GENERAL CONSIDERATIONS: INTERNAL LAYER SOLUTIONS

WE CONSIDER

$$U_t = \varepsilon^2 U_{xx} + Q(U, X) \qquad 0 < X < 1, \ t > 0$$

$$U(0,t) = S_-(0) \qquad U(1,t) = S_+(1)$$

WHERE $Q(U,X)$ HAS THE FOLLOWING PROPERTIES:

$$Q(0,X) = 0 \quad \forall X$$

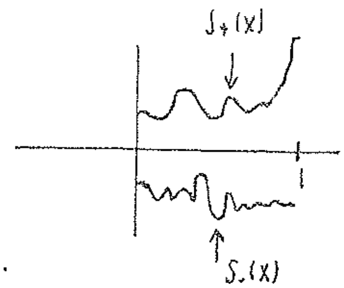
$$Q(S_-, X) = 0 \longrightarrow S_- = S_-(X) \ ; \quad Q(S_+, X) = 0 \longrightarrow S_+ = S_+(X)$$

WE ASSUME THAT

$$S_-(X) < 0 < S_+(X) \qquad \forall X \in [0,1]$$

AND THAT $\quad Q_U(S_\pm, X) < 0 \quad$ WITH $\quad Q_U(0, X) > 0.$

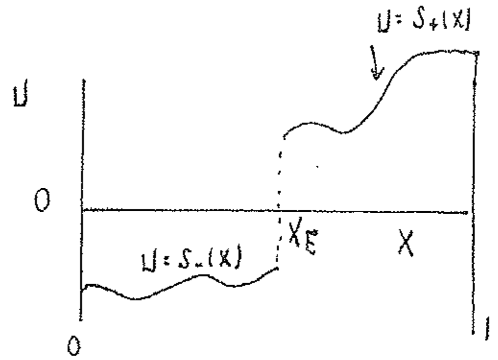
FOR INSTANCE $\quad Q = U\,(S_-(X) - U)(U - S_+(X))$



ASSUME THAT $\exists\ X_E$ SUCH THAT $\displaystyle\int_{S_-(X_E)}^{S_+(X_E)} Q(U, X_E)\, dU = 0.$

THEN, WE WILL TRY TO CONSTRUCT A LAYER NEAR $X = X_E$.

THE OUTER SOLUTIONS THAT ARE MATCHABLE ARE $U = S_+(X)$ AND $U = S_-(X)$.



IN THE INNER REGION WE EXPAND $y = \varepsilon^{-1}(X - X_0)$ WITH

$$U = U_0(y) + \varepsilon U_1(y) + \dots$$

$$X_0 = X_E + \varepsilon X_1 + \dots$$

WE WANT $U(0) = 0$ TO DEFINE LOCATION OF LAYER AS 0 LEVEL SET.

WE OBTAIN

$$U_{0yy} + \varepsilon U_{1yy} + Q(U_0 + \varepsilon U_1, X_0 + \varepsilon y) = 0$$

WITH $X_0 = X_E + \varepsilon X_1$ WE OBTAIN FROM TAYLOR SERIES

$$0 = U_{0yy} + \varepsilon U_{1yy} + Q(U_0, X_E) + \varepsilon U_1 Q_U(U_0, X_E) + \varepsilon(X_1 + y) Q_X(U_0, X_E)$$

COLLECTING POWERS OF ε:

$$U_{0yy} + Q(U_0, X_E) = 0$$

$$U_{1yy} + Q_U(U_0, X_E) U_1 = -(X_1 + y) Q_X(U_0, X_E).$$

NOW THE OUTER SOLUTION HAS

$$S_\pm(X) = S_\pm(X_E + \varepsilon(X_1 + y)) = S_\pm(X_E) + \varepsilon(X_1 + y) S_\pm'(X_E) + \dots$$

THEREFORE OUR TWO PROBLEMS READ:

$$(1)\quad \begin{cases} U_{0yy} + Q(U_0, X_E) = 0 & -\infty < y < \infty \\ U_0 \to S_+(X_E) & \text{AS } y \to \infty \\ U_0 \to S_-(X_E) & \text{AS } y \to -\infty \\ U_0(0) = 0 \end{cases}$$

$$(2)\quad \begin{cases} LU_1 \equiv U_{1yy} + Q_U(U_0, X_E) U_1 = -(X_1 + y) Q_X(U_0, X_E) \\ U_1 \to (X_1 + y) S_+'(X_E) & \text{AS } y \to \infty \\ U_1 \to (X_1 + y) S_-'(X_E) & \text{AS } y \to -\infty \\ U_1(0) = 0. \end{cases}$$

$\exists$ A UNIQUE SOLUTION TO (1) WITH

$$U_0' \to 0 \text{ exponentially as } y \to \pm\infty.$$

NOW $L U_0' = 0$. WE MULTIPLY AND USE LAGRANGE'S IDENTITY

$$\int_{-\infty}^{\infty} U_1 \, L U_0' \, dy - \int_{-\infty}^{\infty} U_0' L U_1 \, dy = \lim_{y\to\infty} \left[U_1 U_0'' - U_0' U_1' \right].$$

NOW $U_1 = O(|y|)$ AS $y \to \pm\infty$, BUT $U_0', U_0'' \to 0$ exponentially.

HENCE $\displaystyle\int_{-\infty}^{\infty} U_0' \, L U_1 \, dy = 0$

$$\longrightarrow \int_{-\infty}^{\infty} (X_1 + y) \, Q_X(U_0, X_E) \, U_0' \, dy = 0.$$

$$\longrightarrow X_1 \int_{-\infty}^{\infty} Q_X(U_0, X_E) \, U_0' \, dy = -\int_{-\infty}^{\infty} y \, Q_X(U_0, X_E) \, U_0' \, dy$$

$$X_1 \int_{S_-(X_E)}^{S_+(X_E)} Q_X(U_0, X_E) \, dU_0 = -\int_{-\infty}^{\infty} y \, Q_X(U_0, X_E) \, U_0' \, dy.$$

NOTICE THAT IF U_0 IS ODD AND $Q_X(U_0; \;) = + Q_X(-U_0; \;)$

THEN $X_1 = 0$.

THIS DETERMINES X_1. TO MAKE THE SOLUTION UNIQUE,

WE WRITE $\quad U_1(y) = U_{1p}(y) + \alpha U_0' \qquad \alpha$ (arbitrary)

where U_{1p} IS A PARTICULAR SOLUTION. SETTING $U_1(0) = 0$ GIVES

$$\alpha = -U_{1p}(0) / U_0'(0).$$

STABILITY OF THIS SOLUTION

WE LET $U_\varepsilon = U_0 + \varepsilon U_1 + \ldots$ AND $y = \frac{X - X_0(\varepsilon)}{\varepsilon}$ WITH $X_0(\varepsilon) = X_E + \varepsilon X_1 + \ldots$

WE LINEARIZE TO OBTAIN $U = U_\varepsilon + e^{\lambda t}\phi$ SO THAT

$$L_\varepsilon \phi = \phi_{yy} + Q_U^\varepsilon \phi = \lambda \phi$$

$$\phi \longrightarrow 0 \text{ AS } |y| \longrightarrow \infty$$

HERE WE HAVE $Q_U^\varepsilon = Q_U\left[U_\varepsilon(y);\ X_0(\varepsilon) + \varepsilon y\right]$.

WE WRITE $Q_U^0 \equiv Q_U(U_0;\ X_E)$.

THE EIGENVALUE PROBLEM IS

$$L_0 \phi = \phi_{yy} + Q_U^0 \phi = (Q_U^0 - Q_U^\varepsilon)\phi + \lambda \phi$$

NOW $L_0 U_0' = 0$ WITH $(U_{0y})_{yy} + Q_U(U_0; X_E) U_{0y} = 0$

THEREFORE MULTIPLYING BY U_0' WE GET

$$0 = \int_{-\infty}^{\infty} (U_0' L_0 \phi - \phi L_0 U_0')\, dy = \int_{-\infty}^{\infty} U_0' (Q_U^0 - Q_U^\varepsilon)\phi + \lambda \int_{-\infty}^{\infty} \phi U_0'\, dy.$$

NOW TO LEADING ORDER $\phi \sim U_0' + O(\varepsilon)$ AND WE EXPAND

$$Q_U^\varepsilon - Q_U^0 = \varepsilon Q_{UU}^0 U_1 + \varepsilon Q_{XU}^0 (X_1 + y) + \ldots$$

THIS YIELDS THAT $\displaystyle\int_{-\infty}^{\infty} \phi U_0' (Q_U^0 - Q_U^\varepsilon)\, dy = -\varepsilon \int_{-\infty}^{\infty} \phi U_0' \left(Q_{UU}^0 U_1 + (X_1 + y) Q_{UX}^0 \right) dy$

THIS YIELDS WITH $\phi \sim U_0'$ THAT

$$(*) \qquad \lambda \int_{-\infty}^{\infty} (U_0')^2\, dy \sim \varepsilon \int_{-\infty}^{\infty} (U_0')^2 \left[Q_{UU}^0 U_1 + (X_1 + y) Q_{UX}^0 \right] dy.$$

NOW RECALL FROM THE CONSTRUCTION OF THE EQUILIBRIUM SOLUTION THAT $\quad U_{1yy} + Q_U^0 U_1 = -(X_1 + y) Q_X^0$

WE DIFFERENTIATE WRT y TO OBTAIN

$$L U_{1y} \equiv (U_{1y})_{yy} + Q_u^0 (U_{1y}) = -\left[Q_{uu}^0 U_{0y} U_1 + (X_1 + y) Q_{uX}^0 U_{0y} \right] - Q_X^0$$

BUT $L U_{0y} = 0$ SO THAT THE SOLVABILITY CONDITION YIELDS:

$$\int_{-\infty}^{\infty} U_0' \, L U_{1y} \, dy = 0 \longrightarrow \int_{-\infty}^{\infty} Q_X^0 U_{0y} \, dy = -\int_{-\infty}^{\infty} (U_0')^2 \left(Q_{uu}^0 U_1 + (X_1 + y) Q_{uX}^0 \right) dy.$$

THIS YIELDS THE EIGENVALUE FROM (*) ON PREVIOUS PAGE

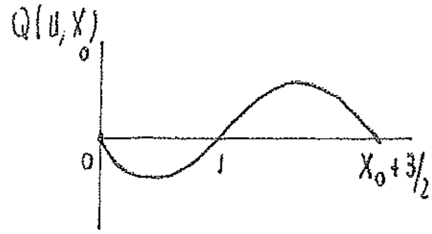
$$\lambda \sim -\varepsilon \frac{\int_{-\infty}^{\infty} Q_X^0 U_{0y} \, dy}{\int_{-\infty}^{\infty} (U_{0y})^2 \, dy} \sim -\varepsilon \frac{\int_{S_-(X_E)}^{S_+(X_E)} Q_X^0 [U_0, X_E] \, dU_0}{\int_{-\infty}^{\infty} (U_{0y})^2 \, dy}.$$

WE CONCLUDE THAT $\lambda = O(\varepsilon)$ AND THAT WE HAVE STABILITY IFF $\displaystyle\int_{S_-(X_E)}^{S_+(X_E)} Q_X [U_0, X_E] \, dU_0 > 0$.

EXAMPLE $\quad U_t = \varepsilon^2 U_{XX} + Q(U,X)$

$$U(0,t) = 0, \qquad U(1,t) = 5/2$$

$$Q(U,X) = U(1-U)\left(U - (X + 3/2)\right).$$



EQUILIBRIUM $\quad$ THE OUTER SOLUTION IS

$$U_0 = \begin{cases} 0, & 0 < X < X_E \\ X + 3/2, & X_E < X < 1 \end{cases} \qquad \begin{aligned} S_-(X) &= 0 \\ S_+(X) &= X + 3/2 \end{aligned}$$

IFF $X_E = 1/2$ THEN $\displaystyle\int_{S_-(X_E)}^{S_+(X_E)} U(1-U)(U-2)\,dU = \int_0^2 U(1-U)(U-2)\,dU = 0.$

THUS $X_E = 1/2$ IS THE EQUILIBRIUM POSITION.

WE THEN CALCULATE

$$Q_X(U_0, X) = -U(1-U) = U(U-1)$$

WE CALCULATE $\displaystyle\int_0^2 Q_X(U_0, X_E)\,dU_0 = \int_0^2 U(U-1)\,dU = \left(\frac{U^3}{2} - \frac{U^2}{2}\right)\Big|_0^2 = 2 > 0.$

$S_-(X_E) = 0, \; S_+(X_E) = 2,$

THUS THE SMALL EIGENVALUE $\lambda = O(\varepsilon)$ SATISFIES $\lambda < 0$

AND HENCE THE EQUILIBRIUM SOLUTION IS STABLE.

DYNAMICS $\quad$ WE SUPPOSE THAT WE HAVE A TRANSITION LAYER THAT MOVES IN TIME FOR THE TIME-DEPENDENT PROBLEM:

$$U(X,\tau) = \bar{U}(Z,\tau) \quad \text{WITH} \quad \tau = \varepsilon t, \quad Z = \frac{X - m(\tau)}{\varepsilon}$$

NOTICE THE SLOW-TIME SCALE $\tau = O(\varepsilon)$ IS CHOSEN ON THE ORDER OF THE SMALL EIGENVALUE $\lambda = O(\varepsilon)$.

WE CALCULATE $\quad U_t = \varepsilon \bar{U}_\tau - \dot{m} \bar{U}_Z$

THEN
$$\varepsilon \bar{U}_\tau - \dot{m} \bar{U}_Z = \bar{U}_{ZZ} + Q(\bar{U}, m) + Q_x(\bar{U}, m)\, \varepsilon Z + \dots$$

THIS LEADS TO THE NONLINEAR EIGENVALUE PROBLEM

$$\begin{cases} \bar{U}_{ZZ} + \dot{m}\, \bar{U}_Z + Q(\bar{U}, m) = 0, & -\infty < Z < \infty \\ \bar{U} \to S_+(m) = m + 3/2 & \text{AS } Z \to +\infty \\ \bar{U} \to S_-(m) = 0 & \text{AS } Z \to -\infty \end{cases}$$

$\bar{U}(0) = 1$ FIXES THE TRANSLATION INVARIANCE AND "FRONT" LOCATION

RECALL THAT $\quad Q(U) = U(1-U)(U - S_+(m))$ WITH $S_+ > 1 \;\; \forall m$ IN $0 < m < 1$.

<u>LEMMA</u> CONSIDER
$$(**)\begin{cases} \bar{U}'' - c\bar{U}' + f(\bar{U}) = 0 & \text{WITH } -\infty < Z < \infty \\ \bar{U}(-\infty) = U_1, \quad \bar{U}(\infty) = U_3 & \text{WITH } f(U) = (\bar{U} - U_1)(\bar{U} - U_2)(\bar{U} - U_3) \end{cases}$$

AND $U_1 < U_2 < U_3$. THEN THE SOLUTION WITH $\bar{U}(0) = (U_1 + U_3)/2$ IS

$$\bar{U}(Z) = \frac{U_1 + U_3 \exp(\lambda Z)}{1 + \exp(\lambda Z)} \quad \text{WITH } \lambda = \frac{U_3 - U_1}{\sqrt{2}} \quad \text{AND} \quad c = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right].$$

<u>PROOF</u> CONSIDER $\quad \bar{U}' = +B(\bar{U} - U_1)(\bar{U} - U_3)$. THEN DIFFERENTIATE

$$\bar{U}'' = +B(2\bar{U} - U_1 - U_3)\bar{U}' = B^2(2\bar{U} - U_1 - U_3)(\bar{U} - U_1)(\bar{U} - U_3).$$

WE CALCULATE

$$\begin{aligned} L(\bar{U}) \equiv \bar{U}'' - c\bar{U}' + f(\bar{U}) &= B^2(2\bar{U} - U_1 - U_3)(\bar{U} - U_1)(\bar{U} - U_3) - cB(\bar{U} - U_1)(\bar{U} - U_3) \\ &\quad + (\bar{U} - U_1)(\bar{U} - U_3)(\bar{U} - U_2) \\ &= (\bar{U} - U_1)(\bar{U} - U_3)\left[B^2(2\bar{U} - U_1 - U_3) - cB - (\bar{U} - U_2)\right] \end{aligned}$$

NOW CHOOSE B^2 AND c TO ELIMINATE TERM IN SQUARE BRACKET.

CHOOSE $\quad 2B^2 = 1 \quad$ AND $\quad -B^2(U_1 + U_3) - cB + U_2 = 0$.

THUS
$$(+)\begin{cases} cB = U_2 - B^2(U_1 + U_3). \\ 2B^2 = 1. \end{cases}$$

WE CONCLUDE THAT IF V SATISFIES

$$(++) \quad V' = B(V - U_1)(V - U_3)$$

THEN V ALSO SATISFIES (**) PROVIDED THAT

$$CB = U_2 - B^2(U_1 + U_3), \quad 2B^2 = 1.$$

IF WE WANT $V \to U_3$ AS $Z \to +\infty$ WE MUST HAVE $B < 0$. THUS,

$B = -1/\sqrt{2}$ AND $C = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right]$ IS THE WAVE SPEED.

WE INTEGRATE (++) TO GET

$$\frac{dV}{(V - U_1)(V - U_3)} = B\,dz \quad \text{WITH} \quad B = -1/\sqrt{2}.$$

THUS $$\frac{1}{U_3 - U_1}\left(\frac{1}{V - U_3} - \frac{1}{V - U_1}\right) dV = B\,dz$$

SINCE $U_1 < V < U_3$ THEN

$$\frac{1}{U_3 - U_1}\log\left(\frac{U_3 - V}{V - U_1}\right) = B(Z - Z_0)$$

SO $$\log\left(\frac{U_3 - V}{V - U_1}\right) = (U_3 - U_1)B(Z - Z_0).$$

OR $$\frac{U_3 - V}{V - U_1} = \exp\Big((U_3 - U_1)B(Z - Z_0)\Big).$$

WE GET $$V(z) = \frac{U_3 + \exp(\lambda_0(z - z_0))\,U_1}{1 + \exp(\lambda_0(z - z_0))} \qquad \lambda_0 = (U_3 - U_1)B = -\frac{(U_3 - U_1)}{\sqrt{2}}.$$

OR EQUIVALENTLY

$$V(z) = \frac{U_1 + U_3\exp(\lambda(z - z_0))}{1 + \exp(\lambda(z - z_0))} \qquad \lambda = -\lambda_0 = \frac{U_3 - U_1}{\sqrt{2}},$$

APPLYING THIS RESULT TO OUR PROBLEM WE HAVE

$$c = -\dot{m}, \quad U_1 = 0, \quad U_2 = 1, \quad U_3 = m + 3/2.$$

THUS $\quad c = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right]$ IS

$$\frac{dm}{d\tau} = -\frac{\sqrt{2}}{2}\left[m - 1/2\right] \quad \text{WITH} \quad m(0) = m_0 \quad \text{INITIAL CONDITION.}$$

WE RECALL THAT WITH $\tau = \varepsilon t$, $m = \frac{1}{2} + \left(m_0 - \frac{1}{2}\right)e^{-\varepsilon t/\sqrt{2}} \to 1/2$ AS $t \to \infty$.

SLOW DRIFT TO EQUILIBRIUM STATE $m_{EQ} = 1/2$ AS $t \to \infty$ WITH $O(\varepsilon)$ speed.

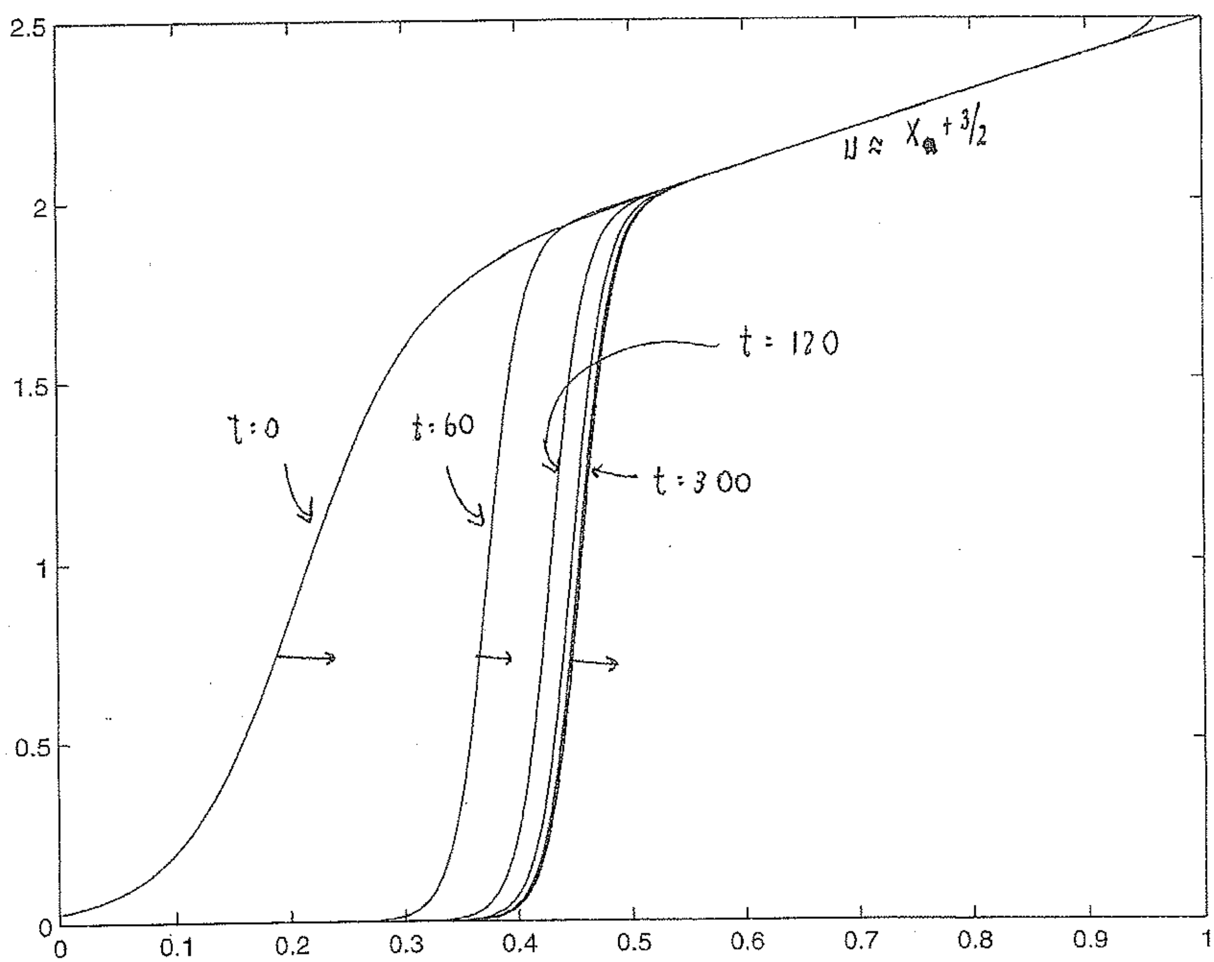
FULL NUMERICAL SOLUTION

$$U_t = \varepsilon^2 U_{xx} + F(U,X) \qquad F(U,X) = U(1-U)\left(U-(X+3/2)\right).$$

$$U(0,t) = 0 \qquad U(1,t) = 5/2$$

$$U(X,0) = \frac{(X+3/2)}{2}\left[1 + \tanh\left((X-.20)/.10\right)\right]$$

$$\varepsilon = .02$$



DEFINE $X_0(t)$ BY

$$U(X,t) = \frac{1}{2}\left[X_0 + 3/2\right] \rightarrow 1/2 \text{ of the jump.}$$

THEN $U[X_0,t] = \frac{1}{2}\left[X_0 + 3/2\right]$

ANALYTICALLY CAN SHOW THAT $\dfrac{dX_0}{d\tau} = -\dfrac{1}{\sqrt{2}}\left[X_0 - 1/2\right] \qquad \tau = \varepsilon t$

SO $X_0 \rightarrow 1/2$ AS $\tau \rightarrow \infty$

INTERNAL LAYERS AND SPATIAL HETEROGENEITY

PROBLEM DIFFUSIVE WAVE PROPAGATION IN A LONG CHANNEL OF VARIABLE CROSS-SECTIONAL AREA $A(X)$ IS MODELED BY A PDE OF THE FORM

$$U_t = \frac{\varepsilon^2}{A(X)} \left[A(X)\, U_X \right]_X + h(U), \qquad 0 < X < 1, \quad t > 0$$

$$U(0,t) = 0, \quad U(1,t) = 2 \qquad h(U) = U(1-U)(U-2)$$

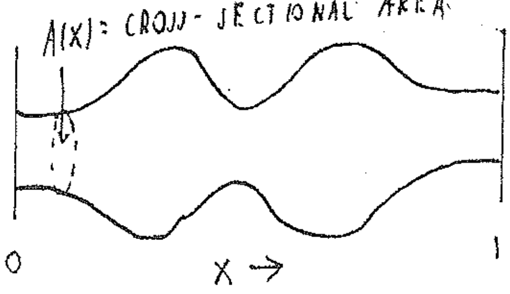
WITH $A(X) > 0$, AND $\varepsilon \to 0^+$. ASSUME THAT $U(X,0) = f(X)$ WITH $f(0) = 0$, $f(1) = 2$ AND $f' > 0$.

(i) SHOW THAT THIS PROBLEM HAS A SLOWLY EVOLVING SHOCK SOLUTION OF THE FORM

$$U(X,t) \sim \mathcal{U}\left[\varepsilon^{-1}(X - m(\tau))\right] \qquad \tau = \varepsilon^2 t$$

WHERE $\mathcal{U}(z)$ IS A HETEROCLINIC CONNECTION, AND $m(\tau)$ SATISFIES AN ODE TO BE FOUND.

(ii) FOR A DOMAIN WITH CROSS-SECTIONAL AREA AS SHOWN BELOW, WHERE ARE THE STABLE EQUILIBRIA OF THE ODE FOR $m(\tau)$? WHAT DO YOU THINK THE TERM "FRONT-PINNING" REFERS TO?



SOLUTION

(i) WE WRITE $\frac{\varepsilon^2}{A}(A u_x)_x = \varepsilon^2 u_{xx} + \varepsilon^2 \frac{A'(x)}{A(x)} u_x$.

THE PDE IS THEN

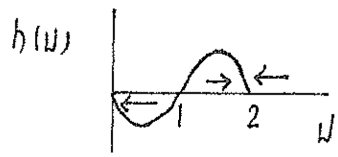
$$u_t = \varepsilon^2 u_{xx} + \varepsilon^2 \frac{A'(x)}{A(x)} u_x + h(u), \quad 0<x<1 \qquad h(u) = u(1-u)(u-2)$$

$$u(0,t) = 0, \quad u(1,t) = 2$$

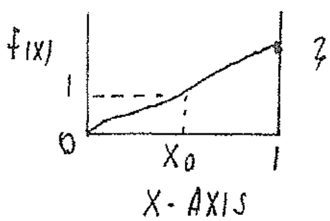
$$u(x,0) = f(x) \quad \text{WITH} \quad f(0)=0,\ f(1)=2 \ \text{AND} \ f'(x)>0.$$

WE FIRST CONSIDER INITIAL TRANSIENT DESCRIBING THE FORMATION OF A FRONT SOLUTION. WE LET $\varepsilon \ll 1$ AND CONSIDER

$$\text{ODE} \left\{ \begin{array}{l} u_t = h(u) \\ u(x,0) = f(x) \end{array} \right.$$



WITH $f(x)$ AS SHOWN BELOW

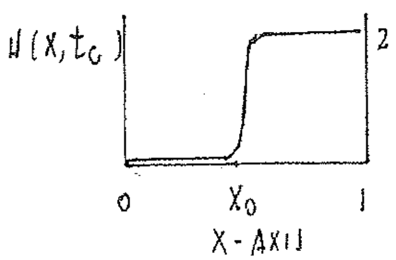


DEFINE x_0 TO BE THE UNIQUE POINT WHERE $f(x_0) = 1$

FROM THE ODE, WE CONCLUDE THAT

$$u \to \left\{ \begin{array}{lll} 0 & \text{IF } 0<x<x_0 & \text{FOR } t \text{ large} \\ 2 & \text{IF } x_0<x<1 & \text{FOR } t \text{ LARGE.} \end{array} \right.$$

THUS, AT THE END OF THE TRANSIENT PERIOD WE HAVE A SOLUTION WITH A "NEAR" DISCONTINUITY AT x_0, WHICH MUST BE SMOOTHED OUT BY $\varepsilon^2 u_{xx} + \varepsilon^2 A' u_x / A$ TERM.



$t_0 = O(1)$, x_0 IS DEFINED BY $f(x_0) = 1$.

NEXT, WE LOOK FOR A SLOWLY EVOLVING FRONT SOLUTION

OF THE FORM

$$u = U\left[\varepsilon^{-1}(x - m(\tau))\right] \quad \text{WITH} \quad \tau = \varepsilon^2 t, \quad y = \varepsilon^{-1}(x - m(\tau))$$

AND $m(0) = x_0$ IS INITIAL CONDITION FROM END OF TRANSIENT PERIOD

AND $U(\infty) = 2, \quad U(-\infty) = 0, \quad U(0) = 1.$

THE EQUATION IS TRANSFORMED VIA:

$$u_x = \varepsilon^{-1} U_y \qquad u_{xx} = \varepsilon^{-2} U_{yy}$$

$$u_t = -\varepsilon^{-1} \frac{dm}{d\tau}\frac{d\tau}{dt} U_y = -\varepsilon^{-1}\frac{dm}{d\tau}\varepsilon^2 U_y = -\varepsilon \mathring{m} U_y \qquad \text{WITH} \quad \mathring{m} = dm/d\tau$$

ALSO,

$$A'(x)/A(x) = A'(m(\tau) + \varepsilon y)/A(m(\tau)+\varepsilon y) = A'(m)/A(m).$$

THE ORIGINAL PDE TRANSFORMS IN THE INNER REGION TO

$$(*)\quad \begin{cases} -\varepsilon \mathring{m} U_y = U_{yy} + \varepsilon \dfrac{A'(m)}{A(m)} U_y + h(U), \quad -\infty < y < \infty \\ U \to 0 \quad \text{AS} \quad y \to -\infty \\ U \to 2 \quad \text{AS} \quad y \to +\infty \\ U(0) = 1 \quad \text{DEFINES THE FRONT} \end{cases}$$

NOW WE EXPAND $U = U_0 + \varepsilon U_1 + \dots$, THIS YIELDS THAT,

$$(1)\quad \begin{cases} U_{0yy} + h(U_0) = 0, \quad -\infty < y < \infty \\ U_0(-\infty) = 0, \quad U_0(\infty) = 2, \quad U_0(0) = 1 \end{cases}$$

$$(2)\quad \begin{cases} U_{1yy} + h'(U_0)U_1 = -\mathring{m} U_{0y} - A'(m) U_{0y}/A(m), \quad -\infty < y < \infty \\ U_1 \to 0 \quad \text{AS} \quad |y| \to \infty \\ U_1(0) = 0. \end{cases}$$

THE SOLUTION TO THE U_0 PROBLEM IS SIMPLY

$$U_0 = 1 + W$$

WHERE

$$W'' + W - W^3 = 0$$

$$W(\infty) = 1, \quad W(-\infty) = -1, \quad W(0) = 0.$$

PUT $W = \tanh(By) \longrightarrow [B\,\mathrm{sech}^2(By)]' + \tanh(By) - \tanh^3 By = 0$

$$-2B\,\mathrm{sech}^2\tanh + \tanh(1-\tanh^2) = 0$$

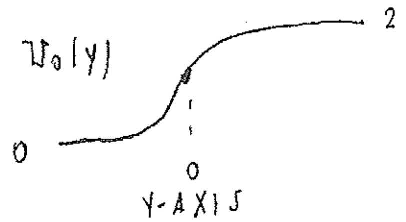
$$-2B^2\,\mathrm{sech}^2\tanh + \tanh\,\mathrm{sech}^2 = 0 \;\rightarrow\; 2B^2 = 1$$

SO $B = 1/\sqrt{2}$.

THEN, $U_0(y) = 1 + \tanh\left(y/\sqrt{2}\right)$

THE FUNCTION U_0 IS THE <u>HETEROCLINIC SOLUTION</u>.

IT LOOKS LIKE



NOW NOTICE THAT THE U_1 EQUATION CAN BE WRITTEN IN OPERATOR FORM AS

$$(3)\quad \begin{cases} \mathcal{L}U_1 \equiv U_{1zz} + h'(U_0)U_1 = -U_{0y}\left(\dot{m} + A'(m)/A(m)\right), & -\infty < y < \infty \\ U_1 \rightarrow 0 \text{ AS } |y| \rightarrow \infty. \end{cases}$$

NOW DIFFERENTIATE WRT y THE U_0 PROBLEM TO GET

$$(4)\quad \begin{cases} \mathcal{L}U_{0y} = (U_{0y})_{yy} + h'(U_0)U_{0y} = 0 \\ U_{0y} \rightarrow 0 \text{ AS } |y| \rightarrow \infty. \end{cases}$$

THUS, WE REQUIRE THAT THE PROBLEM FOR U_1 MUST SATISFY A SOLVABILITY CONDITION THAT

$$\int_{-\infty}^{\infty} -U_{0y}\left(\dot{m} + A'(m)/A(m)\right)U_{0y}\,dy = 0.$$

THE DERIVATION OF THIS CONDITION IS THAT BY LAGRANGE'S IDENTITY

$$\int_{-\infty}^{\infty} (U_{0y}\, \mathcal{L} U_1 - U_1 \mathcal{L} U_{0y})\, dy = (U_{0y} U_{1y} - U_1 U_{0yy}) \Big|_{-\infty}^{\infty}$$

SINCE $U_1 \to 0$ AS $|y| \to \infty$ AND $U_0 \to$ CONSTANTS AS $|y| \to \infty$, AND $\mathcal{L} U_{0y} = 0$

THIS YIELDS
$$\int_{-\infty}^{\infty} U_{0y} \mathcal{L} U_1\, dy = \int_{-\infty}^{\infty} U_{0y} \left[- U_{0y} \left(\dot{m} + A'(m)/A(m) \right) \right] dy = 0.$$

HENCE
$$\left[\dot{m} + A'(m)/A(m) \right] \int_{-\infty}^{\infty} (U_{0y})^2\, dy = 0.$$

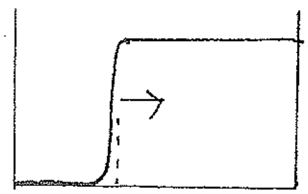
WE CONCLUDE THAT

$$(5) \qquad \frac{dm}{d\tau} = - \frac{A'(m)}{A(m)} \qquad \text{WITH} \quad m(0) = X_0 \quad \text{AND} \quad \tau = \varepsilon^2 t.$$

THE TIME-DEPENDENT SLOW MOVING FRONT SOLUTION SATISFIES

$$(6) \qquad U \sim 1 + \tanh\left[\frac{\varepsilon^{-1}}{\sqrt{2}} \left[X - m(\tau) \right] \right] + O(\varepsilon).$$

THE PICTURE IS



(ii) THE EQUILIBRIA OF THE ODE ARE WHERE

$$A'(m_e) = 0$$

LINEARIZING THE ODE NEAR AN EQUILIBRIUM POINT, WE WRITE

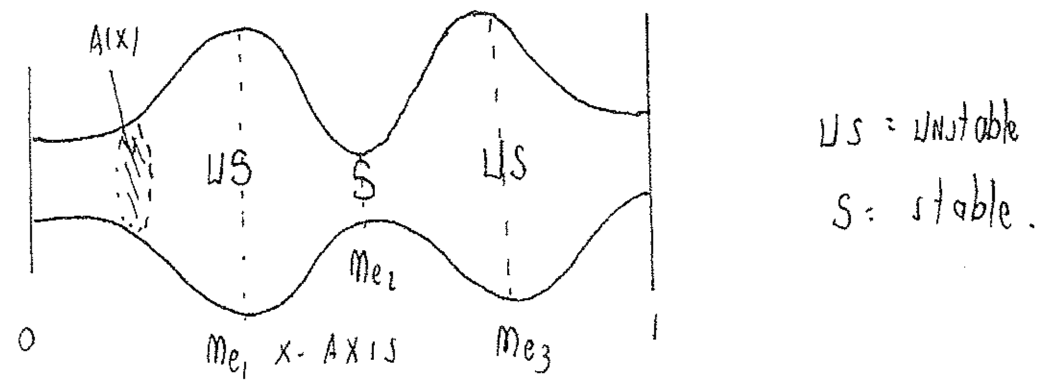
$$m = m_e + \zeta \qquad \zeta \ll 1 \quad \text{TO GET}$$
$$d\zeta/d\tau = - A''(m_e)/A(m_e)\ \zeta.$$

SINCE $A(m_e) > 0$ WE CONCLUDE THAT

m_e is stable when $A''(m_e) > 0 \rightarrow$ LOCAL MIN OF CROSS-SECTIONAL AREA

m_e IS UNSTABLE WHEN $A''(m_e) < 0 \rightarrow$ LOCAL MAX OF CROSS-SECTION.

THUS IF THE CHANNEL CROSS-SECTION (SIDEVIEW) LOOKS LIKE



<u>NOTE:</u>
- EQUILIBRIUM ARE AT DOTTED LINES AT $x = m_{e_1}, m_{e_2}, m_{e_3}$
- m_{e_2} is stable (LOCAL MIN OF $A(x)$), BUT m_{e_1} AND m_{e_3} ARE UNSTABLE

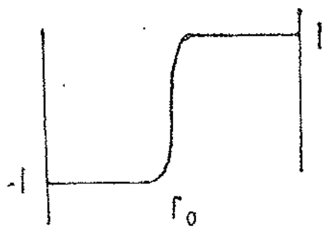
- WE CONCLUDE THAT IF $m(0)$ SATISFIES $m_{e_1} < m(0) < m_{e_2}$ THEN FOR THE ODE $dm/d\tau = -A'(m)/A(m)$ WE HAVE THAT $m \longrightarrow m_{e_2}$ AND THE WAVE IS <u>PINNED</u> AS $t \to \infty$ TO THE LOCAL MINIMUM OF $A(m)$. THE WAVE CANNOT ENTER INTO THE REGION $m > m_{e_2}$.

PROBLEM. WE CONSIDER FIRST THE STEADY-STATE SOLUTION

$$\varepsilon^2\left(u_{rr} + \frac{1}{r}u_r\right) + F(r)\,Q(u) = 0, \quad 0 < r < 1$$

$$u_r = 0 \text{ ON } r = 0, \quad u_r = 0 \text{ ON } r = 1$$

WITH $Q(u) = 2(u - u^3)$. WE LOOK FOR A SOLUTION AS SHOWN. SINCE $Q'(\pm 1) < 0$ IMPLIES THAT $u = \pm 1$ CAN BE USED FOR THE OUTER SOLUTION.



WE NOW CONSTRUCT A LAYER NEAR $r = r_0$.

WE LET $V(\rho) = u[r_0 + \varepsilon\rho]$ AND $\rho = \dfrac{r - r_0}{\varepsilon}$

WE SUBSTITUTE TO OBTAIN

$$V_{\rho\rho} + \frac{\varepsilon V_\rho}{r_0 + \varepsilon\rho} + F(r_0 + \varepsilon\rho)\,Q(V) = 0.$$

WE THEN EXPAND

$$V = V_0 + \varepsilon V_1 + \cdots$$

THIS YIELDS THAT

$$\begin{cases} V_0'' + F(r_0)\,Q(V_0) = 0, & -\infty < \rho < \infty \\ V_0(+\infty) = 1, \quad V_0(-\infty) = -1, \quad V_0(0) = 0 \end{cases}$$

$$\begin{cases} LV_1 \equiv V_1'' + F(r_0)\,Q'(V_0)\,V_1 = -\rho F'(r_0)\,Q(V_0) - V_{0\rho}/r_0 \\ V_1(+\infty) = V_1(-\infty) = 0, \quad V_1(0) = 0. \end{cases}$$

NOW $\exists$ A UNIQUE HETEROCLINIC SOLUTION FOR V_0. LET $S = [F(r_0)]^{1/2}\rho$

SO THAT $V_{0\rho} = V_{0S}\,[F(r_0)]^{1/2}$. HENCE $V_{0SS} + Q(V_0) = 0$, $V_0 = \tanh S$

THIS YIELDS THAT $V_0 = \tanh\left[(F(r_0))^{1/2}\rho\right]$

NOW THE SOLVABILITY CONDITION FOR V_1 IS OBTAINED FROM

$$\mathcal{L} V_0' = 0.$$

THEN WE CONCLUDE $\frac{1}{r_0}\int_{-\infty}^{\infty} V_{0\rho}^2\, d\rho + F'(r_0)\int_{-\infty}^{\infty} \rho\, Q(V_0) V_{0\rho}\, d\rho = 0.$

BUT $\frac{1}{F(r_0)} V_{0\rho\rho} = -Q(V_0) \rightarrow \frac{1}{r_0}\int_{-\infty}^{\infty} V_{0\rho}^2\, d\rho - \frac{F'(r_0)}{F(r_0)}\int_{-\infty}^{\infty} \rho\, V_{0\rho} V_{0\rho\rho}\, d\rho = 0$

NOW $$\int_{-\infty}^{\infty} \rho\, V_{0\rho} V_{0\rho\rho}\, d\rho = \int_{-\infty}^{\infty} \frac{\rho}{2}\frac{d}{d\rho}(V_{0\rho}^2)\, d\rho$$

$$= \frac{\rho}{2} V_{0\rho}^2\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} V_{0\rho}^2\, d\rho = -\frac{1}{2}\int_{-\infty}^{\infty} V_{0\rho}^2\, d\rho.$$

THIS YIELDS THAT $\int_{-\infty}^{\infty} V_{0\rho}^2\, d\rho \left(\frac{1}{r_0} + \frac{F'(r_0)}{2F(r_0)}\right) = 0.$

THEREFORE r_0 FOR THE EQUILIBRIUM PROBLEM SATISFIES

THE TRANSCENDENTAL EQUATION FOR THE EQUILIBRIUM VALUE r_{0E}

$$* \quad \left\{ \frac{1}{r_{0E}} = -\frac{F'(r_{0E})}{2F(r_{0E})} \right. .$$

NEXT WE CONSIDER THE TIME-DEPENDENT PROBLEM

$$U_t = \varepsilon^2\left(U_{rr} + \frac{1}{r}U_r\right) + F(r)Q(U), \quad 0<r<1$$

$$U_r = 0 \quad \text{ON} \quad r = 0, 1.$$

WE LOOK FOR A SOLUTION IN THE FORM

$$U = V(\rho) \quad \text{WITH} \quad \rho = \varepsilon^{-1}(r - r_0(\sigma t)) \quad \text{AND} \quad \sigma = \varepsilon^2 t.$$

THEN $$U_t = V'(\rho)\left[\frac{d\rho}{dt}\right] = V'(\rho)\left(-\varepsilon^{-1}\sigma\, r_0'\right) = -\varepsilon r_0' V'(\rho).$$

THIS LEADS TO THE PDE

$$-\varepsilon r_0' V_\rho = V_{\rho\rho} + \frac{\varepsilon V_\rho}{r_0 + \varepsilon\rho} + F(r_0 + \varepsilon\rho)\, Q(V).$$

WE THEN EXPAND

$$V = V_0 + \varepsilon V_1 + \cdots$$

SO THAT

$$\begin{cases} V_0'' + F(r_0)\, Q(V_0) = 0, \quad -\infty < \rho < \infty \\ V_0(+\infty) = 1, \quad V_0(-\infty) = -1, \quad V_0(0) = 0 \end{cases}$$

$$\begin{cases} \mathcal{L}V_1 = V_1'' + F(r_0)\, Q'(V_0)\, V_1 = -\dfrac{V_0'}{r_0} - \rho F'(r_0) Q(V_0) - r_0' V_0'. \\ V_1(+\infty) = V_1(-\infty) = 0, \quad V_1(0) = 0. \end{cases}$$

THEN $\mathcal{L} V_0' = 0$ YIELDS THE SOLVABILITY CONDITION,

$$r_0' \int_{-\infty}^{\infty} V_0'^2\, d\rho + F'(r_0) \int_{-\infty}^{\infty} \rho\, Q(V_0)\, V_0'\, d\rho + \frac{1}{r_0} \int_{-\infty}^{\infty} V_0'^2\, d\rho = 0.$$

NOW WE PROCEED AS BEFORE TO OBTAIN

$$r_0' \int_{-\infty}^{\infty} V_0'^2\, d\rho = -\frac{1}{r_0} \int_{-\infty}^{\infty} V_0'^2\, d\rho - F'(r_0) \int_{-\infty}^{\infty} \rho\, Q(V_0) V_0'\, d\rho$$

$$r_0' \int_{-\infty}^{\infty} V_0'^2\, d\rho = -\frac{1}{r_0} \int_{-\infty}^{\infty} V_0'^2\, d\rho + \frac{F'(r_0)}{F(r_0)} \int_{-\infty}^{\infty} \rho\, V_0' V_0''\, d\rho$$

$$= \left(-\frac{1}{2} \frac{F'(r_0)}{F(r_0)} - \frac{1}{r_0} \right) \int_{-\infty}^{\infty} V_0'^2\, d\rho.$$

THIS YIELDS THE ODE:

$$(+) \quad \left\{ \frac{dr_0}{d\sigma} = -\left[\frac{1}{r_0} + \frac{1}{2} \frac{F'(r_0)}{F(r_0)} \right], \quad r_0(0) = r_0^0 \right.$$

WITH $\sigma = \varepsilon^2 t$.

NOW IF WE DEFINE

$$j(r_0) = \frac{1}{r_0} + \frac{1}{2}\frac{f'(r_0)}{f(r_0)},$$

THEN $j(r_{0E}) = 0$. WE HAVE THAT r_{0E} IS STABLE FOR THE ODE WHEN $j'(r_{0E}) > 0$.

(iii) NOW LET $f(r) = \exp\left[B\left(r^3/3 - r^2/2\right)\right]$.

THEN $$\frac{f'(r)}{f(r)} = B(r^2 - r)$$

WE GET FROM EQUILIBRIUM CONDITION $\frac{1}{r_0} = -\frac{f'}{2f}$

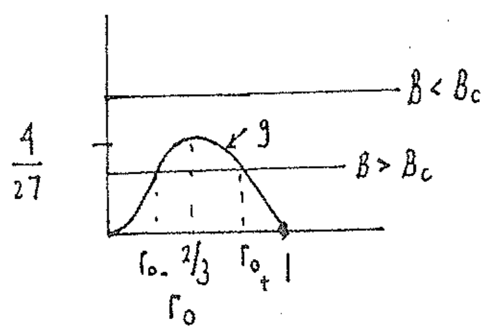
THAT $$\frac{1}{r_0} = +\frac{B}{2}(r_0 - r_0^2) \longrightarrow \frac{2}{B} = r_0^2(1 - r_0).$$

NOW WE NEED $0 < r_0 < 1$. THEN LET $g(r_0) = r_0^2(1 - r_0)$.

WE CALCULATE $g'(r_0) = 2r_0 - 3r_0^2 \longrightarrow g'(r_0) = 0$ AT $r_0 = 2/3$.

NOW $g(2/3) = 4/27$. $\frac{4}{27} = \frac{2}{B_c}$ $B_c = 27/2$.

THUS FOR $B > B_c$, $\exists$ TWO ROOTS $r_{0\pm}$ TO THE EQUILIBRIUM PROBLEM AND NO ROOTS WHEN $B < B_c$.



THUS $\exists$ SADDLE-NODE BIFURCATION AT $B = B_c$, $r_0 = 2/3$

REMARK: LET

$$h(r_0) = \frac{2}{B} - g(r_0).$$

THEN $h'(r_{0-}) = -g'(r_{0-}) < 0$ AND $h'(r_{0+}) = -g'(r_{0+}) > 0$.

FINALLY, THE TIME-DEPENDENT ODE BECOMES

$$\frac{dr_0}{dt} \sim -\varepsilon^2\left(\frac{1}{r_0} + \frac{1}{2}\frac{f'(r_0)}{f(r_0)}\right) \sim -\varepsilon^2\left(\frac{1}{r_0} + \frac{B}{2} r_0 (r_0 - 1)\right)$$

$$\frac{dr_0}{dt} \sim \frac{-B\varepsilon^2}{2r_0}\left(\frac{2}{B} - r_0^2(1-r_0)\right) \sim -\frac{B\varepsilon^2}{2r_0}\, h(r_0).$$

NOTICE IF $B > B_c \longrightarrow \exists$ TWO EQUILIBRIA r_{0+}, r_{0-}

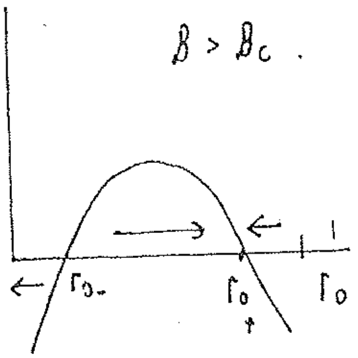
NOTICE $h'(r_{0+}) > 0 \longrightarrow r_{0+}$ is stable

$h'(r_{0-}) < 0 \longrightarrow r_{0-}$ is unstable

IF $B < B_c$ then $dr_0/dt < 0$ AND layer disappears through the origin. IT SHRINKS TO THE ORIGIN AT SOME FINITE TIME.

CASE I

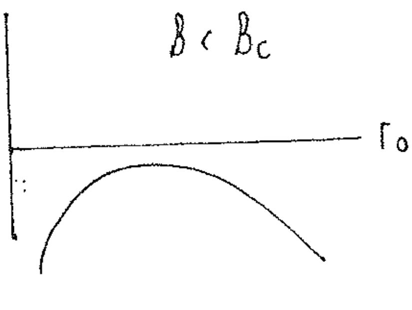
dr_0/dt



$B > B_c$.

- HENCE IF $r_0(0) > r_{0-}$ then $r_0(\sigma t) \to r_{0+}$ AS $t \to \infty$.
- IF $r_0(0) < r_{0-}$, THEN $r_0 \to 0$ AS $t \to T^-$.

CASE II dr_0/dt



$B < B_c$

IN THIS CASE, $r_0 \longrightarrow 0$ AS $t \to T^-$.

FOR ANY $r_0(0)$.

THE TIME FOR COLLAPSE OF THE LAYER IS

$$T = \frac{2}{B\varepsilon^2}\int_0^{r_0(0)} \frac{s}{h(s)}\, ds < \infty.$$