INDENTED CONTOURS

WE NOW GIVE AN EXAMPLE WHERE WE MUST INDENT THE CONTOUR AROUND A POLE ON X-AXIS. WE WISH TO CALCULATE

$$I = \int_0^\infty \frac{\sin x}{x}\,dx. \quad \text{OBSERVE } x=0 \text{ IS REMOVABLE.}$$

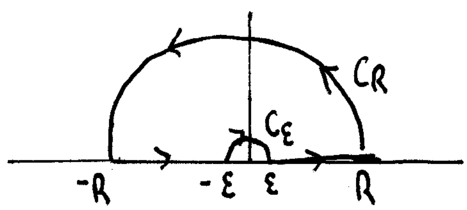
BY SYMMETRY DEFINE $J = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$ SO THAT $I = J/2$.

NOW $J = \mathrm{Im}\left[\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx\right]$ AND NOW $x=0$ IS NO LONGER REMOVABLE.

WE LET $f(z) = e^{iz}/z$ AND CONSIDER CONTOUR SHOWN:

$C_R: \{|z|=R,\ \mathrm{Im}\,z \geq 0\}$

$C_\varepsilon: \{|z|=\varepsilon,\ \mathrm{Im}\,z \geq 0$



BY RESIDUE THM SINCE THERE ARE NO POLES INSIDE CONTOUR,

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \left[\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{C_R} \frac{e^{iz}}{z}\,dz\right] = 0$$

BY JORDAN'S LEMMA, $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\,dz = 0$ AND SO

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx\right] = -\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz$$

$$\to \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = -\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz. \qquad (*)$$

LET $z = \varepsilon e^{i\varphi}$ WITH $0 < \varphi < \pi$. $\quad dz = i\varepsilon e^{i\varphi}d\varphi = iz\,d\varphi \qquad dz/z = i\,d\varphi$

$$\to \lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = \lim_{\varepsilon\to 0} \int_{\pi}^{0} i\,e^{\varepsilon e^{i\varphi}}\,d\varphi = \int_{\pi}^{0} i\,d\varphi = -i\pi.$$

THEREFORE FROM $(*)$

$$J = \mathrm{Im}\left[\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx\right] = \mathrm{Im}\left[-(-i\pi)\right] = \pi.$$

THIS GIVES $\quad I = \frac{J}{2} = \int_0^\infty \frac{\sin x}{x}\,dx = \pi/2.$

NOW A SECOND WAY TO CALCULATE I IS A TRICK DUE TO FEYNMAN. LET

$$F(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax}\, dx, \quad \text{FOR } a \geq 0 \text{ REAL}$$

THEN $I = F(0)$. OBSERVE THAT $F \to 0$ AS $a \to \infty$

WE CALCULATE $F'(a) = -\int_0^\infty (\sin x) e^{-ax} dx = -\mathrm{IM}\left[\int_0^\infty e^{(i-a)x} dx\right]$

$$= -\mathrm{IM}\left[\frac{1}{(i-a)} e^{(i-a)x}\Big|_0^\infty\right] = -\mathrm{IM}\left[\frac{-1}{(i-a)}\right] = -\mathrm{IM}\left[\frac{1}{a-i}\right]$$

$$= -\mathrm{IM}\left[\frac{a+i}{(a-i)(a+i)}\right] = -\mathrm{IM}\left[\frac{a+i}{a^2+1}\right] = -\frac{1}{a^2+1}.$$

SO $F'(a) = -\frac{1}{(a^2+1)}$, $F(\infty) = 0.$

WE INTEGRATE $F(a) = -\tan^{-1} a + \tan^{-1}(\infty) = \frac{\pi}{2} - \tan^{-1}(a)$

FINALLY AT $a = 0$, $F(a) = I = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ SINCE $\tan^{-1}(0) = 0.$