MATH 215: MIDTERM 2: MARCH 20th 2014 (R. Froese, M. Ward)

Closed Book and Notes. 50 minutes. Total 50 points

PROBLEM 1: (20 Points) Let ω be a positive constant, and let $y(t)$ solve

$$y'' + 0.1\,y' + 4y = \sin(\omega t)\,, \qquad t \geq 0\,.$$

i) Calculate the steady-state solution in the form $y = R\sin(\omega t - \phi)$ for some R and ϕ to be found. (Recall: $\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$).

ii) Give a qualitatively accurate plot of the amplitude R versus ω.

iii) Give a qualitatively accurate plot of the steady-state solution y versus t when $\omega = 2$ and when $\omega = 20$. What are the two key differences in your two plots?

PROBLEM 2: (15 Points)

i) Let

$$G(s) = \frac{5}{s(s^2+2s+5)} \quad \text{and} \quad F(s) = e^{-\pi s}G(s).$$

Compute the inverse Laplace transforms $g(t)$ and $f(t)$.

ii) Calculate the solution $y(t)$ to

$$y'' + 2y' + 5y = \begin{cases} 5 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t \end{cases} \qquad \text{with} \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

by using Laplace transforms. (*Hint:* Part i) may be helpful here).

iii) Calculate $\lim_{t\to\infty} y(t)$, explaining your reasoning. (*Hint:* No calculations are needed here).

PROBLEM 3: (15 Points) For $t > 0$, let $y(t)$ solve

$$t^2y'' - 3ty' + 4y = \sqrt{t}\,, \qquad \text{with} \quad y(1) = 4/9\,, \quad y'(1) = 0\,.$$

i) Given that $y_1 = t^2$ solves the homogeneous problem, find the solution to the initial value problem.

ii) Show that the wronskian W between any two solutions to the homogeneous problem must have the form $W = ct^3$ for some constant c independent of t.

MATH 215 QUIZ 2

PROBLEM 1 $\quad y'' + (0.1)y' + 4y = \sin(wt)$

(i) WE PUT $\tilde{y} = Ae^{iwt}$ IN $\tilde{y}'' + 0.1\tilde{y}' + 4\tilde{y} = e^{iwt}$

AND THEN LET $y = \mathrm{IM}(\tilde{y})$. WE GET $-w^2 A + (0.1)iAw + 4A = 1$.

$$\text{SO} \quad A = \frac{1}{(4-w^2) + 0.1wi} = \frac{(4-w^2) - (0.1)iw}{(4-w^2)^2 + (0.01w^2)}$$

$$\text{NOW} \quad y = \mathrm{IM}\left(\left(\frac{(4-w^2) - (0.1)iw}{(4-w^2)^2 + .01w^2}\right)(\cos wt + i\sin wt)\right)$$

THUS THE STEADY-STATE IS

$$(*) \quad y = \frac{1}{(4-w^2)^2 + 0.01w^2}\left[(4-w^2)\sin wt - (0.1)w\cos(wt))\right]$$

NOW PUT $(4-w^2)\sin wt - (0.1)w\cos(wt) = B_0 \sin(wt - \phi)$

$$(4-w^2)\sin(wt) - 0.1w\cos(wt) = B_0\left[\sin(wt)\cos\phi - \cos wt\sin\phi\right]$$

THUS
$$B_0\cos\phi = 4-w^2$$
$$B_0\sin\phi = (0.1)w$$

THUS
$$\tan\phi = (0.1)w/(4-w^2)$$
$$B_0^2\cos^2\phi + B_0^2\sin^2\phi = \left[(4-w^2)^2 + 0.01w^2\right].$$
$$\rightarrow B_0 = \sqrt{(4-w^2)^2 + 0.01w^2}$$

THUS $(*)$ BECOMES
$$y = \frac{1}{(4-w^2)^2 + 0.01w^2}\sqrt{(4-w^2)^2 + 0.01w^2}\,\sin(wt - \phi)$$

THIS YIELDS THAT THE STEADY-STATE SOLUTION IS

$$(+) \quad y = R\sin(wt - \phi) \quad \text{WITH} \quad R = \frac{1}{\sqrt{(4-w^2)^2 + (0.01)w^2}}$$

$$\tan\phi = (0.1w)/4-w^2$$

ii) NOW $R = R(w) = \dfrac{1}{\sqrt{(4-w^2)^2 + 0.01\,w^2}}$

$R(0) = 1/4 \qquad R \approx 1/w^2$ AS $w \to +\infty$.

AND $R(w)$ HAS A MAXIMUM WHEN $g(w) = (4-w^2)^2 + 0.01\,w^2$ HAS A MINIMUM. SET $g'(w) = 0$ TO GET THE MINIMUM OF g.

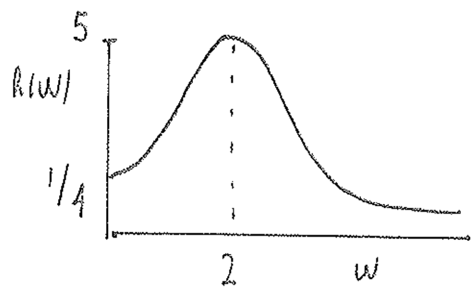
$\longrightarrow 2(4-w^2)(-2w) + 2(.01)w = 0$

$\longrightarrow -2(4-w^2) = (-.01) \longrightarrow 4-w^2 = .005$

THUS $w^2 \approx 4 + .005$

NOW $R(2) = \dfrac{1}{\sqrt{(.01)4}} = \dfrac{1}{(.1)2} = 5.$

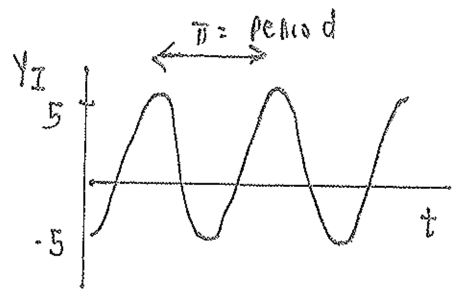
HENCE



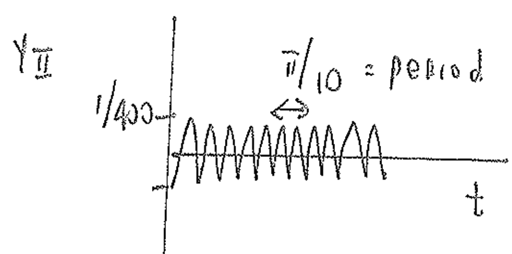
iii) IF $w = 2 \longrightarrow R(2) \cong 5 \Longrightarrow y_{I} \sim 5\sin(2t - \phi)$

IF $w = 20 \longrightarrow R(20)$ IS VERY SMALL $\Longrightarrow y_{II} \sim \dfrac{1}{(20)^2}\sin(20t - \phi)$

$R(20) \sim 1/(20)^2$



large amplitude
low frequency



small amplitude
high frequency.

PROBLEM 2

(i) $G(s) = \dfrac{5}{s(s^2+2s+5)}$ AND $F(s) = e^{-\pi s} G(s)$.

NOW $G(s) = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2s+5}$ SO $5 = A(s^2+2s+5) + s(Bs+C)$

s^2 TERMS $0 = A+B$ $B = -1$

s TERMS $0 = 2A + C$ $C = -2$

CONSTANT : $5 = 5A$ SO $A = 1$

THUS $G(s) = \dfrac{1}{s} - \dfrac{(s+2)}{s^2+2s+5} = \dfrac{1}{s} - \dfrac{[(s+1)+1]}{[(s+1)^2+4]}$

SO $G(s) = \dfrac{1}{s} - \dfrac{(s+1)}{(s+1)^2+4} - \dfrac{1}{2}\left(\dfrac{2}{(s+1)^2+4}\right)$

THUS $g(t) = 1 - \cos(2t)e^{-t} - \frac{1}{2}\sin(2t)e^{-t}$. $(++)$

THEN $f(t) = u_\pi(t)\, g(t-\pi) = u_\pi(t)\left[1 - \cos(2(t-\pi))e^{-(t-\pi)} - \frac{1}{2}\sin(2(t-\pi))e^{-(t-\pi)}\right]$

THIS YIELDS $f(t) = u_\pi(t)\left[1 - \cos(2t)e^{-(t-\pi)} - \frac{1}{2}\sin(2t)e^{-(t-\pi)}\right]$. $(+)$

(ii) WE TAKE LT. WE WRITE

$y'' + 2y' + 5y = 5 - 5u_\pi(t)$ WITH $y(0) = 0$, $y'(0) = 1$.

SO $\left(s^2 Y - s y(0) - y'(0)\right) + 2[sY - y(0)] + 5Y = \dfrac{5}{s} - \dfrac{5e^{-\pi s}}{s}$.

THIS GIVES $(s^2+2s+5)Y = 1 + \dfrac{5}{s} - \dfrac{5e^{-\pi s}}{s}$.

SO $Y = \frac{1}{2}\left(\dfrac{2}{(s+1)^2+4}\right) + G(s) - G(s)e^{-\pi s}$ WITH $G(s) = \dfrac{5}{s(s^2+2s+5)}$

THIS GIVES $y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2}e^{-t}\sin(2t) + g(t) - f(t)$

where g, f given above in $(+)$ AND $(++)$. THUS

$y(t) = \frac{1}{2}e^{-t}\sin(2t) + \left[1 - \cos(2t)e^{-t} - \frac{1}{2}e^{-t}\sin(2t)\right] - u_\pi(t)\left[1 - \cos(2t)e^{-(t-\pi)} - \frac{1}{2}\sin(2t)e^{-(t-\pi)}\right]$.

(iii) FOR $t > \pi$ WE HAVE $y'' + 2y' + 5y = 0$. PUT $y = e^{\lambda t} \rightarrow \lambda^2 + 2\lambda + 5 = 0$ SO $\lambda_\pm = -1 \pm 2i$. THUS SINCE $\mathrm{RE}\,\lambda < 0 \implies y \rightarrow 0$ AS $t \rightarrow \infty$.

PROBLEM 3 $L(y) = t^2 y'' - 3t y' + 4y = \sqrt{t}$ IN $t > 0$

$y(1) = 4/9 \quad y'(1) = 0$

(i) NOTICE $y_1 = t^2$ solves $L(y) = 0$.

WE PUT $y = t^2 v$ so $y' = t^2 v' + 2tv$, $y'' = t^2 v'' + 4tv' + 2v$

$$t^2(t^2 v'' + 4t v' + 2v) - 3t(t^2 v' + 2tv) + 4t^2 v = \sqrt{t}$$

$$t^4 v'' + t^3 v' = t^{1/2} \quad \text{so} \quad v'' + \frac{1}{t} v' = t^{-7/2}. \quad \text{LET } w = v' \text{ so } w' + \frac{1}{t} w = t^{-7/2}$$

INTEGRATING FACTOR is $e^{\ln t} = t$ so $(tw)' = t^{-5/2}$

so $$tw = -\frac{2}{3} t^{-3/2} + C_1$$

so $$w = \frac{C_1}{t} - \frac{2}{3} t^{-5/2} = v'$$

THUS $$v = \frac{4}{9} t^{-3/2} + C_1 \ln t + C_2 \quad \text{FOR } t > 0$$

THUS $y = t^2 v$ is general solution

$$y = \frac{4}{9} t^{1/2} + C_1 t^2 \ln t + C_2 t^2$$

so $y_1 = t^2$ AND $y_2 = t^2 \ln t$ SOLVE HOMOG. problem.

NOW $y(1) = 4/9 \rightarrow 4/9 + C_2 = 4/9$ so $C_2 = 0$

$$y'(1) = 0 \rightarrow \left(\frac{4}{18} t^{-1/2} + C_1 [t + 2t \ln t] \right) \Big|_{t=1} = 0 \quad \text{so} \quad C_1 = -4/18.$$

THUS $$y = 4/9\, t^{1/2} - \frac{4}{18} t^2 \ln t$$ is the solution.

(ii) show that $\overline{W} = y_1 y_2' - y_1' y_2 = C t^3$ FOR SOME C.

<u>METHOD 1</u> we have $y_1 = At^2$, $y_2 = Bt^2 \ln t$ FOR ANY A AND B.

so $\overline{W} = y_1 y_2' - y_1' y_2 = AB[t^2(t + 2t \ln t) - 2t(t^2 \ln t)] = AB\, t^3$.

<u>METHOD 2</u> THE DE is $y'' + p(t) y' + y q(t) = t^{-3/2}$ WITH $p = -3/t$.

RECALL $\overline{W}' + p\overline{W} = 0 \rightarrow \overline{W}' + (-3/t)\overline{W} = 0 \rightarrow \overline{W} = C t^3$ FROM FIRST ORDER ODE.