LAPLACE TRANSFORM

DEFINITION $\hat{f}(s) = \mathcal{L}(f(t)) \equiv \int_0^\infty e^{-st} f(t)\,dt.$ (*)

- $f(t)$ is piecewise continuous
- $\exists$ constants M and C so that $|f(t)| \leq Me^{ct}$.

Then (*) exists for $s \geq a > C$.

Properties

(i) Linearity $\mathcal{L}(f_1 + f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2)$

(ii) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] - f'(0)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

(iii) $\mathcal{L}(e^{at} f(t)) = \hat{f}(s-a)$ with $\hat{f}(s) = \mathcal{L}(f(t))$.

(iv) there is only one p.c. function whose Laplace transform is $\hat{f}(s)$.

PROOF (i) obvious

(ii) $\mathcal{L}(f') = \int_0^\infty f'(t) e^{-st}\,dt = e^{-st} f\Big|_0^\infty + s\int_0^\infty f e^{-st}\,dt = -f(0) + s\mathcal{L}(f)$

$u = e^{-st}$, $du = -se^{-st}\,dt$

$dv = f'\,dt$ $v = f$

(iii) $\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{at} f(t) e^{-st}\,dt = \int_0^\infty f(t) e^{-(s-a)t}\,dt = \hat{f}(s-a).$

TABLE OF TRANSFORMS

(i) $\mathcal{L}(1) = 1/s$ $(s > 0)$

$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \frac{1}{a-s} e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a}$ for $s > a$

(ii) $\mathcal{L}(e^{i\omega t}) = \int_0^\infty e^{-st} e^{i\omega t}\,dt = \frac{1}{s - i\omega}\frac{(s+i\omega)}{(s+i\omega)} = \frac{s + i\omega}{s^2 + \omega^2}.$

so $\mathcal{L}(\cos \omega t) = s/_{s^2 + \omega^2}.$

$\mathcal{L}(\sin \omega t) = \omega/_{s^2 + \omega^2}$

(iii) $\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = a/_{s^2-a^2}$

so $\mathcal{L}(\sinh wt) = \dfrac{w}{s^2 - w^2}$ $\quad s > w$

$\mathcal{L}(\cosh wt) = \frac{1}{2}\mathcal{L}(e^{wt}) + \frac{1}{2}\mathcal{L}(e^{-wt}) = \frac{1}{2}\left(\frac{1}{s-w} + \frac{1}{s+w}\right)$

$\mathcal{L}(\cosh wt) = \dfrac{s}{s^2 - w^2}$

(iv) $\mathcal{L}(t^n)$. Let $n \geq 1$.

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n \, dt = -\frac{t^n}{s} e^{-st}\Big|_0^\infty + n\int_0^\infty t^{n-1} e^{-st} \, dt$$

$u = t^n, \quad du = n t^{n-1} dt$

$dv = e^{-st} dt \quad v = -\frac{1}{s} e^{-st}$

so for $n \geq 1$ $\quad \mathcal{L}(t^n) = \frac{n}{s}\mathcal{L}(t^{n-1})$.

$\hat{f}_n = n\, \hat{f}_{n-1} \quad \hat{f}_0 = 1/s$

so $\hat{f}_1 = 1/s^2$

$\hat{f}_2 = \frac{2}{s}\hat{f}_1 = 2/s^3$

$\hat{f}_3 = \frac{3}{s}\hat{f}_2 = 3\cdot 2/s^4$

Thus $\hat{f}_n = \mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$ $\quad n = 0, 1, 2, \ldots$

$\mathcal{L}(t^2) = 2!/_{s^3} \qquad \mathcal{L}(t) = 1/_{s^2}$.

<u>**REMARK**</u> Calculate $\mathcal{L}(f(t))$ for each of the following:

(i) $f(t) = t^2 + e^{2t} + \sin 3t \qquad \mathcal{L}(f) = 2/_{s^3} + \frac{1}{s-2} + \frac{3}{s^2+3^2}$.

(ii) $f(t) = 4 + 3\sinh 2t \qquad \mathcal{L}(f) = \mathcal{L}(4) + 3\mathcal{L}(\sinh 2t) = 4/_s + 3\cdot\left(\frac{2}{s^2-4}\right)$.

(iii) $\mathcal{L}(e^{2t}\sin 3t) = \dfrac{3}{(s-2)^2+9}$ (iv) $\mathcal{L}(t^2 e^{t}) = \dfrac{2}{(s-1)^3}$

(v) $\mathcal{L}(e^{-t}\cos t) = \dfrac{(s+1)}{(s+1)^2+1}$

THERE IS AN EXPLICIT FORMULA FOR $f(t) = \mathcal{L}^{-1}[\hat{f}(s)]$ BUT IT REQUIRES complex variable theory. As such, we will use a "look-up" table approach.

$\hat{f}(s)$	$f(t)$
$1/(s-2)^2$	te^{2t} using $\mathcal{L}(e^{at}g) = \hat{g}(s-a)$
$1/(s-4)^3 = \frac{1}{2}\left(\frac{2}{(s-4)^3}\right)$	$\frac{1}{2}t^2e^{4t}$
$\dfrac{(s-7)}{(s-7)^2+25}$	$\cos(5t)e^{7t}$
$\dfrac{1}{s^2-4s+9} = \dfrac{1}{(s-2)^2+5} = \dfrac{1}{\sqrt{5}}\left(\dfrac{\sqrt{5}}{(s-2)^2+5}\right)$	$\frac{1}{\sqrt{5}}e^{2t}\sin(\sqrt{5}t)$
$\dfrac{s}{s^2-4s+9} = \dfrac{(s-2)}{(s-2)^2+5} + \dfrac{2}{(s-2)^2+5}$	$e^{2t}\cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}e^{2t}\sin(\sqrt{5}t)$
$\dfrac{1}{s^2-4s+3} = \dfrac{1}{(s-2)^2-1}$	$e^{2t}\sinh t$
$\dfrac{1}{s^2+6} = \dfrac{1}{\sqrt{6}}\dfrac{\sqrt{6}}{s^2+(\sqrt{6})^2}$	$\frac{1}{\sqrt{6}}\sin(\sqrt{6}t)$

OFTEN PARTIAL FRACTIONS IS USEFUL HERE

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

WE get $A(s^2+1) + Bs(s+C) = 1$

s^2: $A+B=0$

set $s=0 \rightarrow A=1$ so $B=-1$

s: $BC=0 \rightarrow C=0$

so $$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

so $\mathcal{L}^{-1}\left(\dfrac{1}{s(s^2+1)}\right) = 1-\cos t.$

APPLICATIONS TO ODE'S <u>WITH CONSTANT COEFFICIENTS</u>

$$a y'' + b y' + c y = g(t) \qquad a, b, c \text{ CONSTANTS} \quad g \text{ piecewise CONTINUOUS}$$
$$y(0),\ y'(0) \text{ given.}$$

WE TAKE $\mathcal{L}(a y'' + b y' + c y) = \mathcal{L}(g(t))$

$$\mathcal{L}(g(t)) \equiv G(s)$$

so $a \mathcal{L}(y'') + b \mathcal{L}(y') + c \mathcal{L}(y) = G(s)$ since $\mathcal{L}$ linear and a, b, c constants.

NOW $a\left[s^2 Y(s) - s y(0) - y'(0) \right] + b \left[s Y(s) - y(0) \right] + c Y(s) = G(s)$

where $Y(s) = \mathcal{L}(y(t))$.

WE SOLVE FOR $Y(s)$ by algebra.

$$Y(s)(a s^2 + b s + c) = a\left[s y(0) + y'(0) \right] + b y(0) + G(s)$$

so $$Y(s) = \frac{a\left[s y(0) + y'(0) \right] + b y(0)}{a s^2 + b s + c} + \frac{G(s)}{a s^2 + b s + c}$$

then invert to get $y(t) = \mathcal{L}^{-1}[Y(s)]$ using look-up table ...

<u>REMARK</u> · does not apply immediately to variable coefficient problems

since if $a = a(t)$ WE <u>DO NOT</u> have $\mathcal{L}(a y'') = a \mathcal{L}(y'')$

· does not apply to <u>NONLINEAR problems</u>

since $\mathcal{L}(y^2) \neq (\mathcal{L}(y))^2$.

OFTEN WHEN SOLVING THE DE'S WE HAVE THE FOLLOWING INVERSION PROBLEM:

NOW SUPPOSE THAT $\hat{f}(s)$ CAN BE EXPRESSED AS RATIO OF TWO POLYNOMIALS P, Q OF FORM

$$\hat{f}(s) = \frac{P(s)}{Q(s)} \qquad \deg Q = n > \deg P.$$

ASSUME THAT $Q(s)$ HAS DISTINCT REAL ZEROES AT $r_1, r_2, \dots, r_n$.

then by partial fractions:

$$\hat{f}(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \dots + \frac{A_n}{s-r_n} \qquad \text{FOR SOME } A_1, \dots, A_n.$$

TO FIND A_K WE CALCULATE

$$A_K = \lim_{s \to r_K} \frac{P(s)(s-r_K)}{Q(s)} = \lim_{s \to r_K} \frac{P(s)}{\frac{Q(s)-Q(r_K)}{s-r_K}} = \frac{P(r_K)}{Q'(r_K)}$$

THUS $A_K = \dfrac{P(r_K)}{Q'(r_K)}$

$$\text{SO} \quad \hat{f}(s) = \sum_{K=1}^{n} \frac{P(r_K)}{Q'(r_K)} \frac{1}{s-r_K}.$$

$$\text{NOW} \quad \left[f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \sum_{K=1}^{n} \frac{P(r_K)}{Q'(r_K)} e^{r_K t}. \right]$$

NOTICE THAT IF ANY ROOT $r_K > 0$ THEN IF $P(r_K) \neq 0$, f GROWS EXPONENTIALLY AS $t \to \infty$.

EXAMPLE solve $y'' - 3y' + 2y = e^{3t}$ WITH $y(0) = 1$, $y'(0) = 0$ BY LT Method.

let $\bar{Y}(s) = \mathcal{L}(y(t))$. so

$$\left(s^2 \bar{Y}(s) - s y(0) - y'(0)\right) - 3\left(s \bar{Y}(s) - y(0)\right) + 2\bar{Y} = \frac{1}{s-3}.$$

THUS $$(s^2 - 3s + 2)\bar{Y}(s) = s - 3 + \frac{1}{s-3}$$

THUS $$\bar{Y}(s) = \frac{s-3}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s-3)}$$

PUT OVER COMMON DENOMINATOR

$$\bar{Y}(s) = \frac{(s-3)^2 + 1}{(s-1)(s-2)(s-3)} \equiv \frac{P(s)}{Q(s)}. \qquad P \equiv (s-3)^2 + 1, \quad Q \equiv (s-1)(s-2)(s-3)$$

TO INVERT

Method 1 use general formula we derived. $Q = 0$ AT $s = 1, 2, 3$.

$$y(t) = \frac{P(1)}{Q'(1)} e^{t} + \frac{P(2)}{Q'(2)} e^{2t} + \frac{P(3)}{Q'(3)} e^{3t}$$

NOW $P(1) = 5$, $P(2) = 2$, $P(3) = 1$

$Q'(1) = 2$, $Q'(2) = -1$, $Q'(3) = 2$.

so $$y(t) = \frac{5}{2} e^{t} - 2e^{2t} + \frac{1}{2} e^{3t}$$

Method 2 (Brute force) partial fractions.

$$\frac{(s-3)^2 + 1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

so $$(s-3)^2 + 1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

let $s = 1 \rightarrow 5 = A(2) \rightarrow A = 5/2$

$s = 2 \rightarrow 2 = B(-1) \rightarrow B = -2$

$s = 3 \rightarrow 1 = 2C \rightarrow C = 1/2$.

EXAMPLE CONSIDER $y'' - y' - 6y = 1 + e^{-t}$ WITH $y(0) = \alpha$, $y'(0) = 0$

where α is a parameter.

FIND the value of α FOR which $y \to -1/6$ AS $t \to \infty$.

Method 1 Use method of undetermined coefficients.

homog: $y = e^{rt} \to r^2 - r - 6 = (r-3)(r+2) = 0 \qquad r = 3, -2.$

inhomog. $L(y_{p_1}) = 1 \to y_{p_1} = -1/6$

$L(y_{p_2}) = e^{-t}$ so $y_{p_2} = Ae^{-t}$ ... must find A.

gen. solution is $y = c_1 e^{-2t} + c_2 e^{+3t} - \frac{1}{6} + Ae^{-t}$

then calculate c_1, c_2 from $y(0) = \alpha$, $y'(0) = 0$. choose α so that $c_2 = 0$.

$\to$ LOTS of Algebra. then $y \to -1/6$ as $t \to \infty$.

Method 2 LT $\to$ FAST WAY.

take L.T. $(s^2 Y - s y(0) - y'(0)) - (sY - y(0)) - 6Y = \frac{1}{s} + \frac{1}{s+1}$.

BUT $y(0) = \alpha$, $y'(0) = 0$ so

$$(s^2 - s - 6) Y = s\alpha - \alpha + \frac{2s+1}{s(s+1)} = \frac{\alpha(s-1)s(s+1) + (2s+1)}{s(s+1)}$$

$$(*) \left\{ \; Y = \frac{\alpha s(s^2-1) + (2s+1)}{s(s+1)(s-3)(s+2)} = \frac{P(s)}{Q(s)} \right.$$

we have $y = \frac{P(0)}{Q'(0)} + \frac{P(-1)}{Q'(-1)} e^{-t} + \frac{P(-2)}{Q'(-2)} e^{-2t} + \frac{P(3)}{Q'(3)} e^{3t}$

to ensure $y \to -1/6$ as $t \to \infty$

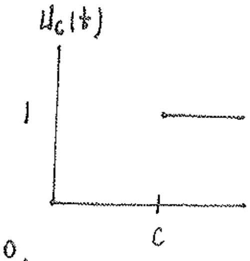
we simply need to set $P(3) = 0$.

thus $\alpha\, 3 \cdot 8 + 7 = 0 \to \alpha = -7/24$.

Advantage of LT is that initial conditions are

BUILT DIRECTLY IN FORMULA $(*)$

STEP FUNCTIONS AND LAPLACE TRANSFORMS



WE DEFINE $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$ FOR $c > 0$

NOW $\mathcal{L}(u_c(t)) = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \dfrac{e^{-cs}}{s}$ FOR $s > 0$.

THUS $\mathcal{L}(u_c(t)) = e^{-cs}/s$ FOR $s > 0$ (AND $c > 0$).

NOW WE SHOW THE FOLLOWING:

LEMMA $\mathcal{L}[u_c(t)\, g(t-c)] = e^{-cs} G(s)$ WITH $G(s) = \mathcal{L}(g(t))$.

PROOF $\mathcal{L}(u_c(t)\, g(t-c)) = \int_0^\infty u_c(t)\, g(t-c) e^{-st}\, dt = \int_c^\infty g(t-c) e^{-st}\, dt.$

NOW let $\tau = t - c$ SO THAT

$$\mathcal{L}(u_c(t)\, g(t-c)) = \int_0^\infty g(\tau) e^{-s(\tau + c)}\, d\tau = e^{-sc} \underbrace{\int_0^\infty g(\tau) e^{-s\tau}\, d\tau}_{G(s)}.$$

THIS YIELDS THAT $\mathcal{L}(u_c(t)\, g(t-c)) = e^{-sc} G(s)$.

EXAMPLES OF USING THIS RESULT. WE NOTE $\boxed{\mathcal{L}^{-1}[e^{-sc} G(s)] = u_c(t)\, g(t-c).}$ (*)

EXAMPLE 1 FIND THE INVERSE L.T. OF EACH OF THE FOLLOWING.

(i) $\dfrac{e^{-s}}{s^2 + 9}$ (ii) $\dfrac{s e^{-3s}}{s^2 + 2s + 10}$

(iii) $\dfrac{1}{(s+1)(s+2)s} e^{-3s}$ (iv) $\dfrac{1}{s^3} e^{-2s}$

SOLUTION

(i) WE WRITE $F(s) = \dfrac{1}{3}\left(\dfrac{3}{s^2 + 3^2}\right) e^{-s}$ AND LET $G(s) = \dfrac{1}{3}\left(\dfrac{3}{s^2 + 3^2}\right)$

THUS $g(t) = \mathcal{L}^{-1}[G(s)] = \dfrac{1}{3} \sin(3t)$. HENCE USING (*) WE HAVE

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{3} \sin[3(t-1)]\, u_1(t) = \begin{cases} 0 & \text{IF } 0 < t < 1 \\ \frac{1}{3} \sin[3(t-1)] & \text{IF } t \geq 1. \end{cases}$$

(ii) $\hat{f}(s) = G(s)e^{-3s}$ WITH $G(s) = \dfrac{s}{s^2+2s+10}$.

COMPLETE SQUARE $G(s) = \dfrac{s}{(s+1)^2+9} = \dfrac{(s+1)}{(s+1)^2+9} - \dfrac{1}{(s+1)^2+9}$

SO $G(s) = \dfrac{(s+1)}{(s+1)^2+3^2} - \dfrac{1}{3}\left(\dfrac{3}{(s+1)^2+3^2}\right)$

THUS $g(t) = -\dfrac{1}{3}e^{-t}\sin(3t) + e^{-t}\cos(3t)$

THEN BY (*) WE HAVE $f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \left(-\dfrac{1}{3}e^{-(t-3)}\sin[3(t-3)] + e^{-(t-3)}\cos(3(t-3))\right)$

$\cdot\, u_3(t)$.

(iii) $\hat{f}(s) = G(s)e^{-3s}$ WITH $G(s) = \dfrac{1}{s(s+1)(s+2)}$.

WE WRITE $G(s) = \dfrac{P(s)}{Q(s)}$ AND DO PF DECOMPOSITION.

$\dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$ SO $1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

LET $s=0 \to A = 1/2$

$s=-1 \to B = -1$ SO $g(t) = \mathcal{L}^{-1}[G(s)] = \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t}$.

$s=-2 \to C = 1/2$

THEN BY (*) $f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \left(\dfrac{1}{2} - e^{-(t-3)} + \dfrac{1}{2}e^{-2(t-3)}\right)u_3(t)$.

(iv) $\hat{f}(s) = \dfrac{1}{s^3}e^{-2s}$. NOW $G(s) = \dfrac{1}{s^3}$. RECALL $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$.

LET $n=2$ SO THAT $\mathcal{L}(t^2) = \dfrac{2}{s^3}$. THEN, $\mathcal{L}(t^2/2) = 1/s^3$.

WE HAVE $g(t) = \mathcal{L}^{-1}[G(s)] = t^2/2$.

THUS $f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \dfrac{(t-2)^2}{2}u_2(t)$.

NOW WITH PIECEWISE CONTINUOUS FUNCTIONS WE NEED TO WRITE THEM IN THE FORM OF STEP FUNCTIONS $U_c(t)$ AND IN THE PRECISE FORM $U_c(t)\, g(t-c)$ SO THAT WE CAN TAKE LT.

<u>EXAMPLE</u> FIND THE LT OF EACH OF THE FOLLOWING FUNCTIONS

(i) $f(t) = \begin{cases} 1, & \text{IF } 0 \le t < T \\ 0, & t \ge T \end{cases}$ (ii) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < T \\ t, & \text{IF } t \ge T. \end{cases}$

(iii) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < T \\ 1, & \text{IF } T < t < 2T \\ 0 & \text{IF } t > 2T \end{cases}$ (iv) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < 1 \\ t^2 - 2t + 2, & \text{IF } t \ge 1 \end{cases}$

(v) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < \pi/2 \\ \sin t, & \text{IF } t \ge \pi/2 \end{cases}$ (vi) $f(t) = \begin{cases} e^{-t} & \text{IF } 0 \le t < T \\ 0, & \text{IF } t \ge T. \end{cases}$

NOW WE PROCEED AS FOLLOWS: RECALL $\mathcal{L}[U_c(t)\, g(t-c)] = e^{-cs} G(s)$.

<u>SOLUTIONS</u>

(i) $f(t) = 1 + \begin{cases} 0, & \text{IF } 0 \le t < T \\ -1, & \text{IF } t \ge T \end{cases} = 1 - U_T(t).$

THUS $\mathcal{L}(f(t)) = F(s) = \dfrac{1}{s} - \dfrac{e^{-sT}}{s}$.

(ii) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < T \\ (t-T) + T, & \text{IF } t \ge T \end{cases}$.

THUS $f(t) = [(t-T) + T]\, U_T(t) = g(t-T)\, U_T(t)$

where $g(t) = t + T$. THUS $F(s) = e^{-Ts} G(s)$ $G(s) = \mathcal{L}(g(t))$.

WE GET $G(s) = 1/s^2 + T/s$

THUS $F(s) = \left(\dfrac{1}{s^2} + \dfrac{T}{s}\right) e^{-Ts}$.

(iii) WE CAN WRITE $f(t) = U_T(t) - U_{2T}(t)$.

THUS $\hat{f}(s) = \dfrac{e^{-sT}}{s} - \dfrac{e^{-2sT}}{s}$.

(iv) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < 1 \\ t^2 - 2t + 2, & t > 1 \end{cases} = \begin{cases} 0, & 0 \le t < 1 \\ (t-1)^2 + 1, & t > 1 \end{cases}$

SO $f(t) = g(t-1)\,U_1(t)$ WITH $g(t) = t^2 + 1$.

THUS $\hat{f}(s) = G(s)e^{-s} = \left[\dfrac{2}{s^3} + \dfrac{1}{s}\right]e^{-s}$ USING $\mathcal{L}(t^2) = 2/s^3$.

(v) $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < \pi/2 \\ \sin t, & \text{IF } t \ge \pi/2 \end{cases}$

WE WRITE $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \sin((t - \pi/2) + \pi/2), & t \ge \pi/2 \end{cases}$

NOW $\sin[(t-\pi/2) + \pi/2] = \sin(t - \pi/2)\cos(\pi/2) + \sin\pi/2 \cos(t - \pi/2) = \cos(t - \pi/2)$.

THUS $f(t) = \begin{cases} 0, & \text{IF } 0 \le t < \pi/2 \\ \cos(t - \pi/2), & \text{IF } t > \pi/2 \end{cases}$

SO $f(t) = g(t - \pi/2)\,U_{\pi/2}(t)$ WITH $g(t) = \cos t$.

THIS YIELDS $\hat{f}(s) = G(s)e^{-\pi s/2}$ WITH $G(s) = \dfrac{s}{s^2+1}$.

THUS $\hat{f}(s) = \mathcal{L}(f(t)) = e^{-\pi s/2}\left(\dfrac{s}{s^2+1}\right)$

(vi) $f(t) = e^{-t} + \begin{cases} 0, & 0 \le t < T \\ -e^{-t}, & t > T \end{cases} = e^{-t} + \begin{cases} 0, & 0 \le t < T \\ -e^{-(t-T)}e^{-T}, & t > T \end{cases}$

THUS $f(t) = e^{-t} + g(t-T)\,U_T(t)$ WHERE $g(t) = -e^{-t}e^{-T}$.

THUS $\hat{f}(s) = \dfrac{1}{s+1} - \dfrac{e^{-T}e^{-Ts}}{s+1}$.

EXAMPLE $\quad y'' + y = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 2, & \text{if } t \ge 1 \end{cases} \qquad y(0) = 0,\ y'(0) = 1.$

SOLUTION WE WRITE THE RHS AS

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 2, & \text{if } t \ge 1 \end{cases} = t + \begin{cases} 0, & \text{if } 0 \le t < 1 \\ 2-t, & \text{if } t \ge 1 \end{cases}.$$

THUS $f(t) = t + u_1(t)(-t+2) = t - u_1(t)(t-2) = t - u_1(t)[(t-1)-1].$

SO $f(t) = t - u_1(t)\, g(t-1)$ WITH $g(t) = t-1$.

WE NOW TAKE LT:

$$s^2\bar{Y} - sy(0) - y'(0) + \bar{Y} = \mathcal{L}(f) = \frac{1}{s^2} - e^{-s}\mathcal{L}(g) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s}\right)$$

THUS $(s^2+1)\bar{Y} = 1 + \frac{1}{s^2} - e^{-s}\left(\frac{1-s}{s^2}\right) = \frac{s^2+1}{s^2} + e^{-s}\left(\frac{s-1}{s^2}\right)$

THUS $\bar{Y} = \frac{1}{s^2} + G(s)e^{-s}$ WITH $G(s) = \frac{(s-1)}{s^2(s^2+1)}$.

WE HAVE $y = t + u_1(t)\, g(t-1)$ where $g(t) = \mathcal{L}^{-1}[G(s)]$.

WE DO PF DECOMPOSITION: $\frac{s-1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$

SO $\frac{s-1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{(As+B)(s^2+1) + s^2(Cs+D)}{s^2(s^2+1)}$

SO $s-1 = (As+B)(s^2+1) + s^2(Cs+D)$

s^3 TERMS: $A + C = 0 \qquad C = -1$

s TERMS: $A = 1$

s^2 TERMS: $B + D = 0$

CONSTANT: $-1 = B$

$B = -1$, $D = 1$

SO $\frac{s-1}{s^2(s^2+1)} = \frac{1}{s} - \frac{1}{s^2} + \frac{(1-s)}{s^2+1}$

THUS $g(t) = 1 - t + \sin t - \cos t$

WE CONCLUDE THAT $y = t + u_1(t)[1 - (t-1) + \sin(t-1) - \cos(t-1)]$.

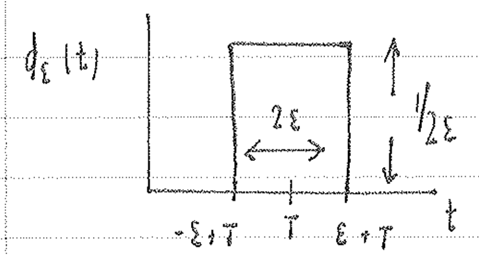
SO $y = t + u_1(t)[-t + 2 + \sin(t-1) - \cos(t-1)] = \begin{cases} t, & \text{if } 0 \le t \le 1 \\ 2 + \sin(t-1) - \cos(t-1), & \text{if } t > 1 \end{cases}$

DIRAC DELTA FUNCTIONS

"IMPULSIVE FORCES":

DEFINE FOR $\varepsilon > 0$ THE FUNCTION $d_\varepsilon(t) = \begin{cases} 1/2\varepsilon, & -\varepsilon \le t-T \le \varepsilon \\ 0, & |t-T| > \varepsilon \end{cases}$

WE PLOT AS FOLLOWS FOR $T > 0$.



AS $\varepsilon \to 0^+$, THE WIDTH OF THE RECTANGLE $\to 0$ WHILE THE HEIGHT $\to +\infty$.

WE HAVE $\lim_{\substack{\varepsilon \to 0^+ \\ t \ne T}} d_\varepsilon(t) = 0$

$\lim_{\substack{\varepsilon \to 0^+ \\ t = T}} d_\varepsilon(t) = \infty$

BUT $\displaystyle\int_{-\varepsilon+T}^{\varepsilon+T} d_\varepsilon(t)\,dt = 1 \quad \forall\, \varepsilon > 0.$

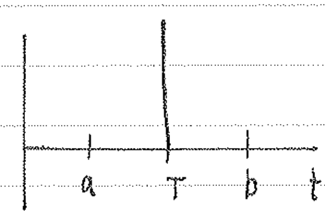
WE SAY THAT $\lim_{\varepsilon \to 0^+} d_\varepsilon(t) = \delta(t-T)$ "DIRAC DELTA"

WHAT THIS MEANS IS MORE COMPLICATED THAN WE EXPLAIN HERE.

WE WILL DEFINE THE DIRAC DELTA FUNCTION "CENTRED AT $t=T$"

BY
$$\delta(t-T) = \begin{cases} 0 & \text{IF } t \ne T \\ \infty & \text{IF } t = T \end{cases}$$

AND $\displaystyle\int_a^b \delta(t-T)\,dt = 1$ IF $b > T$ AND $a < T$.



hammer hitting table at $t=T$

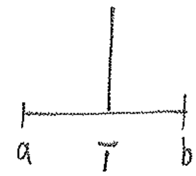
IMPULSIVE FORCE IN MECHANICS SINCE IF $m\,dv/dt = \delta(t-T)$

then by integration $\displaystyle\int_{T^-}^{T^+} m\,dv/dt\,dt = 1 \to m[v] = 1$

"change in momentum".

KEY PROPERTY LET $f(t)$ BE A CONTINUOUS FUNCTION.

THEN $\displaystyle\int_a^b f(t)\,\delta(t-T)\,dt = f(T)$ IF $a<T<b$.



WE HAVE THE "DERIVATION"

$$\int_a^b f(t)\,\delta(t-T)\,dt = \lim_{\varepsilon\to 0^+}\int_a^b f(t)\,d_\varepsilon(t)\,dt = \lim_{\varepsilon\to 0^+}\int_{\varepsilon-T}^{\varepsilon+T} f(t)/2\varepsilon\;dt = f(T)\cdot 1$$

LAPLACE TRANSFORMS

$$\mathcal{L}(\delta(t-T)) = \int_0^\infty \delta(t-T)\,e^{-st}\,dt = e^{-sT} \quad \text{FOR } T>0.$$

NOW SOLVE SOME PROBLEMS WITH DIFFERENTIAL EQUATIONS.

PROBLEM 1 $y'' + 4y = 2\,\delta(t-\pi/4)$ WITH $y(0)=y'(0)=0$.

METHOD 1 USE LAPLACE TRANSFORMS:

$$s^2\bar{Y} - sy(0) - y'(0) + 4\bar{Y} = 2e^{-\pi s/4}$$

WE GET $\displaystyle\bar{Y} = \frac{2}{s^2+4}e^{-\pi s/4}$ RECALL $\mathcal{L}^{-1}[G(s)e^{-cs}] = g(t-c)\,u_c(t)$

THUS $y(t) = \sin[2(t-\pi/4)]\,u_{\pi/4}(t) = \begin{cases} 0, & \text{IF } t<\pi/4 \\ \sin(2t-\pi/2), & \text{IF } t>\pi/4 \end{cases}$

BUT $\sin(2t-\pi/2) = \sin(2t)\cos(\pi/2) - \cos(2t)\sin(\pi/2) = -\cos(2t)$.

THUS $y(t) = \begin{cases} 0, & 0<t<\pi/4 \\ -\cos(2t), & t>\pi/4 \end{cases}$

METHOD 2 WE SOLVE ON EITHER SIDE OF $t=\pi/4$ USING $\delta(t-\pi/4)=0$ IF $t\neq\pi/4$.
THEN GLUE SOLUTION together USING y CONTINUOUS AT $t=\pi/4$

AND $y'(\pi/4^+) - y'(\pi/4^-) = 2$ (*)

NOTE: $\displaystyle\int_{\pi/4-\varepsilon}^{\pi/4+\varepsilon}(y''+4y)\,dy = y'(\pi/4+\varepsilon) - y'(\pi/4-\varepsilon) + 4\int_{\pi/4-\varepsilon}^{\pi/4+\varepsilon} y\,dt = 2.$

let $\varepsilon\to 0$ AND USE y CONTINUOUS. WE get (*)

WITH THIS APPROACH: $y = \begin{cases} c_1 \cos 2t + c_2 \sin 2t, & 0 < t < \pi/4 \\ d_1 \sin 2t + d_2 \cos(2t), & t > \pi/4. \end{cases}$

NOW $y(0) = y'(0) = 0 \longrightarrow c_1 = c_2 = 0.$

BUT NOW $y = \begin{cases} 0, & 0 \le t < \pi/4 \\ d_1 \sin 2t + d_2 \cos 2t, & t > \pi/4 \end{cases}$

IMPOSE CONTINUITY AT $t = \pi/4 \longrightarrow d_1 \sin(\pi/2) + d_2 \cos(\pi/2) = 0 \rightarrow d_1 = 0$

NOW IMPOSE JUMP CONDITION (*) $\longrightarrow 2 d_1 \cos(\pi/2) - 2 d_2 \sin(\pi/4) = 2 \rightarrow d_2 = -1.$

THUS WITH $d_1 = 0,\ d_2 = -1$ $\qquad y = \begin{cases} 0, & 0 \le t < \pi/4 \\ -\cos(2t), & t > \pi/4. \end{cases}$

THIS IS SAME ANSWER AS BEFORE.