AARMS HW # 4

PROBLEM 1 CONSIDER THE COUPLED RD SYSTEM

$$(*)\left\{\begin{array}{l} u_t = u_{xx} + f(u) - 2\delta\, b(x)\, w \\ \tau w_t = D w_{xx} - w + c(x)\, u \end{array}\right. \quad \text{ON } -1<x<1 \quad \text{WITH} \quad w_x = u_x = 0 \ \text{ AT } x=\pm 1.$$

ASSUME THAT $f(u)$ IS SMOOTH WITH $f(0)=0$ AND $f'(0)=1$. IN $(*)$, $\delta>0$ IS A CONSTANT MEASURING THE COUPLING OF u, w.

(i) FOR THE EQUILIBRIUM STATE $(u_e, w_e) = (0,0)$ INTRODUCE THE PERTURBATION $u = u_e + e^{\lambda t}\Phi(x)$, $w = w_e + e^{\lambda t}\Lambda(x)$. FOR $D\to+\infty$, SHOW THAT $\Phi(x)$ IS AN EIGENFUNCTION OF THE NONLOCAL EIGENVALUE PROBLEM

$$(1)\left\{\begin{array}{ll} \Phi'' - \dfrac{\delta\, b(x)}{1+\tau\lambda}\displaystyle\int_{-1}^{1} c(x)\,\Phi(x)\,dx = (\lambda-1)\Phi & -1<x<1 \\ \Phi' = 0 \ \text{ AT } \ x=\pm 1 & \end{array}\right.$$

(ii) LET $\tau=0$ IN (1). SHOW THAT THE EIGENVALUES λ OF (1) MUST BE REAL WHEN $b(x) = K\, c(x)$ FOR SOME CONSTANT K INDEPENDENT OF x.

(iii) FOR $\tau=0$ AND $\delta=0$ SHOW THAT THE EIGENFUNCTIONS OF THE STURM LIOUVILLE PROBLEM OF (1) CAN BE WRITTEN IN TERMS OF EVEN AND ODD EIGENFUNCTIONS AS

$$\hat{\phi}_{2m}(x) = \cos(\pi m x) \qquad \hat{\phi}_{2m+1}(x) = \sin\left[(2m+1)\pi x/2\right]$$

FOR $m = 0,1,2,\ldots$

(iv) suppose that $\Upsilon = 0$ AND

$$b(x) = b_0 + b_1 \cos(\pi x) + b_2 \cos(2\pi x)$$

(2)

$$c(x) = c_0 + c_1 \cos(\pi x) + c_2 \cos(2\pi x).$$

FOR SOME KNOWN VALUES $b_0, b_1, b_2, c_0, c_1, c_2$. LOOK FOR AN eigenFUNCTION FOR (1) IN THE FORM

$$\Phi = S_0 + S_1 \cos(\pi x) + S_2 \cos(2\pi x)$$

DERIVE A 3x3 MATRIX Eigenvalue problem OF the FORM

$$(3) \qquad \left(\Lambda - \delta \mathcal{D}\right) \underline{S} = \sigma \underline{S} \qquad \text{WHERE} \quad \underline{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix}, \ \sigma \equiv \lambda - 1,$$

Λ IS A DIAGONAL MATRIX AND $\mathcal{D}$ IS A 3x3 RANK 1 MATRIX INVOLVING $b_0, b_1, b_2, c_0, c_1, c_2$.

FOR $b(x)$, $c(x)$ OF FORM (2) WHAT HAPPENS TO ALL OTHER eigenFUNCTIONS $\hat{\phi}_{2m}, \hat{\phi}_{2m+1}$ OF STURM LIOUVILLE AS δ IS INCREASED?

(v) PLOT THE Eigenvalues σ OF (3) AS A FUNCTION OF δ ON THE RANGE $0 \leq \delta \leq 5$ FOR THE SPECIFIC CHOICE

$$b_0 = 2.5, \ b_1 = 1.0, \ b_2 = 1.0, \ c_0 = 1.1, \ c_1 = -0.5, \ c_2 = 0.5.$$

PLOT REAL(σ) VS δ

AND IM(σ) VS RE(σ) (WITH δ PARAMetrizing the CURVE IN the PLANE)

WHAT CAN you say about the stability OF $(U_e, W_e) = (0,0)$ FOR $\Upsilon = 0$ AS δ IS VARIED?

(vi) <u>FINALLY</u>, TAKE $b(x) = c(x) = 1 \ \forall x$ AND LET $\tau > 0$.

CONSIDER THEN FOR $\delta \geq 0$,

$$(1') \qquad \Phi'' - \frac{\delta}{1+\tau\lambda} \int_{-1}^{1} \Phi(x)\, dx = (\lambda - 1)\, \Phi .$$

FIND THE RANGE OF τ, δ FOR WHICH $\mathrm{RE}\,\lambda < 0$ FOR ANY EIGENPAIR OF $(1')$.

PROBLEM 1

$$U_t = U_{xx} + F(U) - 2\delta\, b(x)\, W \qquad -1<x<1, \quad W_x = U_x = 0 \text{ AT } x = \pm 1.$$

$$\tau W_t = D W_{xx} - W + C(x)\, U \qquad F(0)=0, \quad F'(0)=1$$

(i) $W_e = U_e = 0$ IS AN EQUILIBRIUM STATE SINCE $F(0)=0$.

WE PUT $U = 0 + e^{\lambda t}\Phi(x)$, $W = e^{\lambda t}\Lambda(x)$ AND LINEARIZE.

TO OBTAIN

$$\lambda \Phi = \Phi_{xx} + F'(0)\,\Phi - 2\delta\, b(x)\,\Lambda \qquad -1<x<1$$

$$\tau\lambda\Lambda = D\Lambda_{xx} - \Lambda + C(x)\,\Phi$$

$$\Lambda_x = \Phi_x = 0 \quad \text{AT } x = \pm 1.$$

NOW ASSUME $D \gg 1$. WE EXPAND $\Lambda = \Lambda_0 + \frac{1}{D}\Lambda_1 + \cdots$ THIS GIVES

$O(D)$: $\Lambda_{0xx} = 0$ WITH $\Lambda_{0x} = 0$ AT $x = \pm 1 \longrightarrow \Lambda_0 = $ CONSTANT

$O(1)$: $\Lambda_{1xx} = \tau\lambda\Lambda_0 + \Lambda_0 - C(x)\,\Phi$ WITH $\Lambda_{1x} = 0$ AT $x = \pm 1$.

WE USE THE "DIVERGENCE" theorem:

$$\int_{-1}^{1} \left(\tau\lambda\Lambda_0 + \Lambda_0 - C(x)\,\Phi\right) dx = \int_{-1}^{1} \Lambda_{1xx}\, dx = \Lambda_{1x}\Big|_{-1}^{1} = 0.$$

THEN SINCE Λ_0 IS A CONSTANT

$$2(\tau\lambda + 1)\,\Lambda_0 = \int_{-1}^{1} C(x)\,\Phi(x)\, dx.$$

NOW PUT INTO Φ EQUATION WITH $F'(0)=1$.

$$\begin{cases} \Phi_{xx} - \dfrac{\delta\, b(x)}{1+\tau\lambda}\displaystyle\int_{-1}^{1} C(x)\,\Phi(x)\, dx = (\lambda - 1)\,\Phi \qquad -1<x<1 \\ \Phi_x = 0 \quad \text{AT } x = \pm 1 \end{cases}$$

THIS IS A <u>NONLOCAL</u> <u>EIGENVALUE</u> <u>PROBLEM</u>.

(ii) DEFINE $L\Phi \equiv \Phi'' - \frac{\delta b(x)}{1+\gamma\lambda}\int_{-1}^{1} c(\xi)\Phi(\xi)\,d\xi = (\lambda-1)\Phi\,; \quad -1<x<1 \qquad \Phi_x = 0 \quad x=\pm 1.$

WE LET Φ, ψ BE ANY TWO FUNCTIONS WITH $\Phi_x = \psi_x = 0$ AT $x=\pm 1$.

WE NOW FIND ADJOINT OPERATOR TO L.

$$\int_{-1}^{1} \psi L\Phi\,dx = \int_{-1}^{1} \psi\left[\Phi'' - \frac{\delta b(x)}{1+\gamma\lambda}\int_{-1}^{1} c(\xi)\Phi(\xi)\,d\xi\right]$$

NOW integrate by parts AND SWITCH ORDER OF integration

$$= \int_{-1}^{1}\left[\psi(\xi)\quad \Phi''(\xi) - \Phi(\xi)\delta c(\xi)\int_{-1}^{1} b(x)\psi(x)\,dx\right]d\xi$$

$$= (\psi\phi' - \psi'\phi')\Big|_{-1}^{1} + \int_{-1}^{1}\left[\psi''(\xi) - \delta c(\xi)\int_{-1}^{1} b(x)\psi(x)\right]\Phi(\xi)\,d\xi.$$

SINCE $\psi' = \phi' = 0$ AT $x=\pm 1$, SWITCHING ξ AND x we get

$$\int_{-1}^{1} \psi L\Phi\,dx = \int_{-1}^{1} \Phi\mathcal{L}\psi\,dx \qquad \mathcal{L}(\psi) = \psi'' - \delta c(x)\int_{-1}^{1} b(\xi)\psi(\xi)\,d\xi$$

THEN $\mathcal{L}$ IS ADJOINT OF L: SIMPLY SWITCH b AND c.

SO IF $b(x) = \kappa\, c(x) \;\; \forall x$ THEN $\mathcal{L} = L$ (self-adjoint).

SO ASSUMING $b(x) = \kappa\, c(x)$ THEN $\int_{-1}^{1} \psi L\Phi\,dx = \int_{-1}^{1} \Phi L\psi\,dx$

NOW SUPPOSE $L\phi = \lambda\phi$ THEN $L\bar{\phi} = \bar{\lambda}\bar{\phi}$ (conjugation).

WE $\int_{-1}^{1}(\bar{\phi}L\phi - \phi L\bar{\phi})\,dx = (\lambda - \bar{\lambda})\int_{-1}^{1}\phi\bar{\phi}\,dx = 0.$

SO $(\lambda - \bar{\lambda})\int_{-1}^{1}|\phi(x)|^2\,dx = 0 \rightarrow \lambda = \bar{\lambda}$ λ IS REAL

(iii) FOR $\tau = 0$ AND $\delta = 0$ WE HAVE

$$\Phi_{xx} + (1-\lambda)\Phi = 0 \quad \text{ON } -1 < x < 1.$$
$$\Phi_x = 0 \text{ AT } x = \pm 1$$

SO $\Phi = \cos\left(\sqrt{1-\lambda}\,(x+1)\right)$ IS $1-\lambda \geq 0$ IS NEEDED.

PUT $\Phi_x = 0$ AT $x = 1 \rightarrow \sqrt{1-\lambda}\,(2) = n\pi \quad n = 0, 1, 2, \ldots$

SO $\lambda_n = 1 - n^2\pi^2/4 \quad n = 0, 1, 2, \ldots$

$\lambda_0 = 1 \qquad \lambda_n < 0$ FOR $n \geq 1$.

SO (u_e, w_e) IS UNSTABLE SINCE $\exists$ AN eigenvalue with $\text{Re}\,\lambda > 0$. SO WITHOUT NONLOCAL term we have instability.

THUS $\Phi_n(x) = \cos\left(\frac{n\pi}{2}(x+1)\right), \quad n = 0, 1, 2, \ldots$

NOW $\Phi_n(x) = \cos\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{n\pi}{2}\right)$

IF n = even $\longrightarrow \Phi_n(x) = A\cos\left(\frac{n\pi x}{2}\right) \qquad n = 2m, \; m = 0, 1, 2, \ldots$

IF n = ODD $\longrightarrow \Phi_n(x) = A\sin\left(\frac{n\pi x}{2}\right) \qquad n = 2m+1, \; m = 0, 1, 2, \ldots$

SO OUR eigenFUNCTION ARE

$$\psi_{2m}(x) = \cos(\pi m x) \qquad \psi_{2m+1}(x) = \sin\left(\frac{(2m+1)\pi x}{2}\right)$$

WITH $m = 0, 1, 2, 3, \ldots$

$\psi_{2m}(x)$ IS EVEN IN x

$\psi_{2m+1}(x)$ IS ODD IN x.

RECALL THAT eigenFUNCTION ARE ORTHOGONAL $\int_{-1}^{1} \Phi_n \Phi_k \, dx = 0 \quad n \neq k.$

(iv) SUPPOSE $q = 0$ AND

$$b(x) = b_0 + b_1 \cos(\pi x) + b_2 \cos(2\pi x)$$

$$c(x) = c_0 + c_1 \cos(\pi x) + c_2 \cos(2\pi x)$$

WE NOTICE THAT THIS IS EQUIVALENT TO WRITING

$$b(x) = b_0 \psi_0(x) + b_1 \psi_2(x) + b_2 \psi_4(x)$$

$$c(x) = c_0 \psi_0(x) + c_1 \psi_2(x) + c_2 \psi_4(x)$$

$$\psi_0 = 1, \; \psi_2 = \cos(\pi x), \; \psi_4 = \cos(2\pi x)$$

BY ORTHOGONALITY WE HAVE

$$\int_{-1}^{1} \psi_j \left(b_0 \psi_0 + b_1 \psi_2 + b_2 \psi_4 \right) dx = 0 \quad \text{IF} \quad j \neq \{0, 2, 4\}$$

NOW THIS MEANS THAT ALL OTHER EIGENFUNCTIONS OF Φ'' ARE INDEPENDENT OF δ. THESE ARE CALLED FIXED EIGENVALUES

SO $\lambda_1 = 1 - \pi^2/4 \qquad \psi_1(x) = \sin(\pi x/2)$

$\lambda_3 = 1 - 3^2\pi^2/4 \qquad \psi_3(x) = \sin(3\pi x/2)$

AND $\lambda_5, \lambda_6, \ldots$ ARE ALL <u>INDEPENDENT</u> OF δ.

THUS THE ONLY Eigenvalues that will be affected by δ IS $\lambda_0, \lambda_2, \lambda_4$. AS SUCH WE PUT

$$\Phi = S_0 + S_1 \cos(\pi x) + S_2 \cos(2\pi x)$$

TO SPAN THIS 3 DIMENSIONAL SUBSPACE.

WE SUBSTITUTE INTO $\quad L\Phi - \delta b(x) \int_{-1}^{1} c(x) \Phi(x)\, dx = (\lambda - 1)\Phi$

WE NEED $\quad \int_{-1}^{1} \psi_0 \, dx = 2, \quad \int_{-1}^{1} \psi_2^2 \, dx = 2/2 = 1 \quad \int_{-1}^{1} \psi_4^2 \, dx = 1.$

THUS

$$S_0 - \pi^2 S_1 \cos(\pi x) - 4\pi^2 S_2 \cos(2\pi x) - \delta\left(b_0 + b_1\cos(\pi x) + b_2\cos(2\pi x)\right)\left[2C_0 S_0 + C_1 S_1 + C_2 S_2\right] = (\lambda - 1)\left(S_0 + S_1\cos(\pi x) + S_2\cos(2\pi x)\right)$$

SO

$$S_0 - \delta\left[2C_0 S_0 + C_1 S_1 + C_2 S_2\right] b_0 = 0$$

$$-\pi^2 S_1 - \delta\left[2C_0 S_0 + C_1 S_1 + C_2 S_2\right] b_1 = 0$$

$$-4\pi^2 S_2 - \delta\left[2C_0 S_0 + C_1 S_1 + C_2 S_2\right] b_2 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\pi^2 & 0 \\ 0 & 0 & -4\pi^2 \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} - \delta\begin{pmatrix} 2C_0 b_0 & C_1 b_0 & C_2 b_0 \\ 2C_0 b_1 & C_1 b_1 & C_2 b_1 \\ 2C_0 b_2 & C_1 b_2 & C_2 b_2 \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = (\lambda - 1)\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix}$$

WE HAVE

$$(\underline{\Lambda} - \delta D)\,\underline{S} = \sigma\,\underline{S}$$

$$\underline{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} \qquad \sigma = \lambda - 1$$

$$D = \begin{pmatrix} 2C_0 b_0 & C_1 b_0 & C_2 b_0 \\ 2C_0 b_1 & C_1 b_1 & C_2 b_1 \\ 2C_0 b_2 & C_1 b_2 & C_2 b_2 \end{pmatrix}$$

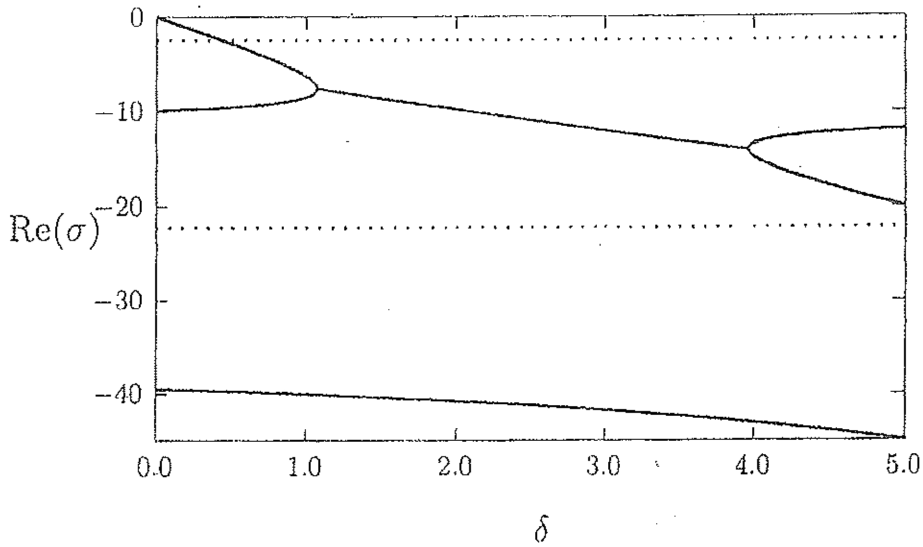
$$\underline{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\pi^2 & 0 \\ 0 & 0 & -4\pi^2 \end{pmatrix} \qquad \underline{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix}$$

ALL OTHER eigenfunctions ARE independent OF δ. THEIR ARE <u>FIXED</u> EigenFUNCTION \ EigenVALUE PAIRS.

(V) FROM SOLVING THE MATRIX EIGENVALUE PROBLEM WE GET: $b_0 = 2.5$, $b_1 = 1 = b_2$

$C_0 = 1.1$, $C_1 = -0.5$

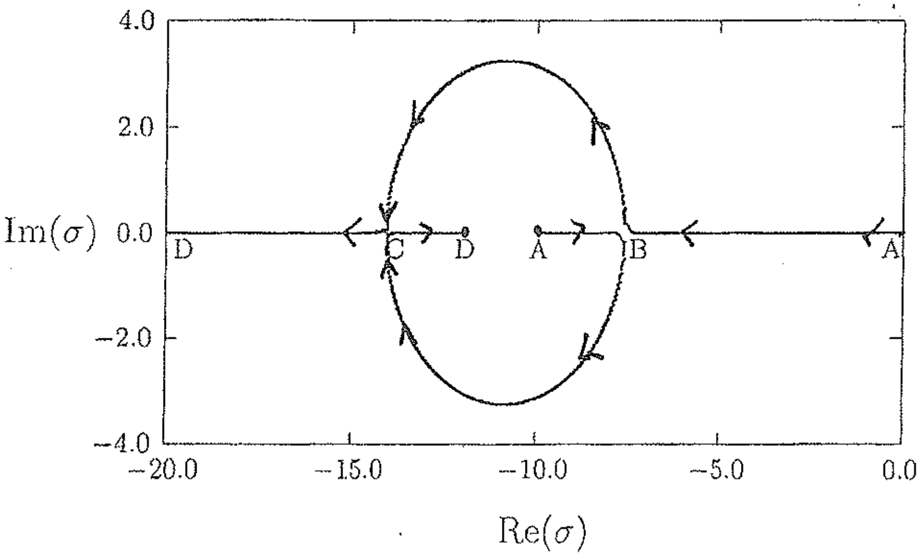
$C_2 = 0.5$



TOP: PLOT OF REAL PART OF EIGENVALUES $RE(\sigma)$ VS δ. THERE ARE TWO COMPLEX EIGENVALUES WHEN $1.076 < \delta < 3.970$. NOTICE THAT $RE\ \sigma < -1$ (i.e. stability) even when δ IS RATHER SMALL.

THE TWO FIXED EIGENVALUES AT $\sigma = -\pi^2/4$ AND $\sigma = -9\pi^2/4$, CORRESPONDING TO ODD EIGENFUNCTIONS ARE THE DOTTED LINES.

BOTTOM: PLOT OF PATH OF EIGENVALUES $IM\ \sigma$ VS. $RE\ \sigma$ AS δ IS INCREASED. POINTS A, B, C, D CORRESPOND TO $\delta = 0$, $\delta = 1.076$, $\delta = 3.97$ AND $\delta = 5.0$.



THE ARROWS INDICATE DIRECTION OF PATH AS δ INCREASES.

FOR STABILITY WE WANT $RE(\lambda) < 0$.

BUT $\lambda = \sigma + 1$

SO WE WANT $RE(\sigma) < -1 \rightarrow$ STABILITY.

WE CONCLUDE THAT THE NONLOCAL TERM "PUSHES" THE UNSTABLE EIGENVALUE OF $\delta = 0$ PROBLEM INTO $Re(\lambda) < 0$ AS δ IS INCREASED.

(vi) FINALLY TAKE $b = c = 1$ $\forall x$. LET $\tau > 0$.

SO $$\Phi'' - \frac{\delta}{1+\tau\lambda} \int_{-1}^{1} \Phi(x)\,dx = (\lambda - 1)\Phi$$

$$\Phi' = 0 \text{ AT } x = \pm 1.$$

<u>NOTICE</u>: <u>ONLY</u> THE EIGENVALUE $\lambda - 1 = 0$ WITH $\Phi_0 = 1$ IS AFFECTED BY NONLOCAL TERM. THE OTHER eigenfunction/eigenvalue PAIRS ARE

$$(*) \begin{cases} \Phi_n(x) = \cos\left(\dfrac{n\pi}{2}(x+1)\right) & n = 1, 2, 3, \ldots \\ \lambda_n = 1 - n^2\pi^2/4 < 0 & \text{FOR } n = 1, 2, 3, \ldots \end{cases}$$

(*) ALSO HOLD FOR $\delta > 0$.

THUS WE NEED ONLY ASK HOW IS $\Phi_0 =$ CONSTANT AFFECTED BY NONLOCAL TERM. THE eigenVALUE WITH $\Phi_0 = A$ $A =$ ANY CONSTANT IS

$$\frac{-2A\delta}{1+\tau\lambda} = (\lambda - 1)A \quad \text{OR} \quad \lambda - 1 = -\frac{2\delta}{1+\tau\lambda}.$$

SO $(1+\tau\lambda)(1-\lambda) = +2\delta$.

WE HAVE $(1+\tau\lambda)(1-\lambda) = 2\delta$

FOR $\delta = 0$, $\lambda = 1$, $\lambda = -1/\tau$.

WE HAVE $\tau\lambda^2 + \lambda(1-\tau) + 2\delta - 1 = 0$. $\lambda^2 - \lambda\left(\frac{\tau-1}{\tau}\right) + \frac{(2\delta-1)}{\tau} = 0$.

THUS IF $\frac{\tau-1}{\tau} < 0$ AND $\frac{2\delta-1}{\tau} > 0 \rightarrow$ $\lambda_1 + \lambda_2 < 0$ $\rightarrow$ RE $\lambda < 0$.
$\lambda_1 \lambda_2 > 0$.

THUS WE HAVE <u>STABILITY</u> <u>IFF</u> $\tau < 1$ AND $\delta > 1/2$.

<u>EG</u>: SUPPOSE WE TAKE $\delta = 1$ AND INCREASE τ.

THEN WE HAVE A HOPF BIFURCATION IF $\tau = 1$.

PROBLEM 2 LET w BE HOMOCLINIC OF $w'' - w + w^{2q-1} = 0$ FOR $q > 1$.

DEFINE $L_0 \Phi = \Phi'' - \Phi + (2q-1) w^{2q-2} \Phi$.

(i) VERIFY $L_0(w^q) = (q^2-1) w^q$.

WE CALCULATE $L_0(w^q) = (w^q)'' - w^q + (2q-1) w^{3q-2}$

NOW $L_0(w^q) = q (w^{q-1} w')' - w^q + (2q-1) w^{3q-2}$

$$L_0(w^q) = q(q-1) w^{q-2} w'^2 + q w^{q-1} w'' - w^q + (2q-1) w^{3q-2}$$

BUT $w'^2/2 = w^2/2 - w^{2q}/2q$ OR $w'^2 = w^2 - w^{2q}/q$ AND $w'' = w - w^{2q-1}$.

WE SUBSTITUTE
$$L_0(w^q) = q(q-1) w^{q-2} \left[w^2 - \frac{w^{2q}}{q} \right] + q w^{q-1} (w - w^{2q-1}) - w^q + (2q-1) w^{3q-2}.$$

THIS GIVES
$$L_0(w^q) = w^{3q-2} \underbrace{\left[(2q-1) - q - (q-1) \right]}_{=0} + w^q \underbrace{\left[q(q-1) + q - 1 \right]}_{= q^2-1}$$

THUS $L_0(w^q) = (q^2-1) w^q$.

IN PARTICULAR FOR $q = 3$, $w'' - w + w^5 = 0$
HAS $L_0(w^3) = 8 w^3$.

(ii) CONSIDER THE NLEP
$$\underbrace{\Phi'' - \Phi + 5 w^4 \Phi}_{L_0 \Phi} - \chi(\lambda) w^3 \frac{\int w^3 \Phi}{\int w^2} = \lambda \Phi$$

THIS IS THE CASE $q = 3$. $L_0(w^3) = 8 w^3$.

WE CALCULATE $\int_{-\infty}^{\infty} (w^3 L_0 \Phi - \Phi L_0(w^3)) dy = 0$

SO
$$\int_{-\infty}^{\infty} w^3 \left(\chi w^3 \frac{\int w^3 \Phi}{\int w^2} + \lambda \Phi \right) dy = \int 8 \Phi w^3 dy$$

THUS
$$\left(\chi \frac{\int w^6}{\int w^2} + \lambda \right) \int w^3 \Phi \, dy = 8 \int \Phi w^3 dy.$$

- FOR EIGENFUNCTION FOR WHICH $\int w^3 \Phi \neq 0$ WE HAVE THAT

$$\chi \frac{\int w^6}{\int w^2} + \lambda = 8.$$

BUT FROM EARLIER IN NOTES WITH $w'' - w + w^p = 0$ WE HAD $\frac{I_2}{I_1} = \frac{2(p+1)}{p+3} = \frac{\int w^{p+1}}{\int w^2}$

SET $p=5$ TO get $\int w^6 / \int w^2 = \frac{2 \cdot 6}{8} = \frac{3}{2}$.

WE CONCLUDE THAT $\frac{3\chi}{2} = 8 - \lambda$ AS DESIRED.

- NOW FOR EIGENFUNCTIONS WITH $\int w^3 \Phi \, dy = 0$, THEY SATISFY $L_0 \Phi = \nu \Phi$.

THE FIRST EIGENFUNCTION WITH $\Phi_0 > 0$ CAN NOT SATISFY $\int w^3 \Phi \, dy = 0$, AND SO IS NOT IN THIS SET. THE ONLY OTHER DISCRETE EIGENVALUE IS $\nu_1 = 0$, $\Phi_1 = w'$.