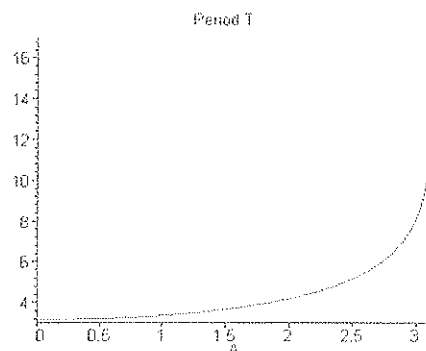SECTION 9.4 # 1, 2, 4

9.5 # 1, 2, 4

FROM BOYCE + DIPRIMA TEXT

28. The characteristic equation of the coefficient matrix is $(r+1)^2 = 0$, with roots $r_1 = r_2 = -1$. Hence the critical point is an asymptotically stable node. On the other hand, the characteristic equation of the perturbed system is $r^2 + 2r + 1 + \epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a stable spiral. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| \ll 1$). The critical point remains a stable node.
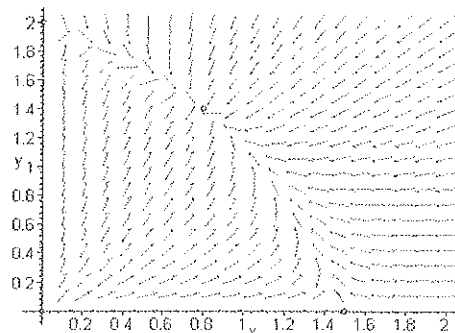
29.(d) Set $k = \sin(\alpha/2) = \sin(A/2)$ and $g/L = 4$.

9.4

$$\frac{dx}{dt} = x(1.5 - x - 0.5y), \qquad \frac{dy}{dt} = y(2 - y - 0.75x)$$

1.(a)

(b) The critical points are solutions of the system of equations

$$x(1.5 - x - 0.5y) = 0$$
$$y(2 - y - 0.75x) = 0.$$

The four critical points are $(0,0)$, $(0,2)$, $(1.5,0)$ and $(0.8,1.4)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0,0)$, the coefficient matrix of the linearized system is

$$J(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2 \, , \, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, ; \, r_2 = 2 \, , \, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0,2)$, the coefficient matrix of the linearized system is

$$J(0,2) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1/2 \, , \, \xi^{(1)} = \begin{pmatrix} 1 \\ -0.6 \end{pmatrix} \, ; \, r_2 = -2 \, , \, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a saddle, which is unstable.
At the critical point $(1.5,0)$, the coefficient matrix of the linearized system is

$$J(1.5,0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1.5 \, , \, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, ; \, r_2 = 0.875 \, , \, \xi^{(2)} = \begin{pmatrix} -0.31579 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a saddle, which is unstable.
At the critical point $(0.8,1.4)$, the coefficient matrix of the linearized system is

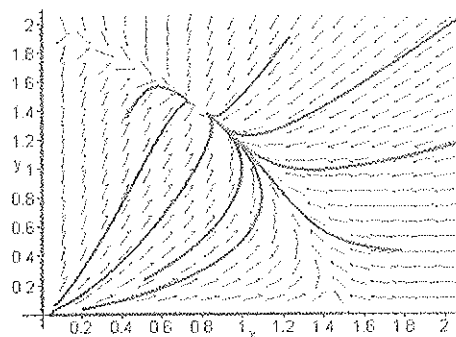$$J(0.8,1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{11}{10} + \frac{\sqrt{51}}{10} \, , \, \xi^{(1)} = \begin{pmatrix} 1 \\ \frac{3-\sqrt{51}}{4} \end{pmatrix} \, ; \, r_2 = -\frac{11}{10} - \frac{\sqrt{51}}{10} \, , \, \xi^{(2)} = \begin{pmatrix} 1 \\ \frac{3+\sqrt{51}}{4} \end{pmatrix}.$$
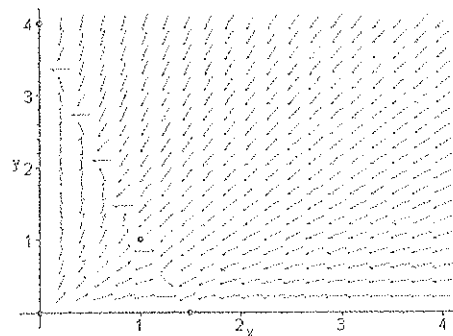
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)

(f) Except for initial conditions lying on the coordinate axes, almost all trajectories converge to the stable node at $(0.8, 1.4)$.

2.(a) $dx/dt = x(1.5 - x - 0.5y)$, $dy/dt = y(2 - 0.5y - 1.5x)$

(b) The critical points are the solution set of the system of equations

$$x(1.5 - x - 0.5y) = 0$$
$$y(2 - 0.5y - 1.5x) = 0.$$

The four critical points are $(0, 0)$, $(0, 4)$, $(1.5, 0)$ and $(1, 1)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2 \,,\, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ;\, r_2 = 2 \,,\, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0,4)$, the coefficient matrix of the linearized system is

$$J(0,4) = \begin{pmatrix} -1/2 & 0 \\ -6 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2 \ , \ \xi^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \ ; \ r_2 = -2 \ , \ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0,4)$ is a stable node,
which is asymptotically stable.
At the critical point $(3/2,0)$, the coefficient matrix of the linearized system is

$$J(3/2,0) = \begin{pmatrix} -3/2 & -3/4 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2 \ , \ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ r_2 = -1/4 \ , \ \xi^{(2)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a stable node, which
is asymptotically stable.
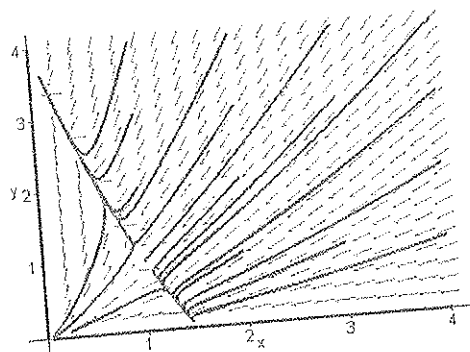At the critical point $(1,1)$, the coefficient matrix of the linearized system is

$$J(1,1) = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{13}}{4} \ , \ \xi^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{13}}{2} \end{pmatrix} \ ; \ r_2 = -\frac{3+\sqrt{13}}{4} \ , \ \xi^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{13}}{2} \end{pmatrix}.$$
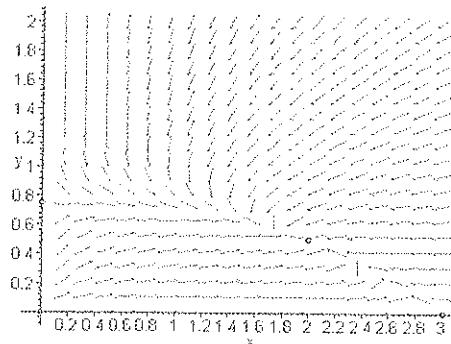
The eigenvalues are of opposite sign, hence $(1,1)$ is a saddle, which is unstable.

(d,e)

(f) Trajectories approaching the critical point $(1,1)$ form a separatrix. Solutions on either side of the separatrix approach either $(0,4)$ or $(1.5,0)$.

4.(a) $\quad dx/dt = x(1.5 - 0.5x - y) \qquad dy/dt = y(0.75 - y - 0.125x)$

(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,x - y) = 0$$
$$y(0.75 - y - 0.125\,x) = 0 \,.$$

The four critical points are $(0,0)$, $(0,3/4)$, $(3,0)$ and $(2,1/2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -x \\ -y/8 & 3/4 - x/8 - 2y \end{pmatrix} .$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix} .$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2 \,, \ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \ r_2 = 3/4 \,, \ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0,3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,3/4) = \begin{pmatrix} 3/4 & 0 \\ -3/32 & -3/4 \end{pmatrix} .$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4 \,, \ \xi^{(1)} = \begin{pmatrix} -16 \\ 1 \end{pmatrix} ; \ r_2 = -3/4 \,, \ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The eigenvalues are of opposite sign, hence the critical point $(0,3/4)$ is a saddle, which is unstable.

At the critical point $(3,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3,0) = \begin{pmatrix} -3/2 & -3 \\ 0 & 3/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2 , \ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \ r_2 = 3/8 , \ \xi^{(2)} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0,3/4)$ is a saddle, which is unstable.

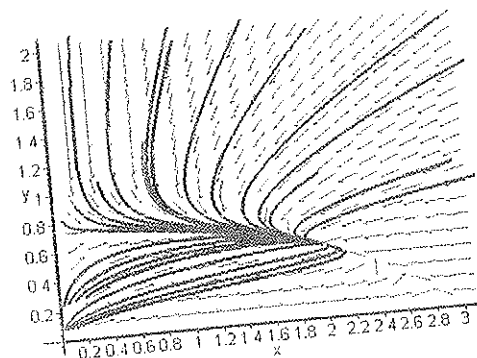At the critical point $(2,1/2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2,1/2) = \begin{pmatrix} -1 & -2 \\ -1/16 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + \sqrt{3}}{4} , \ \xi^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{3}}{8} \end{pmatrix} ; \ r_2 = -\frac{3 + \sqrt{3}}{4} , \ \xi^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}}{8} \end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2,1/2)$ is a stable node, which is asymptotically stable.

(d,e)

(f) Except for initial conditions along the coordinate axes, almost all solutions converge to the stable node $(2,1/2)$.

7. It follows immediately that

$$(\sigma_1 X + \sigma_2 Y)^2 - 4\sigma_1\sigma_2 XY = \sigma_1^2 X^2 + 2\sigma_1\sigma_2 XY + \sigma_2^2 Y^2 - 4\sigma_1\sigma_2 XY$$
$$= (\sigma_1 X - \sigma_2 Y)^2.$$

Since all parameters and variables are positive, it follows that
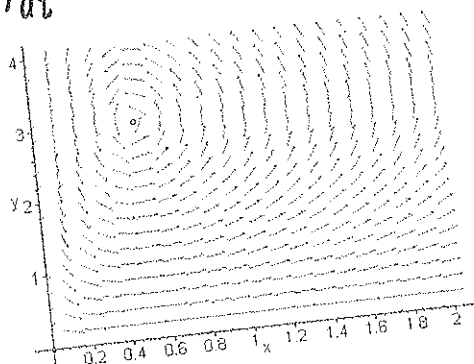
$$(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY \geq 0.$$

Hence the radicand in Eq.(39) is nonnegative.

$$dx/dt = x(1.5 - 0.5y) , \quad dy/dt = y(-0.5 + x)$$

1.(a)

(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5y) = 0$$
$$y(-0.5 + x) = 0.$$

The two critical points are $(0,0)$ and $(0.5,3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

At the critical point $(0,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2 , \; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \; r_2 = -1/2 , \; \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable.
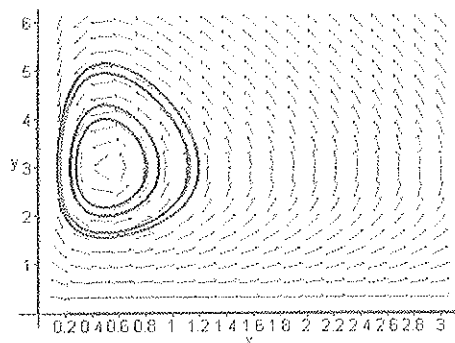At the critical point $(0.5,3)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0.5,3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i\frac{\sqrt{3}}{2} , \; \xi^{(1)} = \begin{pmatrix} 1 \\ -2i\sqrt{3} \end{pmatrix} ; \; r_2 = -i\frac{\sqrt{3}}{2} , \; \xi^{(2)} = \begin{pmatrix} 1 \\ 2i\sqrt{3} \end{pmatrix}.$$
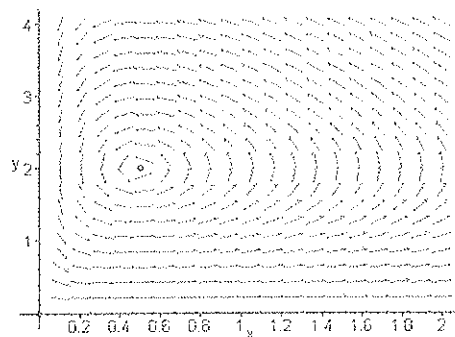
The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.

(d,e)

(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5, 3)$.

2.(a) $dx/dt = x(1 - 0.5y)$ $dy/dt = y(-0.25 + 0.5x)$

(b) The critical points are the solution set of the system of equations

$$x(1 - 0.5\,y) = 0$$
$$y(-0.25 + 0.5\,x) = 0.$$

The two critical points are $(0,0)$ and $(0.5, 2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 - y/2 & -x/2 \\ y/2 & -1/4 + x/2 \end{pmatrix}.$$

At the critical point $(0,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1 \, , \, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, ; \, r_2 = -1/4 \, , \, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable.
At the critical point $(0.5, 2)$, the coefficient matrix of the linearized system is
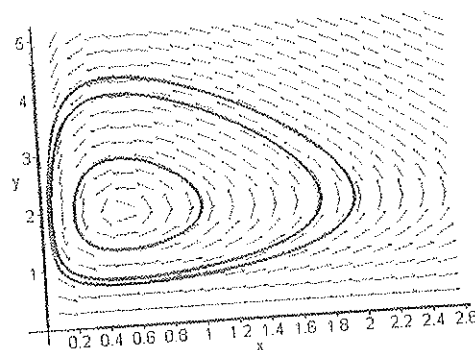
$$J(0.5, 2) = \begin{pmatrix} 0 & -1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2, \; \xi^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}; \; r_2 = -i/2, \; \xi^{(2)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.
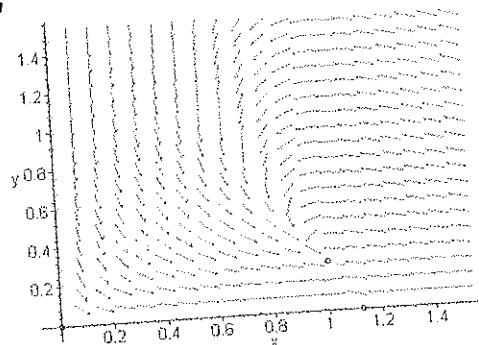
(d,e)

(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5, 2)$.

4.(a)
$$dx/dt = x\left(9/8 - x - y/2\right) \qquad dy/dt = y\left(-1 + x\right)$$

(b) The critical points are the solution set of the system of equations

$$x(9/8 - x - y/2) = 0$$
$$y(-1 + x) = 0.$$

The three critical points are $(0, 0)$, $(9/8, 0)$ and $(1, 1/4)$.

(c) The Jacobian matrix of the vector field is

$$J = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0,0)$, the coefficient matrix of the linearized system is

$$J(0,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8 \,, \, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, ; \, r_2 = -1 \,, \, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable.
At the critical point $(9/8,0)$, the coefficient matrix of the linearized system is

$$J(9/8,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8} \,, \, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, ; \, r_2 = \frac{1}{8} \,, \, \xi^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8,0)$ is a saddle, which is unstable.
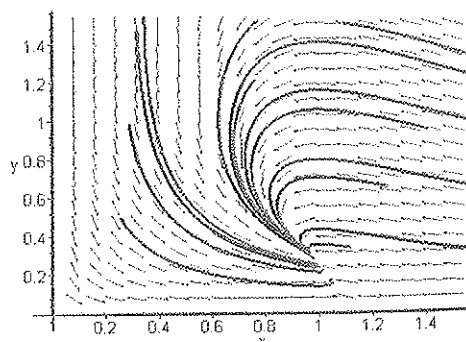At the critical point $(1,1/4)$, the coefficient matrix of the linearized system is

$$J(1,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2 + \sqrt{2}}{4} \,, \, \xi^{(1)} = \begin{pmatrix} -2 + \sqrt{2} \\ 1 \end{pmatrix} \, ; \, r_2 = \frac{-2 - \sqrt{2}}{4} \,, \, \xi^{(2)} = \begin{pmatrix} -2 - \sqrt{2} \\ 1 \end{pmatrix}.$$
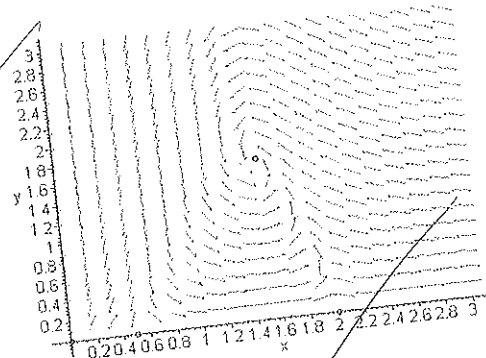
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)

(f) Except for solutions along the coordinate axes, all solutions converge to the critical point $(1, 1/4)$.

5.(a)

(b) The critical points are solutions of the system of equations

$$x(-1 + 2.5x - 0.3y - x^2) = 0$$
$$y(-1.5 + x) = 0.$$

The four critical points are $(0, 0)$, $(1/2, 0)$, $(2, 0)$ and $(3/2, 5/3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -1 + 5x - 3x^2 - 3y/10 & -3x/10 \\ y & -3/2 + x \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & -3/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1, \; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; r_2 = -3/2, \; \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0, 0)$ is a stable node, which is asymptotically stable.
At the critical point $(1/2, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2, 0) = \begin{pmatrix} 3/4 & -3/20 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{3}{4}, \; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; r_2 = -1, \; \xi^{(2)} = \begin{pmatrix} 3 \\ 35 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2, 0)$ is a saddle, which is unstable.