AARMS Course: Homework 3

Problem:

Consider the Brusselator reaction-diffusion system for $U(x,T)$ and $V(x,T)$ where $x \in \Omega \in \mathbb{R}^2$ given by

$$U_T = \epsilon_0^2 \Delta U + E - (B+1)U + U^2 V \,,$$
$$V_T = D\Delta V + BU - U^2 V \,,$$

where $B > 0$, $E > 0$, $D > 0$ are parameters. Assume that $\epsilon_0 \ll 1$.

(1) Assume that that $E = \mathcal{O}(\epsilon_0) \ll 1$, by writing $E = \epsilon_0 E_0$ for some $E_0 = \mathcal{O}(1)$. By introducing a re-scaling of U, V, and T, show that we can transform this system to

$$u_t = \epsilon^2 \Delta u + \epsilon^2 \mathcal{E} - u + f u^2 v \,, \qquad x \in \Omega \tag{0.1 a}$$
$$\tau v_t = \Delta v + \varepsilon^{-2} \left(u - u^2 v \right) \,, \qquad x \in \Omega \,, \tag{0.1 b}$$

for some $\mathcal{O}(1)$ parameters f, τ, and $\mathcal{E}$, and $\epsilon \ll 1$. (Note: ϵ is different than ϵ_0)

(2) For this system for u, and v, construct the quasi-equilibrium spot pattern, similar to that done for the Schnakenberg model in the class notes, for a multi-spot pattern with spots at x_j for $j = 1, \ldots, N$, that effectively sums all of the logarithmic terms in the expansion. (Hint: Be careful in that in the outer region $u \sim \epsilon^2 \mathcal{E}$, which then contributes a term in the second equation.)

Solution:

(1) We let $E = \varepsilon E_0$, and divide the first equation by $B+1$ and the second by D to get

$$\frac{U_T}{B+1} = \frac{\epsilon_0^2}{B+1} \Delta U + \frac{\varepsilon_0 E_0}{B+1} - U + \frac{1}{B+1} U^2 V \,, \tag{0.2 a}$$
$$\frac{1}{D} V_T = \Delta V + \frac{B}{D} U - \frac{1}{D} U^2 V \,, \tag{0.2 b}$$

Now let u_0 and v_0 denote the scalings for U and V. To balance the last two terms on the right side of the V equation we need $Bu_0 = u_0^2 v_0$, which gives $u_0 v_0 = B$. Thus, we next introduce u and v by

$$U = u_0 u \,, \qquad V = v_0 v \,, \qquad \text{with} \qquad u_0 v_0 = B \,.$$

Putting this into (0.2) we get

$$\frac{u_T}{B+1} = \frac{\epsilon_0^2}{B+1} \Delta u + \frac{\varepsilon_0 E_0}{u_0(B+1)} - u + \frac{B}{B+1} u^2 v \,, \tag{0.3 a}$$
$$\frac{1}{D} v_T = \Delta v + \frac{u_0 B}{v_0 D} \left(u - u^2 v \right) \,. \tag{0.3 b}$$

Now we use $v_0 = B/u_0$. We introduce ε, f, τ, and a new time-variable t by

$$\varepsilon = \varepsilon_0 / \sqrt{B+1} \,, \qquad f = \frac{B}{B+1} \,, \qquad \tau = (B+1)/D \,, \qquad T = (B+1) t \,. \tag{0.4}$$

This yields

$$u_t = \varepsilon^2 \Delta u + \frac{\varepsilon E_0}{u_0\sqrt{B+1}} - u + f u^2 v \,, \tag{0.5 a}$$

$$\tau v_t = \Delta v + \frac{u_0^2}{D}\left(u - u^2 v\right) \,. \tag{0.5 b}$$

Finally, we choose $u_0^2/D = \varepsilon^{-2}$, so that

$$u_0 = \sqrt{D}\varepsilon^{-1}\,, \qquad v_0 = B\sqrt{D}\varepsilon\,,$$

This gives our desired system (0.1), where we define $\mathcal{E}$ by

$$\frac{\varepsilon E_0}{u_0\sqrt{B+1}} = \varepsilon^2 \mathcal{E}\,, \qquad \mathcal{E} \equiv E_0/\sqrt{D(B+1)}\,.$$

(2) We now construct a quasi-steady, or quasi-equilibrium, multi-spot pattern in the limit $\epsilon \to 0$. As such we set $yu_t = v_t = 0$ in (0.1). We first formulate the local (or inner) problem that determines the profile of an isolated spot. In the inner region near the j-th spot, we obtain to leading-order in ϵ that $u \sim u_j$ and $v \sim v_j$ where

$$\Delta_y u_j - u_j + f u_j^2 v_j = 0\,, \qquad \Delta_y v_j + u_j - u_j^2 v_j = 0\,, \tag{0.6}$$

on $-\infty < y_1, y_2 < \infty$. We seek a radially symmetric solution to this problem so that $u_j(\rho)$ and $v_j(\rho)$, with $\rho = |y|$, satisfies

$$\begin{gathered} \Delta_\rho u_j - u_j + f u_j^2 v_j = 0\,, \qquad \Delta_\rho v_j + u_j - u_j^2 v_j = 0\,, \qquad 0 < \rho < \infty\,, \\ u_j'(0) = v_j'(0) = 0\,; \qquad u_j \to 0 \quad \text{and} \quad v_j \sim S_j \log\rho + \chi(S_j; f) + o(1) \quad \text{as} \quad \rho \to \infty\,, \end{gathered} \tag{0.7}$$

where f is the bifurcation parameter. This problem is called the *core problem.* Upon integrating the two boundary value problems in (0.7) over $0 < \rho < \infty$, we readily derive that

$$S_j = \int_0^\infty (u_j^2 v_j - u_j)\rho\, d\rho\,. \tag{0.8}$$

The key feature in (0.7) is that we must impose that $v_j \sim S_j \log\rho$ as $\rho \to \infty$, which is appropriate for $\Delta_\rho v_j = (u_j^2 v_j - u_j)$ given that $u_j \to 0$ at infinity. The constant S_j is a parameter at this stage, but it will eventually be determined after the asymptotic matching of the inner and outer solutions for v. In terms of S_j and the bifurcation parameter f, the function $\chi(S_j; f)$ is computed numerically from the limiting process $\lim_{\rho\to\infty} (v_j - S_j \log\rho) = \chi(S_j; f)$. parameter f.

The core problem (0.7) is solved numerically for various f and S_j by approximating the infinite domain by the finite domain $0 \le \rho \le R$, where $R \gg 1$. In this way, $\chi(S_j; f)$ is determined by computing v_j at $\rho = R$. The results shown in Fig. 1 are for $R = 15$. Increasing R did not change these results to a few significant digits. The spot profile $u_j(\rho)$ is seen to develop a volcano shape as S_j increases.

Next, we asymptotically match the far-field behaviours of the inner solutions near each spot to a certain global solution for v, which we will construct. In doing so, we will derive a nonlinear algebraic system of equations for the unknowns S_j, referred to as the "source strengths". Our asymptotic analysis has the important feature

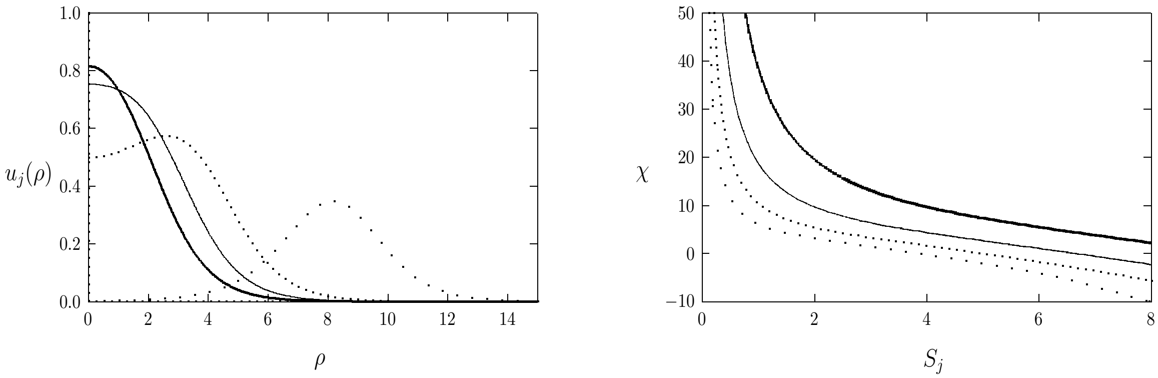

Figure 1. Left panel: the spot solution $u_j(\rho)$ computed numerically from (0.7) when $S_j = 8$ for $f = 0.3$ (heavy solid curve), $f = 0.4$ (solid curve), $f = 0.5$ (dotted curve), $f = 0.6$ (widely spaced dots). As f increases, u_j develops a volcano shape. Right panel: the function $\chi(S_j; f)$ defined in the asymptotic boundary condition in (0.7) for $f = 0.3$, $f = 0.4$, $f = 0.5$, and $f = 0.6$, with the same labelling as in the left panel.

that it retains all of the logarithmic terms in $\nu \equiv -1/\log\epsilon$ as $\epsilon \to 0$, and so our asymptotic approximation for the solution and for the source strengths has an error that is algebraic, rather than logarithmic, in ϵ.

To determine the far-field behaviour of each inner solution we use $v_j \sim S_j \log|y| + \chi(S_j; f) + o(1)$ as $|y| \to \infty$, so that

$$v \sim S_j \log|x - x_j| + \frac{S_j}{\nu} + \chi(S_j; f) \quad \text{as} \quad x \to x_j\,, \quad j = 1, \ldots, N\,; \qquad \nu \equiv -\frac{1}{\log\epsilon}\,, \tag{0.9}$$

which provides the singularity behaviour of the outer solution for v at each x_j.

Next, we study the outer solution for (0.1). In the outer region away from $\mathcal{O}(\epsilon)$ neighborhoods of the spot locations $\{x_1, \ldots, x_N\}$ we have that $\epsilon^2\mathcal{E} - u + fu^2v \sim 0$, so that $u \sim \epsilon^2\mathcal{E} + \mathcal{O}(\epsilon^2)$. By combining the inner and outer approximations for u, we get the leading-order uniformly valid approximation for u given by

$$u \sim \epsilon^2\mathcal{E} + \sum_{j=1}^{N} u_j\left(\epsilon^{-1}|x - x_j|\right)\,. \tag{0.10}$$

We then must estimate the term $\epsilon^{-2}(u - u^2v)$ in the v-equation of (0.1) in the sense of distributions. The evaluation of this term requires care in order to retain both the local contribution near each spot and the global contribution arising from the non-vanishing outer solution for u. In the sense of distributions, we obtain upon using (0.8) that

$$\frac{1}{\epsilon^2}(u - u^2v) \sim \mathcal{E} + 2\pi\sum_{j=1}^{N}\left(\int_0^\infty (u_j - u_j^2v_j)\,\rho\,d\rho\right)\delta(x - x_j) = \mathcal{E} - 2\pi\sum_{j=1}^{N} S_j\,\delta(x - x_j)\,.$$

By using this result together with the matching condition (0.9) for v, the outer problem for v is

$$\Delta v + \mathcal{E} = 2\pi\sum_{j=1}^{N} S_j\delta(x - x_j)\,, \quad x \in \Omega\,; \qquad v \sim S_j\log|x - x_j| + \frac{S_j}{\nu} + \chi(S_j; f) + o(1) \quad \text{as} \quad x \to x_j\,, \tag{0.11}$$

for $j = 1 \ldots, N$. By pre-specifying the form of the non-singular $\mathcal{O}(1)$ term in each singularity condition as $x \to x_j$, we will obtain a nonlinear algebraic system for the source strengths $S_1, \ldots, S_N$.

To solve (0.11) we introduce the Neumann Green's function $G(x;\xi)$ defined as the unique solution to

$$\Delta G = \frac{1}{|\Omega|} - \delta(x-\xi)\,; \qquad \partial_n G = 0\,, \quad x \in \partial\Omega\,, \tag{0.12 a}$$

$$G(x;\xi) \sim -\frac{1}{2\pi} \log|x-\xi| + R(\xi)\,, \qquad \text{as} \quad x \to x_j\,. \tag{0.12 b}$$

Therefore, we can write v as the superposition

$$v = -2\pi \sum_{i=1}^{N} S_j G(x;x_i) + \bar{v}\,,\,, \qquad \text{where} \qquad \sum_{i=1}^{N} S_i = \frac{|\Omega|\mathcal{E}}{2\pi}\,. \tag{0.13}$$

Here $\bar{v}$ is an arbitrary constant that must be determined as part of the analysis. This last condition on the S_j follows from the divergence theorem.

As $x \to x_j$, the matching condition in (0.11), together with the explicit solution for v in (0.13), yields that

$$-2\pi \left[-\frac{S_i}{2\pi} \log|x-x_j| + S_j R(x_j) \right] - 2\pi \sum_{\substack{i=1 \\ i\neq j}}^{N} S_i G_{ji} + \bar{v} \sim S_j \log|x-x_i| + \frac{S_j}{\nu} + \chi(S_j;f)\,,$$

for $j = 1, \ldots, N$. With $G_{ji} \equiv G(x_j;x_i)$, we obtain the $N+1$ nonlinear algebraic equations for S_j and $\bar{v}$ given by

$$\frac{S_j}{\nu} + 2\pi \left(S_j R_j + \sum_{i\neq j}^{N} S_j G_{ji} \right) + \chi(S_j;f) = \bar{v}\,, \quad j = 1, \ldots, N\,; \qquad \sum_{i=1}^{N} S_i = \frac{|\Omega|\mathcal{E}}{2\pi}\,. \tag{0.14}$$