CONVOLUTION

Suppose we can invert separately $\hat{F}(s)$ and $G(s)$.
Is there a formula for inverting the product $H(s) = \hat{F}(s)\, G(s)$?

i.e. can we find

$$h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}[\hat{F}(s)\, G(s)].$$

REMARK THE FORMULA IS <u>NOT</u> $h(t) = f(t)\cdot g(t)$.

INSTEAD IT IS THE <u>CONVOLUTION</u> OF $f(t)$ AND $g(t)$ EXPRESSED AS FOLLOWS:

LEMMA LET $\hat{F}(s) = \mathcal{L}(f(t))$ AND $G(s) = \mathcal{L}(g(t))$.

THEN IF $H(s) = \hat{F}(s)\, G(s)$

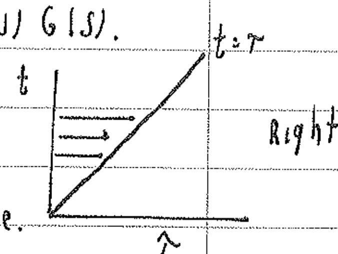
we have
$$\boxed{h(t) \equiv \mathcal{L}^{-1}[H(s)] = \int_0^t f(t-\tau)\, g(\tau)\, d\tau} \qquad (*)$$

(OR EQUIVALENTLY $h(t) = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$.

THE EXPRESSION IN $(*)$ IS CALLED THE <u>CONVOLUTION</u> OF f AND g AND WE HAVE THE SHORTHAND NOTATION

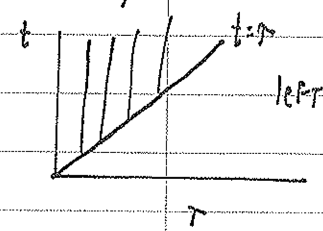
$$h(t) = f * g \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau.$$

<u>PROOF OF $(*)$</u> WE WANT TO SHOW $\mathcal{L}(h(t)) = \hat{F}(s)\, G(s)$.



WE CALCULATE $\mathcal{L}(h) = \int_0^\infty \left(\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right) e^{-st}\, dt.$

This is a 2-D integral represented as shown in right figure.



Instead, let's change direction of integration to left figure.

Then
$$\mathcal{L}(h) = \int_0^\infty \left(\int_\tau^\infty f(t-\tau)\, g(\tau)\, e^{-st}\, dt \right) d\tau$$

$$= \int_0^\infty g(\tau) \left(\int_\tau^\infty f(t-\tau)\, e^{-st}\, dt \right) d\tau$$

let $t' = t - \tau$ so $dt = dt'$ and $e^{-st} = e^{-st'} e^{-s\tau}$.

Then $\mathcal{L}(h) = \int_0^\infty g(\tau) \left(\int_0^\infty f(t') e^{-st'} e^{-s\tau} dt' \right) d\tau$.

This yields $\mathcal{L}(h) = \int_0^\infty g(\tau) e^{-s\tau} \left(\int_0^\infty f(t') e^{-st'} dt' \right) d\tau$

independent of τ

so $\mathcal{L}(h) = \int_0^\infty g(\tau) e^{-s\tau} d\tau \int_0^\infty f(t') e^{-st'} dt' = G(s) \hat{F}(s)$ ∎

EXAMPLE 1 FIND INVERSE TRANSFORM OF

(i) $H(s) = \dfrac{a}{(s^2+a^2) s^2}$ (ii) $H(s) = \dfrac{1}{(s+1)^2} \dfrac{1}{s^2+4}$.

SOLUTION (i) WRITE $H(s) = \underbrace{\dfrac{a}{s^2+a^2}}_{\hat{F}} \underbrace{\dfrac{1}{s^2}}_{G}$ so $f(t) = \sin(at) = \mathcal{L}^{-1}[\hat{F}]$

$g(t) = t = \mathcal{L}^{-1}[G]$

thus $h(t) = f * g = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t \sin[a(t-\tau)] \tau \, d\tau = \int_0^t \sin(a\tau)(t-\tau) d\tau$.

SOLUTION (ii) $H(s) = \hat{F}(s) G(s)$ $\hat{F}(s) = \dfrac{1}{(s+1)^2}$ $G(s) = \dfrac{1}{2}\left(\dfrac{2}{s^2+4}\right)$

so $f(t) = t e^{-t}$ and $g(t) = \frac{1}{2} \sin(2t)$.

Thus $h(t) = f * g = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t (t-\tau) e^{-(t-\tau)} \frac{1}{2} \sin(2\tau) d\tau$.

EXAMPLE FIND THE FUNCTION $y(t)$ THAT IS A SOLUTION TO THE INTEGRAL EQUATION

$$y(t) = 4t - 3\int_0^t y(\tau) \sin(t-\tau) d\tau.$$

SOLUTION Notice that the integral term is a convolution. Hence we LT. let $Y(s) = \mathcal{L}(y(t))$. Then

$$Y(s) = \frac{4}{s^2} - 3\mathcal{L}(y)\mathcal{L}(\sin t) = \frac{4}{s^2} - 3Y\frac{1}{s^2+1}.$$

So $Y\left[1 + \dfrac{3}{s^2+1}\right] = \dfrac{4}{s^2}$

This gives $\bar{Y}(s) = \dfrac{4(s^2+1)}{s^2(s^2+4)} = \dfrac{(s^2+4)}{s^2(s^2+4)} + \dfrac{3s^2}{s^2(s^2+4)}$

so $\bar{Y}(s) = \dfrac{1}{s^2} + \dfrac{3}{s^2+4} = \dfrac{1}{s^2} + \dfrac{3}{2}\left(\dfrac{2}{s^2+4}\right)$

Thus $y(t) = \mathcal{L}^{-1}[\bar{Y}(s)] = t + \dfrac{3}{2}\sin(2t)$.

So $y(t) = t + \dfrac{3}{2}\sin(2t)$ solves integral equation.

<u>Example</u> Let $y'' + 4y' + 4y = g(t)$ with $y(0) = 2$, $y'(0) = -3$ and $g(t)$ is arbitrary. Find the solution using LT.

<u>Solution</u> $(s^2\bar{Y} - s\,y(0) - y'(0)) + 4(s\bar{Y} - y(0)) + 4\bar{Y} = G(s)$

so $(s^2+4s+4)\bar{Y} - 2s + 3 - 8 = G(s)$

so $\bar{Y} = \dfrac{2s+5}{(s+2)^2} + \dfrac{G(s)}{(s+2)^2} = \dfrac{2(s+2)+1}{(s+2)^2} + \dfrac{G(s)}{(s+2)^2}$

Then $\bar{Y}(s) = \dfrac{2}{(s+2)} + \dfrac{1}{(s+2)^2} + \dfrac{G(s)}{(s+2)^2}$

We first write $\bar{Y}(s) = \dfrac{2}{s+2} + \dfrac{1}{(s+2)^2} + \hat{F}(s)G(s)$ $\quad\hat{F}(s) = \dfrac{1}{(s+2)^2}$

so $f(t) = e^{-2t}t = \mathcal{L}^{-1}[\hat{F}]$

invert by convolution theorem

Thus $y(t) = 2e^{-2t} + te^{-2t} + \int_0^t g(\tau)\, f(t-\tau)\, d\tau$

where $f(t) = e^{-2t}t$.

Thus $y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}\, g(\tau)\, d\tau$

is the solution.