INVERSE TRIG FUNCTIONS

INVERSE SIN FUNCTION

LET $z = \sin w$. WE WANT TO DEFINE A BRANCH OF THE INVERSE FUNCTION, DENOTED BY $w = \sin^{-1} z$ FOR WHICH IF z IS REAL AND SATISFIES $-1 < \operatorname{Re}(z) < 1$, $\operatorname{Im} z = 0$ WE HAVE THE USUAL CALCULUS RESULT THAT $-\frac{\pi}{2} < \sin^{-1} x < \pi/2$.

SOLUTION $\quad z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$ SO THAT $e^{iw} - e^{-iw} - 2iz = 0$.

MULTIPLY BY e^{iw} TO GET $e^{2iw} - 2ize^{iw} - 1 = 0$. THIS IS A QUADRATIC IN e^{iw} WHICH HAS THE SOLUTION

$$e^{iw} = iz + (1-z^2)^{1/2}$$

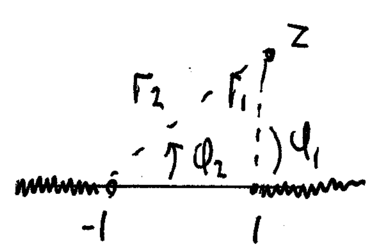
WHERE $(1-z^2)^{1/2}$ IS MULTI-VALUED WITH TWO SOLUTIONS. TAKING log WE HAVE

$$w = -i \log\left[iz + (1-z^2)^{1/2} \right].$$

WE MUST NOW SPECIFY A BRANCH OF log AS WELL AS A BRANCH OF $(1-z^2)^{1/2}$.

WE WILL CHOOSE THE PRINCIPAL BRANCH OF $(1-z^2)^{1/2}$ WHICH IS WRITTEN EITHER AS

$$(1-z^2)^{1/2} = e^{\frac{1}{2}\log(1-z^2)} \qquad (1)$$



OR $\quad (1-z^2)^{1/2} = -i(r_1 r_2)^{1/2} e^{i(\varphi_1 + \varphi_2)/2}$

$$\left.\begin{array}{l} \text{WITH } 0 \le \varphi_1 < 2\pi,\ -\pi \le \varphi_2 < \pi \\ \text{AND } r_1 = |z-1|,\ r_2 = |z+1|. \end{array}\right\} \quad (2)$$

WE READILY FIND THAT THE BRANCH CUTS ARE ON THE SEGMEND $y = 0$ AND $|x| \ge 1$.

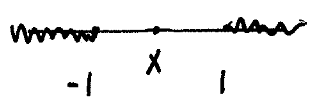
NOW FOR $y=0$ AND $-1<x<1$ IF WE CHOOSE THE PRINCIPAL BRANCH OF $\log$ SO THAT

$$W=-i\,\mathrm{LOG}\left(iz+(1-z^2)^{1/2}\right) \quad \Big\}\ (*)$$

WE EVALUATE AS FOLLOWS ON $y=0$, $|x|<1$

- USING (1) WE GET $(1-z^2)^{1/2}=e^{\frac{1}{2}\mathrm{LOG}(1-x^2)}=e^{\frac{1}{2}\ln(1-x^2)+i0}$

  SO THAT $(1-z^2)^{1/2}=\sqrt{1-x^2}$

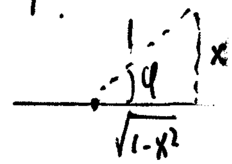
- USING (2) THEN  $\varphi_1=\pi,\ \varphi_2=0,\ r_1=1-x,\ r_2=x+1$

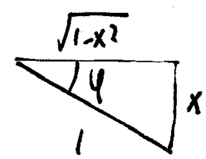
  SO $(1-z^2)^{1/2}=-i\sqrt{1-x^2}\,e^{i\pi/2}=-i(i)\sqrt{1-x^2}$
  $=\sqrt{1-x^2}$

THEREFORE, ON $y=0$, $-1<x<1$

$$W=-i\,\mathrm{LOG}\left(ix+\sqrt{1-x^2}\right)=-i\left[\ln\left|ix+\sqrt{1-x^2}\right|+i\,\mathrm{ARG}\left(ix+\sqrt{1-x^2}\right)\right]$$

WE CALCULATE $\left|ix+\sqrt{1-x^2}\right|=x^2+(1-x^2)=1.$

SO THAT $W=\mathrm{ARG}\left[\sqrt{1-x^2}+ix\right]$  IF $x>0$

THIS IS SIMPLY $W=\sin^{-1}(x),\ -1<x<1$ OR  IF $x<0$

WHICH IS ON THE RANGE $-\pi/2<W<\pi/2$.

NOW FINALLY, WE WANT TO SHOW THAT THE CHOICE OF LOG IN $(*)$ DOES NOT INTRODUCE AN EXTRA BRANCH CUT.

IN PARTICULAR, WE WILL SHOW THAT

$$\mathrm{RE}\left[iz+(1-z^2)^{1/2}\right]\geq 0 \qquad (+)$$

ALWAYS HOLDS AND THAT IF WE HAVE EQUALITY IN (+) THEN

$$\mathrm{IM}\left[iz+(1-z^2)^{1/2}\right]\neq 0.$$

THE PROOF OF THIS IS INDIRECT.

PROOF WE WILL PROVE THAT $\mathrm{RE}\, W \geq 0$ WHEN $W = iz + \sqrt{1-z^2}$.

WE WRITE $(1-z^2)^{1/2} = e^{\frac{1}{2}\left[\ln|1-z^2| + i\,\mathrm{ARG}(1-z^2)\right]}$ WITH $-\pi < \mathrm{ARG}(1-z^2) \leq \pi$

SO $\mathrm{RE}\left((1-z^2)^{1/2}\right) = |1-z^2|^{1/2} \cos\left(\mathrm{ARG}(1-z^2)/2\right) \geq 0.$

SINCE $\cos\varphi \geq 0$ ON $-\pi/2 \leq \varphi \leq \pi/2$. WE CONCLUDE THAT

$$\mathrm{RE}\left((1-z^2)^{1/2}\right) \geq 0$$

NOW LABEL $W = iz + \sqrt{1-z^2}$, $W_- = iz - \sqrt{1-z^2}$.

WE GET $W - W_- = 2\sqrt{1-z^2}$

AND CONCLUDE THAT $\mathrm{RE}\, W - \mathrm{RE}\, W_- = 2\,\mathrm{RE}\left(\sqrt{1-z^2}\right) \geq 0$

SO THAT $\mathrm{RE}\, W_- \leq \mathrm{RE}\, W$. (1)

NOW CALCULATE $W W_- = (iz + \sqrt{1-z^2})(iz - \sqrt{1-z^2}) = -z^2 - (1-z^2) = -1$

SO $W W_- = -1$

NOW $W = -\dfrac{1}{W_-} = -\dfrac{\overline{W}_-}{|W_-|^2}$.

SINCE $\mathrm{RE}\, \overline{W}_- = \mathrm{RE}\, W_-$ WE GET

$$\mathrm{RE}\, W = -\frac{1}{|W_-|^2}\,\mathrm{RE}(W_-) \qquad (2)$$

WE NOW PROVE THAT $\mathrm{RE}\, W \geq 0$ BY CONTRADICTION. SUPPOSE $\exists z$ SUCH THAT $\mathrm{RE}\, W < 0$. THEN BY (1), $\mathrm{RE}\, W_- < 0$. BUT BY (2) WE HAVE $\mathrm{RE}(W_-) > 0$, WHICH IS A CONTRADICTION. AS SUCH WE MUST HAVE $\mathrm{RE}(W) \geq 0$ AND $\mathrm{RE}(W_-) \leq 0$.

FINALLY, IT IS EASY TO SHOW THAT IF $\mathrm{RE}\, W = 0$ WE MUST HAVE $\mathrm{RE}(\sqrt{1-z^2}) = 0 \rightarrow y = 0$ AND $|x| \geq 1$. HOWEVER, $\mathrm{IM}(W) \neq 0$ ON THESE SEGMENTS SO THAT WE NEVER HAVE $\mathrm{IM}\, W = 0$, $\mathrm{RE}\, W = 0$ SIMULTANEOUSLY. WE CONCLUDE THAT THE "LOG(..)" DOES <u>NOT</u> INTRODUCE ADDITIONAL BRANCH CUTS.

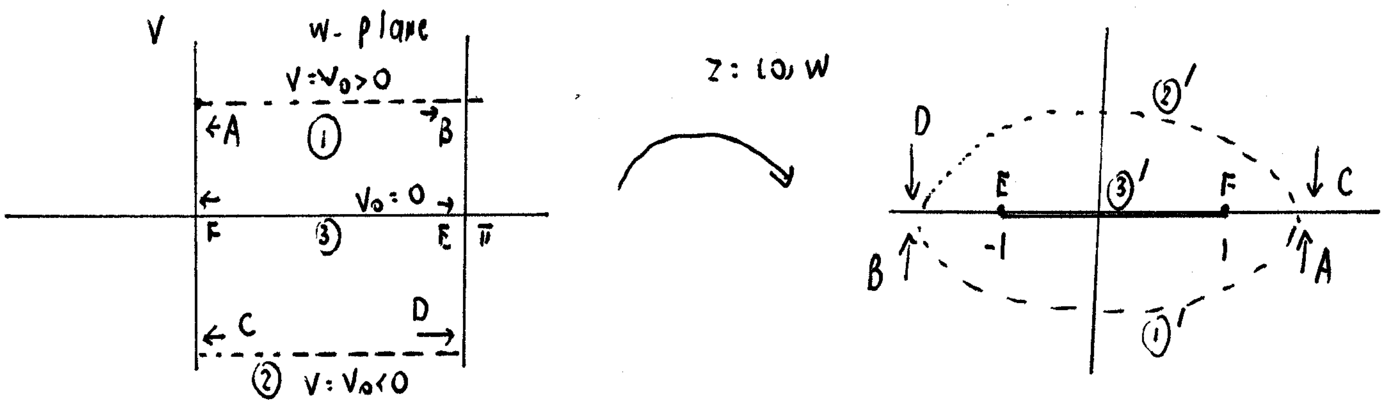
INVERSE TRIG FUNCTION $z = \cos w$ WITH $0 \le \mathrm{Re}\, w \le \pi$.

NOW $\cos(u+iv) = \cos u \cosh v - i \sin u \sinh v$.

SO WITH $z = x + iy$ WE HAVE

$$x = \cos u \cosh v$$

$$y = -\sin u \sinh v$$



- CONSIDER LINE ① WITH $v = v_0 > 0$, $0 < u < \pi$

$$\rightarrow \quad \begin{aligned} x &= \cos u \cosh v_0 \\ y &= -\sin u \sinh v_0 \le 0 \end{aligned} \quad \longrightarrow \quad \frac{x^2}{\cosh^2 v_0} + \frac{y^2}{\sinh^2 v_0} = 1 \text{ ellipse } ①'$$

WITH $y \le 0$ SINCE $0 < u < \pi$.

- CONSIDER LINE ② WITH $v = v_0 < 0$, $0 < u < \pi$

$$\rightarrow \quad \begin{aligned} x &= \cos u \cosh v_0 \\ y &= -\sin u \sinh v_0 \ge 0 \end{aligned} \quad \longrightarrow \quad \frac{x^2}{\cosh^2 v_0} + \frac{y^2}{\sinh^2 v_0} = 1 \text{ ellipse } ②'$$

WITH $y \ge 0$ SINCE $0 < u < \pi$, $v_0 < 0$

- CONSIDER LINE ③ WITH $v = 0$, $0 < u < \pi$

THEN $y = 0$ AND $x = \cos u \in (-1, 1) \rightarrow$ LINE ③'.

THEREFORE $z = \cos w$ GIVES A 1-1 MAPPING OF $0 < \mathrm{Re}\, w < \pi$

TO THE CUT PLANE $z \in \mathbb{C} \setminus \{(-\infty, -1), (1, \infty)\}$.

IN THIS WAY, IN CONSTRUCTING THE 1-1 INVERSE FUNCTION

$$w = \cos^{-1} z$$

WE WANT BRANCH CUTS ON REAL AXIS WITH $(-\infty, -1)$ AND $(1, \infty)$.

NOW PROCEED ANALYTICALLY.

$$Z = \cos W = \frac{e^{iW} + e^{-iW}}{2}$$

THUS $e^{2iW} - 2z\, e^{iW} + 1 = 0$. QUADRATIC IN e^{iW}

THIS GIVES $$e^{iW} = \frac{2z + \sqrt{4z^2 - 4}}{2} = z + \sqrt{z^2 - 1},$$

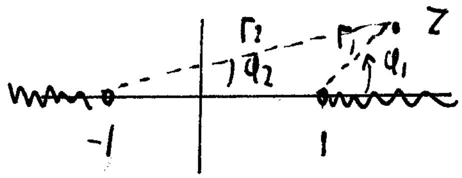
WHERE $\sqrt{z^2-1}$ IS MULTI-VALUED (NO NEED TO WRITE $\pm$ IN QUADRATIC FORMULA SOLUTION). THIS YIELDS

$$W = -i \log\left(z + \sqrt{z^2-1} \right). \qquad (*)$$

WE CHOOSE A BRANCH OF $\log$ AND A BRANCH OF $\sqrt{z^2-1}$.

THE BRANCH OF $\sqrt{z^2-1}$ HAS A CUT ON $|x| \geq 1$ WITH $y = 0$. SO WE CAN WRITE

$$(z^2-1)^{1/2} = (r_1 r_2)^{1/2}\, e^{i(\varphi_1 + \varphi_2)/2} \quad \text{WITH} \quad -\pi \leq \varphi_2 < \pi, \quad 0 \leq \varphi_1 < 2\pi$$



OR EQUIVALENTLY $$(z^2-1)^{1/2} = \left[-(1-z^2) \right]^{1/2} = i\, e^{\frac{1}{2} \mathrm{LOG}(1-z^2)}.$$

MOREOVER, WE CHOOSE $\mathrm{LOG}(\ldots)$ IN $(*)$. THIS GIVES,

$$W = -i\, \mathrm{LOG}\left(z + \sqrt{z^2-1} \right). \qquad (+)$$

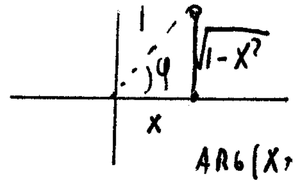
• NOW EVALUATE ON SEGMENT $y = 0$ WITH $-1 < x < 1$. THEN $\varphi_1 = \pi$, $\varphi_2 = 0$ AND $(z^2-1)^{1/2} = i\sqrt{1-x^2}$. THUS ON THIS SEGMENT

$$W = -i\, \mathrm{LOG}\left[x + i\sqrt{1-x^2} \right] = -i \left[\ln \left| x + i\sqrt{1-x^2} \right| + i\, \mathrm{ARG}\left(x + i\sqrt{1-x^2} \right) \right]$$

WE CALCULATE $\left| x + i\sqrt{1-x^2} \right| = \sqrt{x^2 + (1-x^2)} = 1.$

THUS YIELDS,

$$W = -i(i)\,\mathrm{ARG}\left(x + i\sqrt{1-x^2}\right).$$

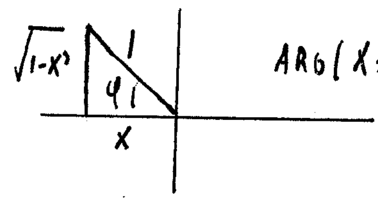


IF $x > 0$

$\mathrm{ARG}(x + i\sqrt{1-x^2}) = \varphi = \cos^{-1} x$

THUS $W = \cos^{-1} x$

AND SATISFIES $0 \le W \le \pi$.



$\mathrm{ARG}(x + i\sqrt{1-x^2}) = \varphi = \cos^{-1} x$

FINALLY, AS PROVED INDIRECTLY FOR $\sin^{-1} z$ FORMULA WE CAN SHOW THAT THE $\mathrm{LOG}(..)$ IN (+) DOES <u>NOT</u> INTRODUCE ADDITIONAL BRANCH CUTS. AS SUCH THERE IS NO SOLUTION TO

$$\mathrm{IM}\left(z + \sqrt{z^2-1}\right) = 0 \quad \text{AND} \quad \mathrm{RE}\left(z + \sqrt{z^2-1}\right) \le 0$$

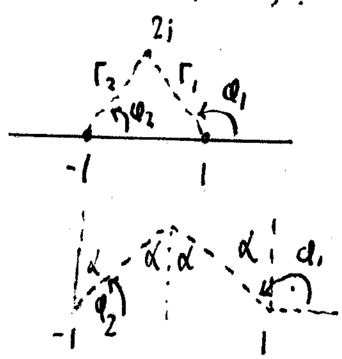
FOR OUR CHOICE OF THE FUNCTION $\sqrt{z^2-1}$. WE CONCLUDE THAT

$$W = -i\,\mathrm{LOG}\left(z + \sqrt{z^2-1}\right) \equiv \cos^{-1} z.$$

THIS IS CALLED THE "PRINCIPAL VALUE" FOR $\cos^{-1} z$.

<u>EXAMPLE</u> CALCULATE $W = \cos^{-1}(2i)$.

LET $z = 2i$



WE GET $r_1 = \sqrt{1+4} = \sqrt{5}$

$r_2 = \sqrt{5}$

SO $\varphi_1 = \pi/2 + \alpha$
$\varphi_2 = \pi/2 - \alpha$ $\rightarrow \varphi_1 + \varphi_2 = \pi$.

THUS $\sqrt{z^2-1} = (\sqrt{5}\,\sqrt{5})^{1/2}\, e^{i\pi/2} = i\sqrt{5}$.

NOW $W = -i\,\mathrm{LOG}(2i + i\sqrt{5}) = -i\,\mathrm{LOG}(i(2+\sqrt{5}))$

$$W = -i\left[\ln(2+\sqrt{5}) + i\pi/2\right]$$

$$W = \pi/2 - i\ln(2+\sqrt{5}).$$