSUME $U_{x_2} = V_{x_2} = 0$ ON $x_2 = 0, d$.

(i) CONSIDER A solution $V(x_1)$, $U(x_1)$ of steady-state problem.
$$\varepsilon^2 V_{x_1 x_1} - V + V^3/U^q = 0 \qquad -\infty < x_1 < \infty.$$
$$U_{x_1 x_1} - U + \frac{1}{\varepsilon} V^3/U^s = 0$$

IN INNER REGION PUT $y = \varepsilon^{-1} x_1$, $\bar{V}(y) = V(\varepsilon y)$, $\bar{U}(y) = U(\varepsilon y)$.

WE OBTAIN THAT $\frac{1}{\varepsilon^2} \bar{U}'' - \bar{U} + \frac{1}{\varepsilon} \frac{\bar{V}^3}{\bar{U}^s} = 0 \rightarrow \bar{U} = \bar{U}_0 + \varepsilon \bar{U}_1 + \cdots$
$$\bar{V}'' - \bar{V} + \bar{V}^3/\bar{U}^q = 0 \rightarrow \bar{V} = \bar{V}_0 + \varepsilon \bar{V}_1 + \cdots$$

WE HAVE $\bar{U} = \bar{U}_0$ AND $\bar{V}_0'' - \bar{V}_0 + \bar{V}_0^3/\bar{U}_0^q = 0$. $\bar{U}_0$ CONSTANT TO BE FOUND.

PUT $\bar{V}_0 = \bar{U}_0^{q/2} w(y)$ SO THAT $w'' - w + w^3 = 0$, $-\infty < y < \infty$
$$w'(0) = 0, \quad w(0) > 0$$
$$w \to 0 \text{ AS } |y| \to \infty.$$
SO $w = \sqrt{2}\,\mathrm{sech}(y)$

NOW IN OUTER REGION $V \sim 0$ AND
$$U_{x_1 x_1} - U = -\frac{1}{\varepsilon} \frac{V^3}{U^s} \rightarrow -\bar{U}_0^{3q/2 - s} \int_{-\infty}^{\infty} w^3\, dy \;\; \delta(x_1)$$

THUS IN OUTER REGION
$$\left\{ \begin{array}{ll} U_{x_1 x_1} - U = -\bar{U}_0^{3q/2 - s} \int_{-\infty}^{\infty} w^3\, dy \;\; \delta(x_1) & \text{IN } -\infty < x_1 < \infty \\ U \to 0 \text{ AS } |x_1| \to \infty & \end{array} \right.$$

DEFINE $\qquad G_{x_1x_1} - G = -\delta(x_1)$

$$G \to 0 \quad \text{AS} \quad |x_1| \to \infty .$$

THE SOLUTION IS
$$G = \begin{cases} Ae^{-x_1} & \text{IF } x_1 > 0 \\ Ae^{x_1} & \text{IF } x_1 < 0 \end{cases}$$

NOW $\quad G_{x_1}\Big|_{0^+} - G_{x_1}\Big|_{0^-} = -1 \;\rightarrow\; -2A = -1 \;$ SO $\; A = +1/2 .$

THUS $\qquad G(x_1) = \frac{1}{2} e^{-|x_1|} .$

WE HAVE $\qquad U = U_0^{3\theta/2 - s} \int_{-\infty}^{\infty} w^3\,dy \;\; G(x_1) .$

NOW THE MATCHING CONDITION IS $\; U(0) = U_0 .$

HENCE, $\qquad U_0 = U_0^{3\theta/2 - s} \int_{-\infty}^{\infty} w^3\,dy \;\; G(0) \qquad$ WITH $\quad G(0) = 1/2$

SO
$$U_0^{3\theta/2 - (s+1)} = \frac{2}{\int_{-\infty}^{\infty} w^3\,dy} = \frac{2}{\sqrt{2}\,\pi} = \frac{\sqrt{2}}{\pi}$$

IN CONCLUSION, THE PULSE IS

$$(1) \begin{cases} V_e \sim U_0^{\theta/2}\, w(y) & y = x_1/\varepsilon \qquad w = \sqrt{2}\,\operatorname{sech} y \\ U_e \sim U_0 \dfrac{G(x_1)}{G(0)} = U_0\, e^{-|x_1|} \end{cases}$$

WHERE
$$U_0 = \left(\sqrt{2}/\pi\right)^{1/\left(3\theta/2 - (s+1)\right)} .$$

(ii) NOW WE LINEARIZE

$$V = V_e + e^{\lambda t}\,\Phi(x_1/\varepsilon)\,e^{imx_2} \qquad y = x_1/\varepsilon$$

$$U = U_e + e^{\lambda t}\,\Lambda(x_1)\,e^{imx_2}$$

WE OBTAIN FROM LINEARIZED PROBLEM

$$\lambda\Phi = \Phi'' - \Phi + \frac{3V_e^2}{U_e^q}\Phi - q\frac{V_e^3}{U_e^{q+1}}\Lambda(0) - \varepsilon^2 m^2\Phi \qquad -\infty < y < \infty$$

$$\tau\lambda\Lambda = \Lambda_{x_1x_1} - m^2\Lambda - \Lambda + \frac{1}{\varepsilon}\left[\frac{3V_e^2}{U_e^s}\Phi - \frac{V_e^3}{U_e^{s+1}}\Lambda\right]$$

THE FIRST EQUATION GIVES

$$(2) \qquad \Phi'' - \Phi + 3w^2\Phi - q\,U_0^{q/2-1}\,w^3\Lambda(0) = (\lambda + \varepsilon^2 m^2)\Phi.$$

NOW FROM 2ND EQUATION

$$\Lambda_{x_1x_1} - [1 + m^2 + \tau\lambda]\Lambda = -\frac{1}{\varepsilon}\left[\frac{3V_e^2}{U_e^s}\Phi - s\frac{V_e^3}{U_e^{s+1}}\Lambda\right]$$

IN THE OUTER REGION, THEN IN SENSE OF DELTA FUNCTIONS

$$\frac{1}{\varepsilon}\left(\frac{3V_e^2\Phi}{U_e^s} - \frac{V_e^3\Lambda}{U_e^{s+1}}\right) \longrightarrow \left(3U_0^{q-s}\int_{-\infty}^{\infty} w^2\Phi\,dy - s\Lambda(0)\,U_0^{3q/2-(s+1)}\int_{-\infty}^{\infty} w^3\,dy\right)\delta(x_1)$$

THUS

$$(3) \quad \begin{cases} \Lambda_{x_1x_1} - \varphi_\lambda^2\Lambda = -\left[3U_0^{q-s}\int_{-\infty}^{\infty} w^2\Phi\,dy - s\Lambda(0)\,U_0^{3q/2-(s+1)}\int_{-\infty}^{\infty} w^3\,dy\right)\delta(x_1) \\ \Lambda \to 0 \quad \text{AS} \quad |x_1| \to \infty \end{cases}$$

WHERE $\varphi_\lambda = \sqrt{1+m^2+\tau\lambda}$.

DEFINE $$G_{\lambda x_1 x_1} - \varphi_\lambda^2 G_\lambda = -\delta(x_1)$$

$$G \to 0 \quad \text{AS} \quad |x_1| \to \infty.$$

THEN $$G_\lambda(x_1) = \frac{1}{2\varphi_\lambda} e^{-\varphi_\lambda |x_1|}$$ IS EXACT SOLUTION

PROVIDED THAT $\mathrm{RE}(\varphi_\lambda) > 0$ (THIS BRANCH MUST BE CHOSEN).

THUS $$\Lambda(x_1) = \left[3\,\Pi_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi\, dy - s\Lambda(0)\, \Pi_0^{3q/2-(s+1)} \int_{-\infty}^{\infty} w^3\, dy \right] G_\lambda(x_1)$$

NOW EVALUATE AT $X = X_1$ AND SOLVE FOR $\Lambda(0)$:

$$\Lambda(0) = \left[3\,\Pi_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi\, dy - s\Lambda(0)\, 2 \right] G_\lambda(0).$$

HERE WE USED $\Pi_0^{3q/2-(s+1)} \int_{-\infty}^{\infty} w^3\, dy = 2$ FROM EQUILIBRIUM PROBLEM

THUS $$\Lambda(0)\left(1 + 2sG_\lambda(0)\right) = 3\,G_\lambda(0)\, \Pi_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi\, dy.$$

BUT $G_\lambda(0) = 1/2\varphi_\lambda$ SO $$\Lambda(0) = \frac{3\,G_\lambda(0)\, \Pi_0^{q-s}}{1 + 2G_\lambda(0)s} \int_{-\infty}^{\infty} w^2 \Phi\, dy$$

$$\Lambda(0) = \frac{\left(3\,\Pi_0^{q-s}/2\varphi_\lambda\right)}{1 + 1/\varphi_\lambda\, s} \int_{-\infty}^{\infty} w^2 \Phi\, dy$$

SO $$\Lambda(0) = \frac{3}{2}\, \frac{\Pi_0^{q-s}}{s + \varphi_\lambda} \int_{-\infty}^{\infty} w^2 \Phi\, dy$$

NOW PUT THIS INTO (2) SO THAT

$$\Phi'' - \Phi + 3w^2\Phi - \frac{3\sigma}{2}\frac{\Pi_0^{3\sigma/2-(s+1)}}{s+q_\lambda} w^3 \int_{-\infty}^{\infty} w^2 \Phi\, dy = \lambda \Phi + \varepsilon^2 m^2 \Phi$$

THEN USING $\Pi_0^{3\sigma/2-(s+1)} = 2 \Big/ \int_{-\infty}^{\infty} w^3 dy$ WE FINALLY get

$$L_0 \Phi - \chi w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi\, dy}{\int_{-\infty}^{\infty} w^3\, dy} = \lambda \Phi + \varepsilon^2 m^2 \Phi$$

$$L_0 \Phi \equiv \Phi'' - \Phi + 3w^2\Phi.$$

WITH $$\chi = \frac{3\sigma}{s+q_\lambda} = \frac{3\sigma}{s+\sqrt{1+m^2+\tau\lambda}}$$

(iii) RECALL THAT $L_0(w^2) = 3w^2$. NOW

$$L_0 \Phi = \chi w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi\, dy}{\int_{-\infty}^{\infty} w^3\, dy} + (\lambda + \varepsilon^2 m^2)\Phi$$

$$L_0 w^2 = 3w^2.$$

USE LAGRANGE'S IDENTITY

$$\int_{-\infty}^{\infty} \left(w^2 L_0 \Phi - \Phi L_0(w^2)\right) dy = 0$$

so $$\int_{-\infty}^{\infty} \left(\chi w^5 \frac{\int w^2 \Phi\, dy}{\int w^3 dy} + (\lambda + \varepsilon^2 m^2) w^2 \Phi - 3w^2\Phi \right) dy = 0.$$

so $$\int_{-\infty}^{\infty} w^2 \Phi\, dy \left(\chi \frac{\int_{-\infty}^{\infty} w^5\, dy}{\int_{-\infty}^{\infty} w^3\, dy} + (\lambda + \varepsilon^2 m^2) - 3 \right) = 0.$$

FOR EIGENFUNCTIONS FOR WHICH $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$,

THEN
$$\lambda = 3 - \varepsilon^2 m^2 - \chi \frac{\int_{-\infty}^{\infty} w^5 \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} \qquad \text{WITH} \quad w = \sqrt{2}\,\text{sech}\, y.$$

NOW WITH $\chi = \dfrac{3\delta}{S + \sqrt{1+m^2+\tau\lambda}}$ AND $\dfrac{\int_{-\infty}^{\infty} w^5 \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \dfrac{3}{2}$.

THUS
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9\delta}{2\left[S + \sqrt{1+m^2+\tau\lambda}\right]}$$

NOW LET $\tau = 0$. THEN WE WANT TO PLOT λ VS. m FOR

$$(*) \left\{ \lambda = 3 - \frac{9\delta}{2\left[S+\sqrt{1+m^2}\right]} - \varepsilon^2 m^2. \right.$$

NOW IF WE SET $\lambda = 0$ THEN $m_{up} \sim \sqrt{3}/\varepsilon$.

THE LOWER EDGE OF BAND IS AT

$$3 - \frac{9\delta}{2\left[S+\sqrt{1+m^2}\right]} \approx 0.$$

THUS
$$3\delta = 2\left[S + \sqrt{1+m^2}\right]$$

OR
$$\frac{3\delta}{2} = S + \sqrt{1+m^2} \quad \text{GIVES LOWER BAND.}$$

WE WRITE $\frac{3q}{2} - (s+1) = \sqrt{1+m^2} - 1 = \eta(m)$ $\eta(0) = 0,\ \eta'(m) > 0$

$\eta \to +\infty$ as $m \to \infty$.

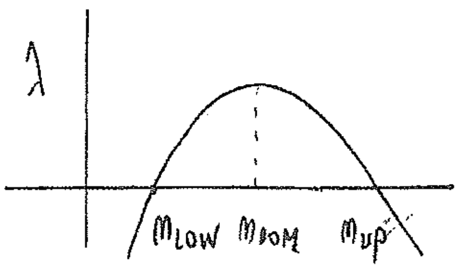
THIS HAS A ROOT IN $m > 0$ ONLY IF $\frac{3q}{2} - (s+1) > 0$.

WE CALCULATE $m_{low} = \sqrt{(3q/2 - s)^2 - 1} > 0$.

NOTICE $\lambda < 0$ IF $m >> O(1/\varepsilon)$

$\lambda < 0$ IF $m = 0$ WHEN $\frac{3q}{2} - (s+1) > 0$ SINCE $\lambda\Big|_{m=0} = \frac{3}{(s+1)}\left(s+1 - \frac{3q}{2}\right)$

THUS WE WILL GET



$m_{low} = \sqrt{\left(\frac{3q}{2} - s\right)^2 - 1}$

$m_{up} \doteq \sqrt{3}/\varepsilon$.

(A PLOT IS given BELOW FOR A PARAMETER SET $q = 2,\ s = 0$).

TO FIND MOST UNSTABLE MODE WRITE $\lambda = \lambda(m)$ AS

$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9q}{2}\left[s + \sqrt{1+m^2}\right]^{-1}$$

SET $$\frac{d\lambda}{dm} = -2\varepsilon^2 m + \frac{9q}{2}\left[s + \sqrt{1+m^2}\right]^{-2} m(1+m^2)^{-1/2} = 0$$

SO $$\frac{4\varepsilon^2}{9q} = \left[s + \sqrt{1+m^2}\right]^{-2}(1+m^2)^{-1/2}$$

NOW $$\frac{9q}{4\varepsilon^2} = \left[s + \sqrt{1+m^2}\right]^2 \sqrt{1+m^2}.$$

SET $B = \sqrt{1+m^2}$. THEN,

$$(S+B)^2 B = \frac{9\delta}{4\varepsilon^2}.$$

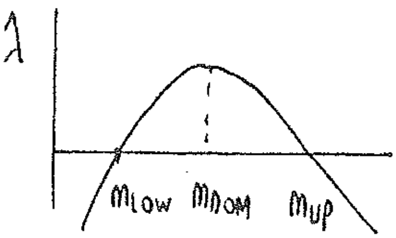
SINCE $\varepsilon \ll 1$ THEN $B \gg 1 \rightarrow \quad B^3 \sim \frac{9\delta}{4\varepsilon^2}$.

THUS $\quad B = \sqrt{1+m^2} \sim \left(\frac{9\delta}{4\varepsilon^2}\right)^{1/3}$.

FOR $m \gg 1$ $\quad \sqrt{1+m^2} \sim m$. THUS $\quad m_{DOM} \sim \left(\frac{9\delta}{4\varepsilon^2}\right)^{1/3}$

OR $\quad m_{DOM} \sim \left(\frac{9\delta}{4}\right)^{1/3} \varepsilon^{-2/3}$ AS $\varepsilon \to 0$.

OUR PICTURE IS



$m_{LOW} = O(1)$ AS $\varepsilon \to 0$

$m_{UP} \sim \sqrt{3}/\varepsilon$

$m_{DOM} \sim \left(\frac{9\delta}{4}\right)^{1/3} \varepsilon^{-2/3}$ AS $\varepsilon \to 0$.

NOW FOR THE NEUMANN PROBLEM ON $x_2 = 0, d$ WE NEED

$$m d = n\pi \qquad n = 0, 1, 2, 3, \ldots$$

WE HAVE <u>DISCRETE</u> VALUES $\quad m = \frac{n\pi}{d}$.

THUS $\forall d > 0$ independent OF ε, $\exists$ integers n inside the BAND OF INSTABILITY AS $\varepsilon \to 0$. THE STRIPE IS ALWAYS UNSTABLE TO BREAKUP INTO SPOTS FOR ANY $d > 0$, WITH d INDEPENDENT OF ε. ONLY FOR A THIN DOMAIN WITH $d = O(\varepsilon)$ CAN THE STRIPE BE STABLE.

FOR $q = 2$, $S = 0$, $\varepsilon = .02$ OUR ASYMPTOTIC RESULTS GIVE

$m_{LOW} \cong 2.828$ $\quad m_{UP} \cong 86.603$, $\quad m_{DOM} \cong 22.407$

FULL NUMERICAL RESULTS FOR $\lambda = \lambda(m)$ CONFIRM THIS PREDICTION.