

AARMS Course: Homework 2

Problem :

Consider the following problem in the 2-D circular disk $\Omega = \{\mathbf{x} \mid |\mathbf{x}| \leq 2\}$ that contains three small holes

$$\Delta u = 0, \quad \mathbf{x} \in \Omega \setminus \sum_{j=1}^3 \Omega_{\varepsilon_j}, \quad (0.1)$$

$$u = 4 \cos(2\theta), \quad |\mathbf{x}| = 2, \quad (0.2)$$

$$u = \alpha_j, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j}, \quad j = 1, \dots, 3. \quad (0.3)$$

- (1) Apply the theory for summing infinite logarithmic expansions to first derive a linear system for the source strengths γ_j for $j = 1, \dots, 3$. (Hint: You will need to know the Green's function for a disk, which can be obtained by the method of images.)
- (2) Now assume that each of the three holes has an elliptical shape with semi-axes ϵ and 2ϵ . Assume that the holes are centered at the cartesian coordinate locations $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$, and $\mathbf{x}_3 = (-1/4, 0)$. Take the boundary values $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 2$. By solving the system in (1) numerically, output the source strengths γ_j for $j = 1, \dots, 3$ versus ϵ for $0 < \epsilon < 0.15$.

Solution:

We let the holes be centered at x_1, \dots, x_N . In the outer region, defined away from Ω_{ε_j} for $j = 1, \dots, N$, we expand

$$u(\mathbf{x}; \varepsilon) \sim U_{0H}(\mathbf{x}) + U_0(\mathbf{x}; \boldsymbol{\nu}) + \sigma(\varepsilon)U_1(\mathbf{x}; \boldsymbol{\nu}) + \dots, \quad (0.4)$$

where we assume that $\sigma \ll \nu^m$ for any integer $m > 0$. Since the holes have a common shape, we have that $\nu = -1/\log(\varepsilon d)$ where d is the common logarithmic capacitance of the holes. In (0.4), $U_{0H}(\mathbf{x})$ is the smooth function satisfying the unperturbed problem in the unperturbed domain Ω

$$\Delta U_{0H} = 0, \quad \mathbf{x} \in \Omega; \quad U_{0H} = f, \quad \mathbf{x} \in \partial\Omega. \quad (0.5)$$

Substituting (0.4) into (0.1) and (0.3), and letting $\Omega_{\varepsilon_j} \rightarrow \mathbf{x}_j$ as $\varepsilon \rightarrow 0$, we get that U_0 satisfies

$$\Delta U_0 = 0, \quad \mathbf{x} \in \Omega \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_N\}, \quad (0.6 a)$$

$$U_0 = 0, \quad \mathbf{x} \in \partial\Omega, \quad (0.6 b)$$

$$U_0 \text{ is singular as } \mathbf{x} \rightarrow \mathbf{x}_j, \quad j = 1, \dots, N. \quad (0.6 c)$$

The singularity behavior for U_0 as $\mathbf{x} \rightarrow \mathbf{x}_j$ will be found below by matching the outer solution to the far-field behavior of the inner solution to be constructed near each Ω_{ε_j} .

In the j^{th} inner region near Ω_{ε_j} we introduce the inner variables \mathbf{y} and $v(\mathbf{y}; \varepsilon)$ by

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j), \quad v(\mathbf{y}; \varepsilon) = u(\mathbf{x}_j + \varepsilon\mathbf{y}; \varepsilon). \quad (0.7)$$

We then expand $v(\mathbf{y}; \varepsilon)$ as

$$v(\mathbf{y}; \varepsilon) = \alpha_j + \nu\gamma_j v_{cj}(\mathbf{y}) + \mu_0(\varepsilon)V_{1j}(\mathbf{y}) + \dots, \quad (0.8)$$

where $\gamma_j = \gamma_j(\boldsymbol{\nu})$ is a constant to be determined. Here $\mu_0 \ll \nu^k$ as $\varepsilon \rightarrow 0$ for any $k > 0$. In (0.8), the logarithmic gauge function ν is defined by

$$\nu = -1/\log(\varepsilon d), \quad (0.9)$$

where d is specified below. By substituting (0.7) and (0.8) into (0.1) and (0.2), we conclude that $v_{cj}(\mathbf{y})$ is the unique solution to

$$\Delta_{\mathbf{y}} v_{cj} = 0, \quad \mathbf{y} \notin \Omega_j; \quad v_{cj} = 0, \quad \mathbf{y} \in \partial\Omega_j, \quad (0.10 a)$$

$$v_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1), \quad \text{as } |\mathbf{y}| \rightarrow \infty. \quad (0.10 b)$$

Here $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$, and the logarithmic capacitance, d , is determined by the shape of Ω_j . Since the holes were assumed to have the same shape then d is independent of j .

Writing (0.10 b) in outer variables and substituting the result into (0.8), we get that the far-field expansion of v away from each Ω_j is

$$v \sim \alpha_j + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|, \quad j = 1, \dots, N. \quad (0.11)$$

Then, by expanding the outer solution (0.4) as $\mathbf{x} \rightarrow \mathbf{x}_j$, we obtain the following matching condition between the inner and outer solutions:

$$U_{0H}(\mathbf{x}_j) + U_0 \sim \alpha_j + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|, \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_j, \quad j = 1, \dots, N. \quad (0.12)$$

In this way, we obtain that U_0 satisfies (0.6) subject to the singularity structure

$$U_0 \sim \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| + o(1), \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_j, \quad j = 1, \dots, N. \quad (0.13)$$

Observe that in (0.13) both the singular and regular parts of the singularity structure are specified. Therefore, (0.13) will effectively lead to a linear system of algebraic equations for γ_j for $j = 1, \dots, N$.

The solution to (0.6 a) and (0.6 b), with $U_0 \sim \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|$ as $\mathbf{x} \rightarrow \mathbf{x}_j$, can be written as

$$U_0(\mathbf{x}; \boldsymbol{\nu}) = -2\pi\nu \sum_{i=1}^N \gamma_i G(\mathbf{x}; \mathbf{x}_i), \quad (0.14)$$

where $G(\mathbf{x}; \mathbf{x}_j)$ is the Green's function satisfying

$$\Delta G = -\delta(\mathbf{x} - \mathbf{x}_j), \quad \mathbf{x} \in \Omega; \quad G = 0, \quad \mathbf{x} \in \partial\Omega, \quad (0.15 a)$$

$$G(\mathbf{x}; \mathbf{x}_j) \sim -\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_j| + R(\mathbf{x}_j; \mathbf{x}_j) + o(1), \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_j. \quad (0.15 b)$$

Here $R_{jj} \equiv R(\mathbf{x}_j; \mathbf{x}_j)$ is the regular part of G .

Finally, we expand (0.14) as $\mathbf{x} \rightarrow \mathbf{x}_j$ and equate the resulting expression with the required singularity behavior (0.13) to get

$$\nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| - 2\pi\nu\gamma_j R_{jj} - 2\pi\nu \sum_{\substack{i=1 \\ i \neq j}}^N \gamma_i G(\mathbf{x}_j; \mathbf{x}_i) = \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|, \quad j = 1, \dots, N. \quad (0.16)$$

In this way, we get the following linear algebraic system for γ_j for $j = 1, \dots, N$:

$$-\gamma_j (1 + 2\pi\nu R_{jj}) - 2\pi\nu \sum_{\substack{i=1 \\ i \neq j}}^N \gamma_i G_{ji} = \alpha_j - U_{0H}(\mathbf{x}_j), \quad j = 1, \dots, N. \quad (0.17)$$

Here $G_{ji} \equiv G(\mathbf{x}_j; \mathbf{x}_i)$ and $\nu_j = -1/\log(\varepsilon d_j)$. We summarize the asymptotic construction as follows:

For $\varepsilon \ll 1$, the outer expansion from (0.4) is

$$u \sim U_{0H}(\mathbf{x}) - 2\pi\nu \sum_{i=1}^N \gamma_i G(\mathbf{x}; \mathbf{x}_i), \quad \text{for } |\mathbf{x} - \mathbf{x}_j| = \mathcal{O}(1). \quad (0.18 a)$$

The inner expansion near Ω_{ε_j} with $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)$, is

$$u \sim \alpha_j + \nu\gamma_j v_{cj}(\mathbf{y}), \quad \text{for } |\mathbf{x} - \mathbf{x}_j| = \mathcal{O}(\varepsilon). \quad (0.18 b)$$

Here $\nu = -1/\log(\varepsilon d)$, d is defined in (0.10 b), $v_{cj}(\mathbf{y})$ satisfies (0.10), U_{0H} satisfies the unperturbed problem (0.5), while $G(\mathbf{x}; \mathbf{x}_j)$ and $R(\mathbf{x}_j; \mathbf{x}_j)$ satisfy (0.15). Finally, the constants γ_j for $j = 1, \dots, N$ are obtained from the N dimensional linear algebraic system (0.17).

For the problem under consideration we have $f = 4 \cos(2\theta) = 4(\cos^2 \theta - \sin^2 \theta) = x^2 - y^2$ on $(x^2 + y^2)^{1/2} = 4$. Thus, the solution to the unperturbed problem (0.5) is simply

$$U_{0H}(x, y) = x^2 - y^2. \quad (0.19)$$

Next, the Green's function satisfying (0.15) and its regular part are calculated from the method of images as

$$G(\mathbf{x}; \mathbf{x}_j) = -\frac{1}{2\pi} \log \left(\frac{2|\mathbf{x} - \mathbf{x}_j|}{|\mathbf{x} - \mathbf{x}'_j| |\mathbf{x}_j|} \right), \quad R_{jj} \equiv R(\mathbf{x}_j; \mathbf{x}_j) = -\frac{1}{2\pi} \log \left[\frac{2}{|\mathbf{x}_j - \mathbf{x}'_j| |\mathbf{x}_j|} \right]. \quad (0.20)$$

Here \mathbf{x}'_j is the image point of \mathbf{x}_j in the unit disk of radius two.

Next, we note that since each of the holes has an elliptic shape with semi-axes ε and 2ε , then from the Table of the class notes their common logarithmic capacitance is $d = 3/2$. The holes are assumed to be centered at $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$ and $\mathbf{x}_3 = (-1/4, 0)$, and have the constant boundary values $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 2$.

Therefore, upon defining $\nu = -1/\log(3\varepsilon/2)$ we obtain from (0.17) that γ_j for $j = 1, \dots, 3$ is the solution of the linear system

$$-\gamma_1 [1 + 2\pi\nu R_{11}] - 2\pi\nu [\gamma_2 G(\mathbf{x}_1; \mathbf{x}_2) + \gamma_3 G(\mathbf{x}_1; \mathbf{x}_3)] = 1, \quad (0.21 a)$$

$$-\gamma_2 [1 + 2\pi\nu R_{22}] - 2\pi\nu [\gamma_1 G(\mathbf{x}_2; \mathbf{x}_1) + \gamma_3 G(\mathbf{x}_2; \mathbf{x}_3)] = -1/4, \quad (0.21 b)$$

$$-\gamma_3 [1 + 2\pi\nu R_{33}] - 2\pi\nu [\gamma_1 G(\mathbf{x}_3; \mathbf{x}_1) + \gamma_2 G(\mathbf{x}_3; \mathbf{x}_2)] = 31/16. \quad (0.21 c)$$

Here R_{jj} and $G(\mathbf{x}_j; \mathbf{x}_i)$ are to be evaluated from (0.20).

We solve this linear system numerically for γ_j as a function of ε . The curves $\gamma_j(\varepsilon)$ as a function of ε are plotted in Fig. 1. We observe that the leading-order approximation to (0.21), valid for $\nu \ll 1$, is simply $\gamma_1 = -1$, $\gamma_2 = 1/4$ and $\gamma_3 = -31/16$. From Fig. 1 we observe that this approximation, which neglects interaction effects between the holes, is rather inaccurate unless ε is very small.

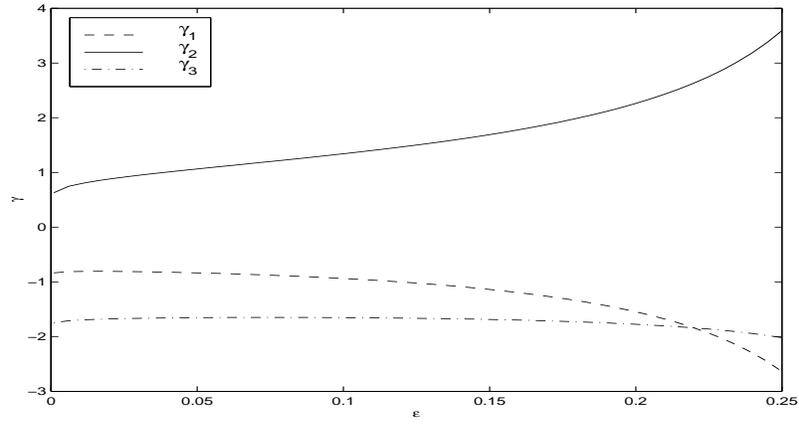


Figure 1. Plot of $\gamma_j = \gamma_j(\epsilon)$ for $j = 1, 2, 3$ obtained from the numerical solution to (0.21).