MATH 301 HW 2 SOLUTIONS

SECTION 3.3 # 4

PROVE $\operatorname{LOG}(e^{z}) = z$ IFF $-\pi < \operatorname{IM} z \leq \pi$.

SOLUTION $e^{z} = e^{x+iy} = e^{x}[\cos y + i \sin y]$

WE CALCULATE $|e^{z}| = e^{x}$, $\operatorname{Arg}(e^{z}) = y$ ONLY IF $-\pi < y \leq \pi$.

NOW $\operatorname{LOG} w = \ln |w| + i \operatorname{ARG}(w)$ WITH $-\pi < \operatorname{ARG} w \leq \pi$.

- LET $w = e^{z}$, THEN $|w| = e^{x}$. WE LET $-\pi < y \leq \pi$ SO THAT $\operatorname{ARG} w = y$.

  THEN $\operatorname{LOG} e^{z} = \ln(e^{x}) + iy = x + iy = z$ WHEN $-\pi < y \leq \pi$.

- NOW IF $z = x + iy$, FOR z TO BE ON THE RANGE OF $\operatorname{LOG}(e^{z})$

  WE NEED $\operatorname{IM} z = y \in (-\pi, \pi]$.

SECTION 3.3 # 5b)

SOLVE $\operatorname{LOG}(z^{2}-1) = i\pi/2$.

SOLUTION OBSERVE THAT $\operatorname{LOG}(w) = \frac{i\pi}{2}$ HAS THE UNIQUE SOLUTION

$w = e^{i\pi/2} = i$. THEREFORE $w = z^{2} - 1 = i$. THIS GIVES,

$$z^{2} = 1 + i = \sqrt{2}\, e^{i\pi/4}.$$

WE GET $$z = \pm\, 2^{1/4} e^{i\pi/8}.$$

SECTION 3.3 # 11

FIND A BRANCH OF $\log(z^{2} + 2z + 3)$ THAT IS ANALYTIC AT $z = -1$ AND FIND ITS DERIVATIVE AT $z = -1$.

SOLUTION LET $w = z^{2} + 2z + 3$. THEN,

$$\log w = \ln |w| + i[\operatorname{ARG} w + 2k\pi] \qquad k = 0, \pm 1, \pm 2 \ldots$$

WE WILL TRY TO SEE IF $\operatorname{LOG}(w)$ (i.e. SET $k = 0$) SATISFIES DESIRED PROPERTIES.

$\operatorname{LOG}(w) = \ln |w| + i \operatorname{ARG}(w)$ IS ANALYTIC IN w-PLANE EXCEPT WHEN $\operatorname{RE} w \leq 0$ WITH $\operatorname{IM} w = 0$. SET $w = u + iv$.

THEREFORE $W = +u$, $u \leq 0$ REAL IS THE BRANCH CUT.

WE FIND ITS IMAGE IN Z-PLANE. WE HAVE

$$+u = z^2 + 2z + 3 \quad \text{OR} \quad z^2 + 2z + (3 - u) = 0.$$

SOLVING THE QUADRATIC $\quad z = \dfrac{-2 \pm \sqrt{4 - 4(3 - u)}}{2} = -1 \pm \sqrt{-2 + u}$.

OBSERVE THAT THERE IS NO $u < 0$ FOR WHICH $z = -1$. HENCE

$z = -1$ IS <u>NOT</u> AN IMAGE POINT ON THE BRANCH CUT $\text{RE } W \leq 0$ WITH $\text{IM } W = 0$.

TO FIND BRANCH CUTS IN Z-PLANE PUT $z = x + iy$ INTO $z^2 + 2z + (3 - u) = 0$.

THEN $\quad x^2 - y^2 + 2ixy + 2(x + iy) + 3 - u = 0$.

SEPARATING REAL AND IMAGINARY PARTS

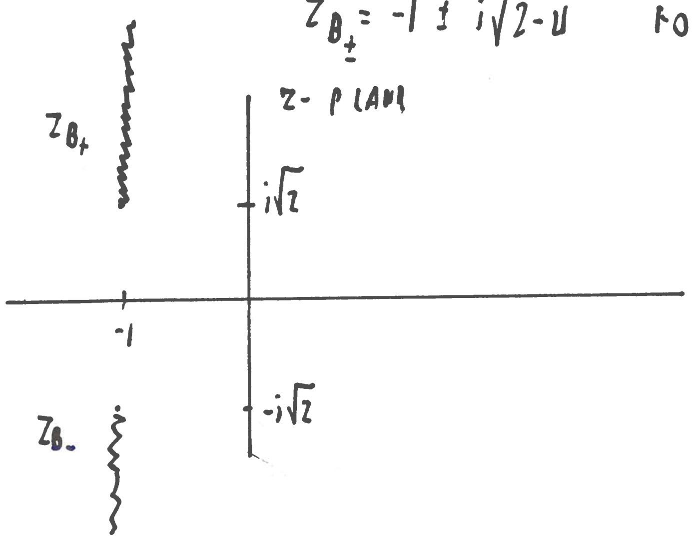
$$x^2 - y^2 + 2x + 3 - u = 0 \quad \underline{\text{AND}} \quad y(x+1) = 0 \rightarrow y = 0 \text{ OR } x = -1.$$

IF $y = 0 \rightarrow x^2 + 2x + 3 - u = 0 \rightarrow x = -1 \pm \sqrt{-2 + u}$ HAS NO REAL SOLUTION IF $u < 0$.

SO WE MUST HAVE $x = -1 \rightarrow 1 - y^2 - 2 + 3 - u = 0 \rightarrow y^2 = 2 - u$ OR $y = \pm i\sqrt{2 - u}$.

WE CONCLUDE THAT BRANCH CUTS z_B ARE PARAMETRIZED AS

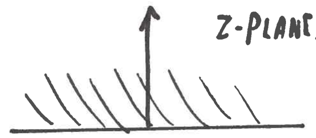
$$z_{B_\pm} = -1 \pm i\sqrt{2 - u} \quad \text{FOR } u \leq 0$$



NOW SINCE $f(z) = \text{LOG}(z^2 + 2z - 3)$ IS THE COMPOSITION OF TWO ANALYTIC FUNCTIONS AT $z = -1$ WE HAVE BY CHAIN RULE

$$f'(z)\Big|_{z=-1} = \frac{(2z + 2)}{(z^2 + 2z - 3)}\Bigg|_{z=-1} = 0.$$

SECTION 3.3 # 15 FIND A 1-1 ANALYTIC MAPPING OF UPPER-HALF PLANE $\operatorname{Im} z > 0$ ONTO INFINITE HORIZONTAL STRIP $H \equiv \{ u + iv \mid -\infty < u < \infty,\ 0 < v < 1 \}$.



SOLUTION WE CONSIDER THE MAP

$$W = C \operatorname{LOG} Z$$

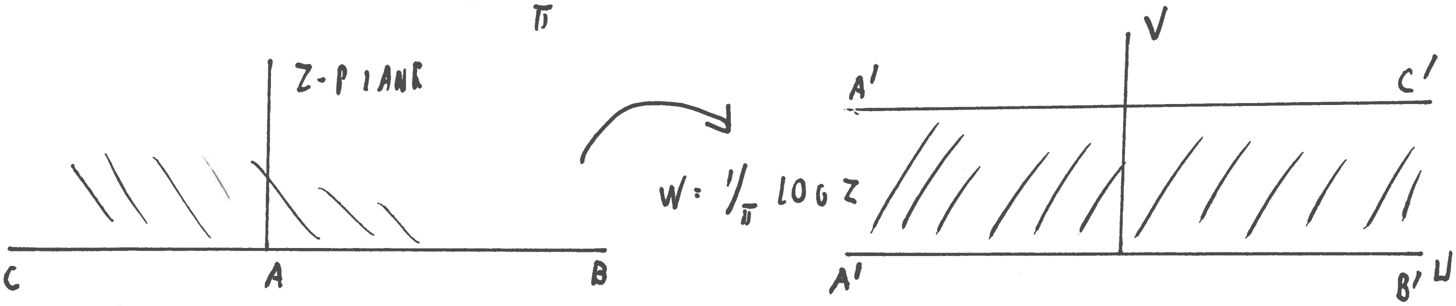
WHERE $C > 0$ IS A REAL CONSTANT TO BE FOUND.

LET $Z = x + iy = |z| e^{i\varphi}$ SO THAT

$$W = C \ln|z| + C i \operatorname{Arg}(z).$$

NOW $W = u + iv$, $u = C \ln|z|$, $v = C \operatorname{Arg}(z)$. FIXING $|z| = r$ AND $0 < \operatorname{Arg} z < \pi$, FOR UPPER HALF-PLANE IT FOLLOWS THAT THE RAY $|z| = r$ WITH $0 < r < \infty$ AND $\operatorname{Arg} z$ FIXED MAPS TO $u \in (-\infty, \infty)$ WITH $v = C \operatorname{Arg}(z)$. AS SUCH, IF $\operatorname{Arg} z = \pi \rightarrow v = C\pi$, SO WE WANT $C = 1/\pi$.

THE MAP IS $W = \frac{1}{\pi} \operatorname{LOG}(Z)$. THE IMAGE OF A FEW POINTS IS SHOWN.



SECTION 3.5 3c)

FIND PRINCIPAL VALUE OF $(1+i)^{1+i}$.

SOLUTION $z = (1+i)^{1+i} = e^{(1+i)\,\mathrm{LOG}\,(1+i)}$ IS PRINCIPAL BRANCH.

NOW $\mathrm{LOG}(1+i) = \ln\sqrt{2} + i\pi/4$. THIS GIVES

$$z = e^{(1+i)[\frac{1}{2}\ln 2 + i\pi/4]} = e^{\frac{1}{2}\ln 2 - \pi/4}\, e^{i\pi/4}\, e^{i/2 \ln 2}$$

$$z = \sqrt{2}\, e^{\pi i/4}\, e^{-\pi/4 + (i/2)\ln 2}.$$

SECTION 3.5 15b)

FIND A BRANCH OF $f(z) = (z^2+4)^{1/2}$ THAT IS ANALYTIC IN THE COMPLEX PLANE EXCEPT ALONG $\mathrm{RE}\, z = 0,\ -2 \leq \mathrm{IM}\, z \leq 2$

SOLUTION THERE ARE TWO APPROACHES. METHOD 1 CHOOSE A BRANCH OF log.
METHOD 2 : RANGE OF ANGLES METHOD.

METHOD 1 WE WANT ANALYTICITY IN $|z| > 2$. WE WRITE

$$f(z) = z\,[1 + 4/z^2]^{1/2} = z\, e^{\frac{1}{2}\log(1+4/z^2)}.$$

WE WILL TRY PRINCIPLE BRANCH OF log AND SO CHOOSE

$$f(z) = z\, e^{\frac{1}{2}\mathrm{LOG}(1+4/z^2)}.$$

THIS HAS A BRANCH CUT(S) WHEN $\mathrm{RE}\left[1 + \frac{4}{z^2}\right] \leq 0$ WITH $\mathrm{IM}\left[1 + \frac{4}{z^2}\right] = 0$.

PUT $z = x + iy$ SO set $1 + \dfrac{4}{z^2} = 1 + \dfrac{4}{(x+iy)^2} = u$ WITH $u \leq 0$.

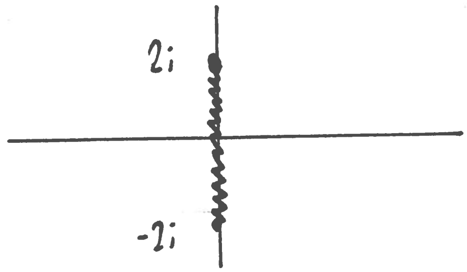
THIS YIELDS $\dfrac{4}{(x+iy)^2} = u - 1$ OR $(x+iy)^2 = \dfrac{4}{u-1} \rightarrow x^2 - y^2 + 2ixy = \dfrac{4}{u-1}$.

SO IN Z-PLANE BRANCH CUT SATISFIES $x^2 - y^2 = \dfrac{4}{u-1}$ AND $xy = 0$, WITH $u \leq 0$ REAL

IF $y=0 \rightarrow x^2<0$ WHICH IS IMPOSSIBLE. SO MUST HAVE $x=0$ SO THAT

$$y = \pm \frac{2}{\sqrt{1-u}} \quad \text{FOR } -\infty < u \le 0.$$

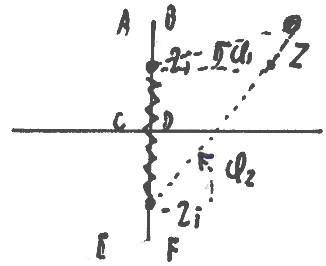
THIS YIELDS THE BRANCH CUTS IN Z PLANE BETWEEN $\pm 2i$



$$f(z) = z\, e^{\frac{1}{2}\log(1+4/z^2)}$$

IS ANALYTIC IN $|z|>2$.

METHOD 2 (RANGE OF ANGLES)



WRITE $z-2i = r_1 e^{i\varphi_1}$

$z+2i = r_2 e^{i\varphi_2}$

SO $f(z) = (z^2+4)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$

WE WANT A BRANCH CUT ON $-2 \le \mathrm{Im}(z) \le 2$.

CHOOSE THE RANGE $\varphi_1 \in (-\pi/2, 3\pi/2]$, $\varphi_2 \in (-\pi/2, 3\pi/2]$

	φ_1	φ_2	$e^{i(\varphi_1+\varphi_2)/2}$	
A	$\pi/2$	$\pi/2$	i	$=$
B	$\pi/2$	$\pi/2$	i	
C	$3\pi/2$	$\pi/2$	-1	NOT EQUAL $\rightarrow \exists$ JUMP ACROSS ANY POINT WITH $-2 \le \mathrm{Im}\,z \le 2$
D	$-\pi/2$	$\pi/2$	1	
E	$3\pi/2$	$3\pi/2$	$-i$	$=$
F	$-\pi/2$	$-\pi/2$	$-i$	

$\rightarrow$ BRANCH CUT IS $\mathrm{Re}(z)=0$, $|\mathrm{Im}\,z|<2$

SO $f(z) = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$ WITH

$\varphi_1 \in (-\pi/2, 3\pi/2]$, $\varphi_2 \in (-\pi/2, 3\pi/2]$

WILL BE ANALYTIC IN $|z|>2$.

SECTION 3.5 15 d)

FIND A BRANCH OF $(z^3-1)^{1/3}$ THAT IS ANALYTIC IN $|z|>1$.

SOLUTION WE WILL DO THE CHOICE OF $\log$ APPROACH.

WE WRITE $f(z) = z\left[1-1/z^3\right]^{1/3} = z\, e^{1/3 \log(1-1/z^3)}$

AND CHOOSE PRINCIPAL BRACH OF $\log$:

DEFINE $\hat{f}(z) = z\, e^{1/3\, \mathrm{LOG}(1-1/z^3)}$

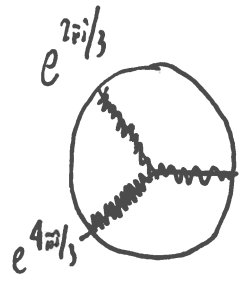
WHERE IS THIS CHOICE ANALYTIC? WHERE ARE BRANCH CUTS?

OBSERVE $\hat{f}(z)$ IS ANALYTIC EXCEPT WHEN

$$\mathrm{RE}\left(1-\frac{1}{z^3}\right) = u < 0 \quad \text{AND} \quad \mathrm{IM}\left[1-\frac{1}{z^3}\right] = 0.$$

WE SOLVE $z^3 = \dfrac{1}{1-u}$ TO OBTAIN $z = z_B = \left(\dfrac{1}{1-u}\right)^{1/3} e^{2\pi i k/3}$, $k = 0,1,2$.

OBSERVE THAT WITH $-\infty < u < 0$ WE HAVE $|z_B| < 1$ SO THAT BRANCH CUTS ARE INSIDE UNIT DISK.



THUS $\hat{f}(z)$ IS ANALYTIC IN $|z|>1$.

EXTRA PROBLEM

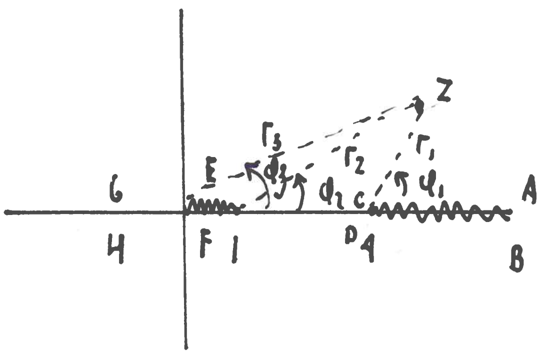
USING RANGES OF ANGLES METHOD FIND A BRANCH OF

$$f(z) = \sqrt{z(z-1)(z-4)}$$

THAT HAS BRANCH CUTS ALONG $0 \le z \le 1$, $z \ge 4$.

SOLUTION LET $z-4 = r_1 e^{i\varphi_1}$, $z-1 = r_2 e^{i\varphi_2}$, $z = r_3 e^{i\varphi_3}$

SO THAT $w = f(z) = (r_1 r_2 r_3)^{1/2} e^{i(\varphi_1+\varphi_2+\varphi_3)/2}$



WE TRY $\varphi_1 \in (0, 2\pi]$, $\varphi_2 \in (0, 2\pi]$, $\varphi_3 \in (0, 2\pi]$.

OUR TABLE IS

POINT	φ_1	φ_2	φ_3	$e^{i(\varphi_1+\varphi_2+\varphi_3)/2}$	
A	0	0	0	1	JUMP
B	2π	2π	2π	$e^{3\pi i} = -1$	
C	π	0	0	$e^{i\pi/2} = i$	$\neq$
D	π	2π	2π	$e^{5\pi i/2} = i$	
E	π	π	0	$e^{\pi i} = -1$	JUMP
F	π	π	2π	$e^{4\pi i/2} = 1$	
G	π	π	π	$e^{3\pi i/2} = i$	
H	π	π	π	$e^{3\pi i/2} = i$	

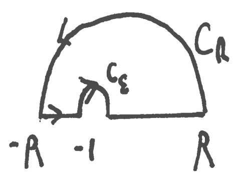
WE CONCLUDE THAT $f(z)$ HAS JUMPS ACROSS AB AND EF. THESE ARE WHERE BRANCH CUTS ARE, AS DESIRED

THUS THE RANGE OF ANGLES IS $\varphi_j \in (0, 2\pi]$, $j = 1, 2, 3$.

PROBLEM CALCULATE $P.V. \int_{-\infty}^{\infty} \frac{\cos(2x)}{x^3+1}\,dx$.

SOLUTION LET $I = P.V. \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^3+1}\,dx$.

SINGULARITIES ARE AT $z^3=-1$ SO $z=-1,\ e^{\pi i/3},\ e^{-\pi i/3}$. OBSERVE $z=-1$ IS ON THE CONTOUR AND $z=e^{\pi i/3}$ IS IN $\mathrm{Im}(z)>0$. WE CONSIDER INDENTED CONTOUR



$$\lim_{\substack{R\to\infty\\ \varepsilon\to0^+}}\left[\int_{-R}^{-1-\varepsilon} \frac{e^{2ix}}{x^3+1}\,dx + \int_{-1+\varepsilon}^{R} \frac{e^{2ix}}{x^3+1}\,dx + \int_{C_\varepsilon} \frac{e^{2iz}}{z^3+1}\,dz + \int_{C_R} \frac{e^{2iz}}{z^3+1}\,dz\right]$$

$$= 2\pi i\ \mathrm{RES}\left[\frac{e^{2iz}}{z^3+1};\ e^{\pi i/3}\right]. \qquad (*)$$

NOW $\lim_{R\to\infty}\int_{C_R} \frac{e^{2iz}}{z^3+1}\,dz = 0$ BY JORDAN'S LEMMA.

ON C_ε, $z=-1+\varepsilon e^{i\varphi}$ AND $z^3+1 = (z+1)(z^2-z+1)$ SO THAT

ON C_ε $\quad \frac{e^{2iz}}{z^3+1} = \frac{e^{2iz}}{(z+1)(z^2-z+1)} \longrightarrow \frac{e^{-2i}}{\varepsilon e^{i\varphi}(3)}$ AS $\varepsilon\to0$.

THUS, $\lim_{\varepsilon\to0}\int_{C_\varepsilon} \frac{e^{2iz}}{z^3+1}\,dz = \lim_{\varepsilon\to0}\int_{\pi}^{0} \frac{e^{-2i}}{3\varepsilon e^{i\varphi}}\, i\varepsilon e^{i\varphi}\,d\varphi = -\frac{i}{3}\pi e^{-2i}$

FROM (*) WE OBTAIN $I - \frac{i\pi}{3}e^{-2i} = 2\pi i\ \mathrm{RES}\left[\frac{e^{2iz}}{z^3+1};\ e^{\pi i/3}\right] = \frac{2\pi i}{3(e^{2\pi i/3})}\, e^{2i(e^{\pi i/3})}$

THUS $I = \frac{i\pi}{3}e^{-2i} + \frac{2\pi i}{3} e^{-2\pi i/3} e^{2i[\cos(\pi/3)+i\sin(\pi/3)]}$

NOW $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, SO $I = \frac{i\pi}{3}[\cos(2) - i\sin(2)] + \frac{2\pi i}{3}e^{-2\pi i/3}e^{-\sqrt{3}+i}$.

MULTIPLYING OUT $I = \frac{\pi}{3}\sin(2) + \frac{i\pi}{3}\cos(2) + \frac{2\pi i}{3}e^{-\sqrt{3}}\left[\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right]\left[\cos 1 + i\sin\right.$

FINALLY WE TAKE THE REAL PART TO GET

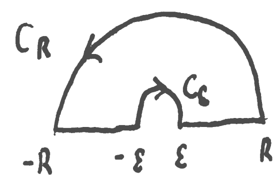
$$P.V.\int_{-\infty}^{\infty}\frac{\cos 2x}{x^3+1}\,dx = \frac{\pi}{3}\sin(2) + \frac{2\pi e^{-\sqrt{3}}}{3}\left[\frac{1}{2}\sin(1) + \frac{\sqrt{3}}{2}\cos(1)\right].$$

PROBLEM LET $a>0$, $b>0$ BE REAL. CALCULATE

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx.$$

SOLUTION WE CONSIDER THE INDENTED CONTOUR INTEGRAL FOR

$$I = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{iax}}{x(x^2+b^2)}\,dx.$$



SINGULARITY AT $z=0$ AND $z=ib$

IN UPPER 1/2 PLANE. $z=ib$ IS SIMPLE POLE.

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0^+}} \left[\int_{-R}^{-\varepsilon} \frac{e^{iax}}{x(x^2+b^2)}\,dx + \int_{\varepsilon}^{R} \frac{e^{iax}}{x(x^2+b^2)}\,dx + \int_{C_\varepsilon} \frac{e^{iaz}}{z(z^2+b^2)}\,dz + \int_{C_R} \frac{e^{iaz}}{z(z^2+b^2)}\,dz \right]$$

$$= 2\pi i \,\text{Res}\left[\frac{e^{iaz}}{z(z^2+b^2)}; ib\right] \quad (*)$$

NOW SINCE $a>0$ WE HAVE $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = 0$ BY JORDAN'S LEMMA. NOW

ON C_ε, $z = \varepsilon e^{i\varphi}$ SO $\displaystyle\frac{e^{iaz}}{z(z^2+b^2)} = \frac{e^{ia\varepsilon e^{i\varphi}}}{\varepsilon e^{i\varphi}(b^2+\varepsilon^2 e^{2i\varphi})} \cong \frac{1}{b^2 \varepsilon e^{i\varphi}}$ AS $\varepsilon \to 0^+$.

THEREFORE $\displaystyle\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = \lim_{\varepsilon\to 0}\left[\int_{\pi}^{0} \frac{1}{b^2 \varepsilon e^{i\varphi}}\, i\varepsilon e^{i\varphi}\,d\varphi\right] = -\frac{i\pi}{b^2}$. THEREFORE,

$(*)$ YIELDS, $\displaystyle I - i\pi/b^2 = 2\pi i \,\text{Res}\left[\frac{e^{iaz}}{z(z^2+b^2)}; ib\right] = 2\pi i \left[\frac{e^{-ab}}{ib(2ib)}\right]$

SO $\displaystyle I = \frac{\pi i}{b^2} - \frac{i\pi e^{-ab}}{b^2}$. WE NOW TAKE THE IMAGINARY

PARTS TO CONCLUDE $\displaystyle \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{b^2}\left(1-e^{-ab}\right)$ FOR $a>0$, $b>0$.

EXTRA PROBLEM

LET $f(z) = \dfrac{1}{\sin z} - \dfrac{1}{z}$. IT IS EASY TO SHOW THAT POSSIBLE SINGULARITIES ARE AT $z = 0$, $n\pi$, $n = \pm 1, \pm 2 \ldots$

THEN $f(z) = \dfrac{z - \sin z}{z \sin z}$. AS $z \to 0$ $f(z) \simeq \dfrac{z - (z - z^3/6 + \cdots)}{z(z - z^3/6 + \cdots)} \simeq \dfrac{z^3/6}{z^2} \simeq z/6$.

WE CONCLUDE THAT $z = 0$ IS A REMOVABLE SINGULARITY AND $\lim_{z \to 0} f(z) = 0$.

NOW $f(z)$ HAS SIMPLE POLES AT $z = n\pi$, $n = \pm 1, \pm 2, \ldots$, SO WE LABEL

$$a_n = n\pi, \quad n = \pm 1, \pm 2, \ldots$$

NOW

$$\mathrm{RES}[f; a_n] = \mathrm{RES}\left[\frac{P}{Q}; a_n\right] = \frac{P(a_n)}{Q'(a_n)} = (-1)^n.$$

$P = (z - \sin z)/z$ $\quad P(a_n) = 1$

$Q = \sin z$ $\quad Q(a_n) = 0$

$Q'(a_n) = \cos(n\pi) = (-1)^n$

LABEL $b_n = (-1)^n$.

NOW BY ENCLOSING IN A LARGE SQUARE C_N WITH VERTICES $\pm(N + 1/2) \pm i(N + 1/2)$ WE HAVE BY MITTAG-LEFFLER PROCEDURE

$$(*) \quad \lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_N} \frac{f(w)}{w(w-z)} \, dw = \frac{f(z)}{z} - \frac{f(0)}{z} + \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{b_n}{a_n(a_n - z)} + \sum_{n=-N}^{-1} \frac{b_n}{a_n(a_n - z)} \right)$$

THE INTEGRAL ON THE LEFT VANISHES IF WE HAVE $|f(z)| \leq M$ ON C_N. THIS HOLDS SINCE $|1/z|$ IS BOUNDED AND $\left|\dfrac{1}{\sin z}\right| = \dfrac{2}{|e^{iz} - e^{-iz}|} \leq \dfrac{2}{\left||e^{iz}| - |e^{-iz}|\right|} \leq K$ ON C_N (SEE NOTES ON WATSON TRANSFORM FOR PROOF). SETTING $f(0) = 0$, $a_n = n\pi$, $b_n = (-1)^n$ IN $(*)$ WE HAVE

$$f(z) = \sum_{n=1}^{\infty} \frac{z(-1)^n}{n\pi(z - n\pi)} + \sum_{n=-\infty}^{-1} \frac{z(-1)^n}{n\pi(z - n\pi)}.$$

REPLACE $n = -m$ IN SECOND SUM THEN PUT $m \mapsto -n$.

$$= \sum_{n=1}^{\infty} \frac{z(-1)^n}{n\pi(z - n\pi)} + \sum_{n=1}^{\infty} \frac{z(-1)^n}{-n\pi(z + n\pi)}$$

SO

$$f(z) = z \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \left(\frac{1}{z - n\pi} - \frac{1}{z + n\pi} \right) = 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2\pi^2}.$$

NOW SET $f(z) = \sin z - \frac{1}{z}$ SO THAT

$$[\sin z]^{-1} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2\pi^2}.$$

NOW SET $z = 1$.

$$[\sin 1]^{-1} - 1 = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1 - n^2\pi^2}.$$

$$\text{SO} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - 1} = \frac{1 - [\sin 1]^{-1}}{2} \cong -0.0942$$