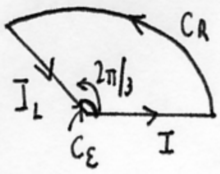
MATH 301 M. WARD FINAL APRIL 2006

PROBLEM 1 i $\quad I = \int_0^\infty \frac{\ln x}{1+x^3}\, dx.$

CONSIDER $\int_C \frac{\log z}{z^3+1}\, dz$

NOW ON C_R: $\left| \int_{C_R} \frac{\log z}{z^3+1}\, dz \right| \le \frac{d(\ln R)R}{R^3} \to 0$ AS $R \to \infty$



NOW ON C_ε: $\left| \int_{C_\varepsilon} \frac{\log z}{z^3+1}\, dz \right| \le d|\ln \varepsilon| \varepsilon \to 0$ AS $\varepsilon \to 0$.

THEREFORE $\quad I + I_L = 2\pi i \text{ RES}\left[\frac{\log z}{1+z^3} ; e^{\pi i/3} \right]$ AS $R \to \infty, \varepsilon \to 0$.

NOW ON I_L: $z = xe^{2\pi i/3}$, $dz = e^{2\pi i/3}\, dx$ SO THAT

$$I_L = e^{2\pi i/3} \int_\infty^0 \frac{(\ln x + 2\pi i/3)}{(x^3+1)}\, dx = -e^{2\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} J \qquad J = \int_0^\infty \frac{dx}{1+x^3}.$$

HENCE $\quad I(1 - e^{2\pi i/3}) - \frac{2\pi i}{3} e^{2\pi i/3} J = 2\pi i \frac{(i\pi/3)}{3e^{2\pi i/3}} = -\frac{2\pi^2}{9} e^{-2\pi i/3}$

MULTIPLY BY $e^{-2\pi i/3}$: $\quad I(e^{-2\pi i/3} - 1) - \frac{2\pi i}{3} J = -\frac{2\pi^2}{9} e^{-4\pi i/3}.$

NOW TAKE REAL PARTS TO EXTRACT I.

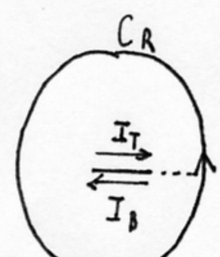
$$I \text{ RE}\left(e^{-2\pi i/3} - 1\right) = -\frac{2\pi^2}{9} \text{RE}\left(e^{-4\pi i/3}\right)$$

$$\to \quad I\left(\cos(-2\pi/3) - 1\right) = -\frac{2\pi^2}{9} \cos\left(-4\pi/3\right) \Rightarrow I\left(-3/2\right) = -\frac{2\pi^2}{9}\left(\frac{-1}{2}\right)$$

THUS $\quad \underline{I = -2\pi^2/27} = \int_0^\infty \ln x/(1+x^3)\, dx.$

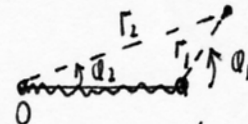
PROBLEM 1(ii) $\quad$ CONSIDER A KEY-HOLE CONTOUR FOR

$\int_C f(z)\, dz \quad$ WITH $\quad f(z) = z^{1/3}(z-1)^{2/3}$



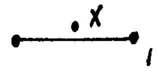
HENCE $\quad I_T + I_B + \lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$ BY RESIDUE THEOREM.

WE TAKE BRANCH CUT FROM $z = 0$ TO $z = 1$



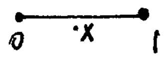
$$f(z) = r_1^{2/3} r_2^{1/3} e^{2i\theta_1/3 + i\theta_2/3} \qquad 0 \le \theta_1 < 2\pi, \quad 0 \le \theta_2 < 2\pi$$

NOW ON I_T: $\quad \theta_1 = \pi, \ \theta_2 = 0 \ \longrightarrow \ f(z) = x^{1/3}(1-x)^{2/3} e^{2\pi i/3}$



$$\longrightarrow \quad I_T = \left(\int_0^1 x^{1/3}(1-x)^{2/3}\,dx \right) e^{2\pi i/3}$$

NOW ON I_B: $\quad \theta_1 = \pi, \ \theta_2 = 2\pi \ \longrightarrow \ f(z) = x^{1/3}(1-x)^{2/3} e^{4\pi i/3}$



$$\longrightarrow \quad I_B = \left(\int_1^0 x^{1/3}(1-x)^{2/3}\,dx \right) e^{4\pi i/3}$$

HENCE $\quad \left(e^{2\pi i/3} - e^{4\pi i/3} \right) I + \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$

NOW RECALL $\quad (1+h)^p \approx 1 + ph + \frac{p(p-1)}{2} h^2 + \ldots \quad$ AS $\quad h \to 0.$

CONSISTENT WITH BRANCH CUT

HENCE, FOR $z >> 1 \ \longrightarrow \ f(z) \simeq z^{1/3} z^{2/3} \left(1 - \frac{1}{z}\right)^{2/3} \cong z \left(1 - \frac{2}{3z} - \frac{1}{9z^2} + \ldots \right)$

THUS $\quad a_{-1} = -1/9 \ \longrightarrow \ \lim_{R\to\infty} \int_{C_R} f(z)\,dz = -2\pi i/9.$

BUT $\quad e^{4\pi i/3} = e^{-2\pi i/3} \quad \longrightarrow \quad \left(e^{2\pi i/3} - e^{-2\pi i/3} \right) I = 2\pi i/9$

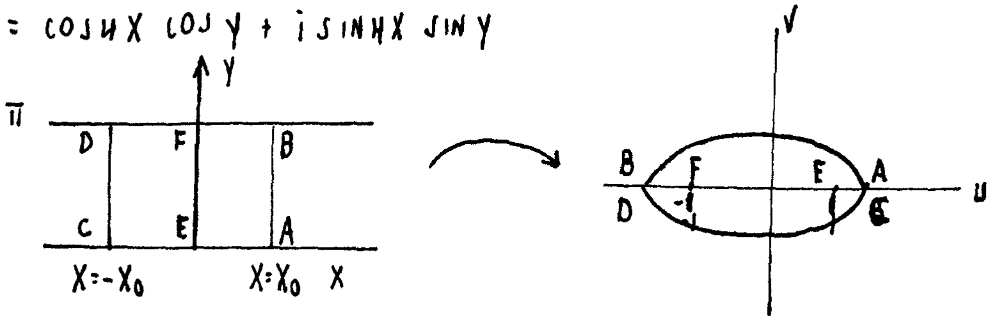
SO $\quad \sin(2\pi/3)\, I = \pi/9 \ \longrightarrow \ \frac{\sqrt{3}}{2} I = \frac{\pi}{9} \quad$ OR $\quad \int_0^1 x^{1/3}(1-x)^{2/3}\,dx = \frac{2\pi}{9\sqrt{3}}.$

PROBLEM 2 WE WRITE $W = \cosh Z$ AS

$$U + iV = \cosh(X+iY) = \cosh X \cos Y + i \sinh X \sin Y$$

SO $U = \cosh X \cos Y$

$V = \sinh X \sin Y$



FOR $X = X_0$: $\dfrac{U^2}{\cosh^2 X_0} + \dfrac{V^2}{\sinh^2 X_0} = 1$ WITH $V \geq 0 \rightarrow$ TOP PART OF ELLIPSE

FOR $X = -X_0$: WE GET LOWER PART OF ELLIPSE WITH $V < 0$ SINCE $\sinh(-X_0) < 0$.

FOR $X = 0$ WE GET $U = \cos Y$, $V = 0$ WHICH GIVES LINE BETWEEN -1 AND 1.

NOW SINCE $\cosh X_0 > 0$ AND $\sinh X_0 \geq 0$ ARE MONOTONICALLY INCREASING IN X_0 WE GET THAT $0 < \operatorname{Im} Z < \pi$ MAPS TO W-PLANE WITH $(-\infty, -1)$ AND $(1, \infty)$ REMOVED.

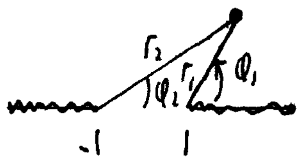
TO MAKE IT SINGLE-VALUED WE PUT A BRANCH CUT ALONG THESE SEGMENTS.

NOW $\quad W = \cosh Z = \dfrac{e^Z + e^{-Z}}{2}$ SO $e^{2Z} - 2We^Z + 1 = 0$

SO $\quad e^Z = \dfrac{2W \pm \sqrt{4W^2 - 4}}{2} = W + \sqrt{W^2 - 1}$ ($\pm$ IRRELEVANT SINCE $\sqrt{\ }$ MULTI-VALUED)

HENCE $\quad Z = \log\left(W + \sqrt{W^2-1}\right)$.

WE WRITE $\quad \sqrt{W^2 - 1} = (r_1 r_2)^{1/2} e^{i(\varphi_1 + \varphi_2)/2}$



WITH $0 < \varphi_1 < 2\pi$, $-\pi < \varphi_2 < \pi$.

WE THEN TAKE PRINCIPAL BRANCH OF $\log(\ldots)$.

HENCE $\quad Z = \operatorname{LOG}\left(W + \sqrt{W^2-1}\right) = \cosh^{-1}(W)$

WE CALCULATE $\quad \sqrt{W^2-1}\Big|_{W=0} = (1)e^{i\pi/2} = e^{i\pi/2}$

SO $\quad Z = \operatorname{LOG}\left(e^{i\pi/2}\right) = i\pi/2 = \cosh^{-1}(0)$,

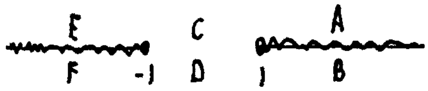
IS THE BRANCH REQUESTED.

REMARKS (EXTRA OBSERVATIONS). (THIS IS SUPPLEMENTARY TO THE TEST)

(i) RECALL THAT $\mathrm{LOG}\,\zeta$ HAS A BRANCH CUT ON $-\infty < \zeta < 0$, ζ REAL.

WE NOW ASK WHETHER $\mathrm{LOG}\left(W + \sqrt{W^2-1}\right)$ IS EVER SUCH THAT $W + \sqrt{W^2-1}$ = Negative Real.

WE TAKE THE POINTS SHOWN AND CALCULATE:



AT A: $\phi_1 = \phi_2 = 0 \to Z = \mathrm{LOG}(\ \ + \sqrt{\ \ -1})$ $U > 1$

AT B: $\phi_1 = 2\pi,\ \phi_2 = 0 \to Z = \mathrm{LOG}\left(U - \sqrt{U^2-1}\right)$ $U > 1$

AT C, D: $\phi_1 = \pi,\ \phi_2 = 0 \to Z = \log\left(U + i\sqrt{1-U^2}\right)$ $-1 < U < 1$

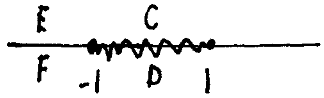
AT E: $\phi_1 = \pi,\ \phi_2 = \pi \to Z = \mathrm{LOG}\left(U - \sqrt{U^2-1}\right)$ $U < -1$

AT F: $\phi_1 = \pi,\ \phi_2 = -\pi \to Z = \mathrm{LOG}\left(U + \sqrt{U^2-1}\right)$ $U < -1$

SO AT BOTH E, F WE ARE EVALUATING $Z = \mathrm{LOG}\,\zeta$ WITH ζ REAL AND NEGATIVE. THUS $-\infty < U < -1$ LIES ALSO ON BRANCH CUT OF $Z = \mathrm{LOG}\left(W + \sqrt{W^2-1}\right)$.

(ii) ANOTHER WAY OF DEFINING A SINGLE-VALUED INVERSE IS TO TAKE

$Z = \mathrm{LOG}\left(W + \sqrt{W^2-1}\right)$ WITH



SO $\sqrt{W^2-1} = (r_1 r_2)^{1/2} e^{i(\phi_1+\phi_2)/2}$ $\quad 0 < \phi_1 < 2\pi,\ 0 < \phi_2 < 2\pi.$

THEN AT

C: $\phi_1 = \pi,\ \phi_2 = 0 \to Z = \mathrm{LOG}\left(U + i\sqrt{1-U^2}\right)$ $-1 < U < 1$

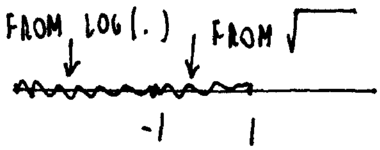
D: $\phi_1 = \pi,\ \phi_2 = 2\pi \to Z = \mathrm{LOG}\left(U - i\sqrt{1-U^2}\right)$ $-1 < U < 1$

E: $\phi_1 = \pi,\ \phi_2 = \pi \to Z = \mathrm{LOG}\left(U - \sqrt{U^2-1}\right)$ $U < -1$

F: $\phi_1 = \pi,\ \phi_2 = \pi \to Z = \mathrm{LOG}\left(U - \sqrt{U^2-1}\right)$ $U < -1$

NOTICE THAT AT E AND F WE HAVE $Z = \mathrm{LOG}\,\zeta$ WITH $\zeta < 0$ REAL.

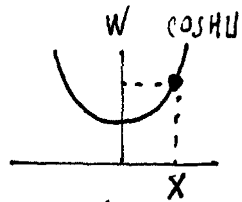
HENCE THE AXIS $-\infty < U < -1$ ALSO CORRESPONDS TO BRANCH CUT OF $\mathrm{LOG}(\cdot)$. THUS, OVERALL, THE BRANCH CUT IS



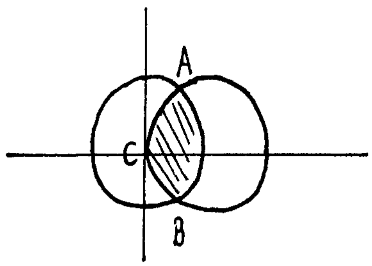
NOTICE FOR $U > 1$ $\phi_1 = \phi_2 = 0$ SO

$Z = \mathrm{LOG}\left(U + \sqrt{U^2-1}\right)$

WHICH CORRESPONDS TO + ROOT IN $W = \cosh X$ WHEN X IS REAL.



PROBLEM 3



$$x^2+y^2 = (x-1)^2+y^2 = 1$$

SO $x^2=(x-1)^2 \longrightarrow x=\frac{1}{2},\ y=\pm\frac{\sqrt{3}}{2}$.

LET $A:\ z=e^{\pi i/3} \longrightarrow w=0$

$B:\ z=e^{-\pi i/3} \longrightarrow w=\infty$ $\Rightarrow$ BOTH CIRCLES MAPPED TO LINES THROUGH THE ORIGIN.

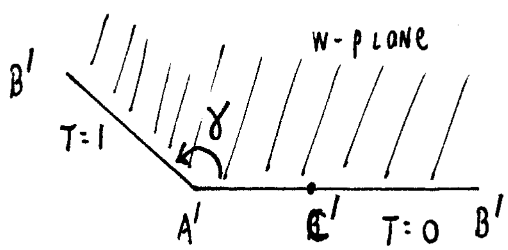
WE TAKE THE MAP $w=\beta\left(\dfrac{z-e^{\pi i/3}}{z-e^{-\pi i/3}}\right)$

WE MAP C $\cancel{\text{TO}}\Rightarrow z=0 \longrightarrow w=1$. THUS $1=\beta e^{+2\pi i/3}$ OR $\beta=e^{-2\pi i/3}$.

THIS YIELDS $w=e^{-2\pi i/3}\left(\dfrac{z-e^{\pi i/3}}{z-e^{-\pi i/3}}\right)$

THEN $|z|=1$ IS MAPPED TO POSITIVE REAL AXIS IN W-PLANE.

NOTICE THAT WE MUST GET A REGION AS SHOWN SINCE WE HAVE PRESERVED ORIENTATION



TO FIND WEDGE ANGLE γ:

TAKE A POINT $z=1$ ON $|z|=1$.

THEN $w=e^{-2\pi i/3}\left(\dfrac{1-\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{1-\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)=e^{-2\pi i/3}\dfrac{e^{-\pi i/3}}{e^{\pi i/3}}$

HENCE, $w=e^{-4\pi i/3}=e^{2\pi i/3}\Rightarrow \arg w = 2\pi/3=\gamma$.

HENCE $T=A\varphi$ SO $1=A\,2\pi/3$ OR $A=3/2\pi$.

HENCE $T=\dfrac{3}{2\pi}\varphi$ $\quad \varphi=\arg w=\tan^{-1}\left(\dfrac{v}{u}\right)=\begin{cases}\operatorname{ATAN}\left(\frac{v}{u}\right) & \text{IF } v,u\geq 0\\ \pi-\operatorname{ATAN}\left(\frac{v}{u}\right) & \text{IF } u<0,v>0\end{cases}$

NOW SEPARATE REAL AND IMAGINARY PARTS:

$$u+iv=e^{-2\pi i/3}\left(\frac{z-e^{i\pi/3}}{z-e^{-i\pi/3}}\right)\left(\frac{\bar{z}-e^{i\pi/3}}{\bar{z}-e^{i\pi/3}}\right)=e^{-2\pi i/3}\left(\frac{z\bar{z}-e^{i\pi/3}(z+\bar{z})+e^{2\pi i/3}}{|z-e^{-i\pi/3}|^2}\right)$$

$$u+iv=\frac{|z|^2e^{-2\pi i/3}-2xe^{-i\pi/3}+1}{|z-e^{i\pi/3}|^2}.$$

NOW $$\frac{V}{U} = \frac{-|z|^2 \sin(2\pi/3) + 2\sin(\pi/3)\, x}{|z|^2 \cos(2\pi/3) - 2\cos(\pi/3)\, x + 1} = \frac{\frac{\sqrt{3}}{2}\left(-(x^2+y^2)+2x\right)}{\frac{1}{2}\left(2-2x-(x^2+y^2)\right)}$$

HENCE $$\frac{V}{U} = \sqrt{3}\left[\frac{x^2+y^2-2x}{x^2+y^2+2x-2}\right],$$

WHICH GIVES $$T = \frac{3}{2\pi}\tan^{-1}\left(\sqrt{3}\left(\frac{x^2+y^2-2x}{x^2+y^2+2x-2}\right)\right)$$

CHECK: ON $x^2+y^2=1 \rightarrow T = \frac{3}{2\pi}\tan^{-1}(-\sqrt{3}) = \frac{3}{2\pi}\left(\frac{2\pi}{3}\right) = 1$

ON $(x-1)^2+y^2=1 \rightarrow T = \frac{3}{2\pi}\tan^{-1}(0) = 0.$

(ii) TO GET A BOUNDED REGION THE BILINEAR MAP MUST NOT HAVE A POLE ON THE BOUNDARY OF C (OTHERWISE WE WOULD GET A LINE). HOWEVER, THE ANGLE AT A AND B MUST BE PRESERVED BY A CONFORMAL MAP. SINCE THIS ANGLE IS $2\pi/3 \neq 180°$, WE CANNOT MAP TO $|w|=1$.

PROBLEM 4

$$u_t = D_1 u_{xx} + D_2 u_{yy}$$

$$u(x,y,0) = \delta(x)\,\delta(y)$$

LET $\quad U(k_1,k_2,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\,e^{-ik_1x - ik_2y}\,dx\,dy = \hat{f}(u)$

THEN $\quad \frac{d}{dt}U = -\left(D_1k_1^2 + D_2k_2^2\right)U$

$$U(k_1,k_2,0) = \hat{f}\left(\delta(x)\delta(y)\right) = 1.$$

HENCE $\quad U = e^{-D_1k_1^2t}\,e^{-D_2k_2^2t}$

THEN $\quad u(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D_1k_1^2t - D_2k_2^2t}\,e^{ik_1x + ik_2y}\,dk_1\,dk_2$

HENCE, $\quad u(x,y,t) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-D_1k_1^2t + ik_1x}\,dk_1\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-D_2k_2^2t + ik_2y}\,dk_2\right)$

$$\Longrightarrow \quad u(x,y,t) = \hat{f}^{-1}\left(e^{-D_1k_1^2t}\right)\hat{f}^{-1}\left(e^{-D_2k_2^2t}\right)$$

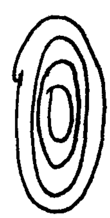
RECALLING $\quad \hat{f}^{-1}\left(e^{-Dk^2t}\right) = \frac{1}{\sqrt{4\pi Dt}}\,e^{-x^2/4Dt} \Rightarrow u(x,y,t) = \frac{e^{-x^2/4D_1t - y^2/4D_2t}}{4\pi t\sqrt{D_1D_2}}$

THEREFORE, $\quad u(x,y,t) = \frac{1}{4\pi t\sqrt{D_1D_2}}\,e^{-\frac{1}{4t}\left(\frac{x^2}{D_1} + \frac{y^2}{D_2}\right)}$

NOW NOTICE AT EACH FIXED t WE HAVE $u =$ CONSTANT

WHEN $\quad x^2/D_1 + y^2/D_2 =$ CONSTANT, which is an ellipse.

SINCE $0 < D_1 < D_2$ WE HAVE



ELLIPSES FOR THE ISOTHERMS.

THIS IS PERFECTLY REASONABLE SINCE IF $D_1 < D_2$

THE "MATERIAL" DIFFUSES FASTER IN Y-DIRECTION THAN X-DIRECTION.

PROBLEM 5 i)

$$y''' + 2y'' + 4y' + 6y = \sin t \qquad t \geq 0.$$

$$y(0) = y'(0) = y''(0) = 0$$

LET $Y(s) = \mathcal{L}(y(t))$. RECALL $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$.

HENCE $$(s^3 + 2s^2 + 4s + 6)Y = \frac{1}{s^2+1}$$

OR $$Y(s) = \frac{1}{(s^2+1)\,p(s)} \qquad p(s) = s^3 + 2s^2 + 4s + 6.$$

WE HAVE THAT $Y(s)$ HAS poles at $s = \pm i$ AND AT ZEROES OF $p(s) = 0$. SINCE $p(\pm i) \neq 0$, $s = \pm i$ ARE SIMPLE POLES.

THEREFORE, IF WE CAN PROVE THAT $p(s) = 0$ ONLY FOR $\mathrm{Re}(s) < 0$ WE WILL HAVE THAT $y(t)$ is BOUNDED AS $t \to \infty$.

RECALL SINCE $\deg(p(s)) = 3$ WE HAVE THE NYQUIST CRITERION

$$\frac{1}{2\pi}\left(3\pi + \Delta_\Gamma \arg(p(iy))\right) = N_0(p) = \#\text{ ZEROES OF } p(s) = 0 \text{ IN } \mathrm{RE}(s) > 0.$$

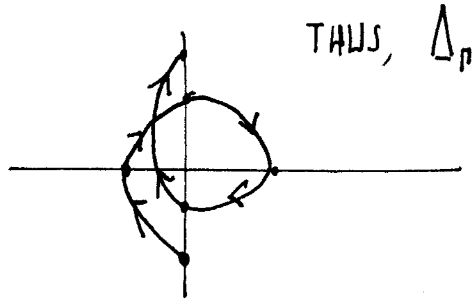
THUS WE MUST SHOW THAT $\Delta_\Gamma \arg(p(iy)) = -3\pi$ FOR $N_0(p) = 0$.

WE CALCULATE $p(iy) = (6 - 2y^2) + i(4y - y^3)$ $\qquad y_{R_\pm} = \pm\sqrt{3}$, $y_{I_\pm} = \pm 2$ AND $y_I = 0$.

	y	RE P	IM P
A	∞	< 0	< 0
B	y_{I_+}	< 0	$= 0$
C	y_{R_+}	$= 0$	> 0
D	y_I	> 0	$= 0$
E	y_{R_-}	$= 0$	< 0
F	y_{I_-}	< 0	$= 0$
G	$-\infty$	< 0	> 0

AT A: $\dfrac{\mathrm{IM}}{\mathrm{RE}} \to +\infty$

AT G: $\dfrac{\mathrm{IM}}{\mathrm{RE}} \to -\infty$



THUS, $\Delta_\Gamma \arg(p(iy)) = -3\pi$

$\Longrightarrow N_0(p) = 0$,

AND $y(t)$ BOUNDED AS $t \to \infty$.

NOW FOR $t \gg 1$ THE POLES WITH THE LARGEST REAL PART DETERMINE THE APPROXIMATE SOLUTION. THESE POLES ARE AT $S = \pm i$.

HENCE $$y(t) \approx \mathrm{RES}\left[\frac{e^{st}}{(s^2+1)\,p(s)}\,;\, i\right] + \mathrm{RES}\left[\frac{e^{st}}{(s^2+1)\,p(s)}\,;\, -i\right]$$

THIS GIVES $$y(t) \approx \frac{e^{it}}{2i\,p(i)} + \frac{e^{-it}}{(-2i)\,p(-i)}$$

LET $$\chi = \frac{e^{it}}{2i\,p(i)}.$$ SINCE $\overline{p(i)} = p(-i)$

WE HAVE $$y(t) \approx \chi + \overline{\chi} = 2\,\mathrm{RE}(\chi).$$

WE CALCULATE $$\chi = \frac{-ie^{it}}{2\,p(i)} = \frac{-i e^{it}}{2}\,\frac{1}{(-i+4i+6-2)} = \frac{-e^{it+i\pi/2}}{2}\,\frac{1}{(4+3i)}$$

SO $$\chi = \frac{e^{i(t-\pi/2)}}{2}\,\frac{(4-3i)}{25} = \frac{e^{i(t-\pi/2)}}{50}\,(4-3i)$$

$$\mathrm{RE}\,\chi = \frac{1}{50}\left(4\cos\left(t-\pi/2\right) + 3\sin\left(t-\pi/2\right)\right)$$

HENCE $$y(t) \approx \frac{1}{25}\left(4\cos\left(t-\pi/2\right) + 3\sin\left(t-\pi/2\right)\right)$$

OR $$y(t) \approx \frac{1}{25}\left(4\sin t - 3\cos t\right)$$

PROBLEM 5 ii)

$$y' + y = F(t) \qquad F(t) : \begin{cases} 0 & 0 \le t \le 1 \\ 2 & 1 \le t \le 2 \end{cases} \qquad \hat{F}_0(s) = \int_0^2 F(t) e^{-st} dt.$$

$$y(0) = 0$$

WE TAKE LAPLACE TRANSFORMS TO GET $\displaystyle Y(s) = \frac{\hat{F}_0(s)}{(s+1)(1-e^{-2s})}$

RECALL THAT $\hat{F}_0(s)$ IS ANALYTIC SINCE IT HAS FINITE RANGE OF INTEGRATION.

WE HAVE SIMPLE POLES AT $s=-1$ AND $s = K\pi i \quad K = 0, \pm 1, \pm 2, \ldots$

HENCE
$$y(t) = \mathcal{L}^{-1}[Y(s)] = \text{RES}\left[\frac{\hat{F}_0(s)e^{st}}{(s+1)(1-e^{-2s})}; -1\right] + \text{RES}\left[\frac{\hat{F}_0(s)e^{st}}{(s+1)(1-e^{-2s})}; 0\right]$$
$$+ \sum_{K=1}^{\infty} \text{RES}\left[\frac{\hat{F}_0(s)e^{st}}{(s+1)(1-e^{-2s})}; iK\pi\right] + \sum_{K=-\infty}^{-1} \text{RES}\left[\frac{\hat{F}_0(s)e^{st}}{(s+1)(1-e^{-2s})}; iK\pi\right]$$

USING THE P/Q METHOD FOR RESIDUES WE OBTAIN $\left(\text{NOTE: } \frac{d}{ds}(1-e^{-2s})\Big|_{K\pi i} = +2e^{-2K\pi i} = 2\right)$

$$y(t) = \frac{\hat{F}_0(-1)e^{-t}}{(1-e^{2})} + \frac{\hat{F}_0(0)}{2} + \sum_{K=1}^{\infty} \frac{\hat{F}_0(iK\pi)e^{iK\pi t}}{2(1+iK\pi)} + \sum_{K=-1}^{-\infty} \frac{\hat{F}_0(iK\pi)e^{iK\pi t}}{2(1+iK\pi)}$$

DEFINE $\displaystyle Z_K = \frac{\hat{F}_0(iK\pi)e^{iK\pi t}}{2(1+iK\pi)} \implies y(t) = \frac{\hat{F}_0(-1)e^{-t}}{(1-e^{2})} + \frac{\hat{F}_0(0)}{2} + \sum_{K=1}^{\infty} \underbrace{(Z_K + \overline{Z_K})}_{2\,\text{RE}(Z_K)}.$

THEN $\displaystyle 2\,\text{RE}\,Z_K = \frac{1}{(1+K^2\pi^2)} \text{RE}\left[\hat{F}_0(iK\pi)e^{iK\pi t}(1-iK\pi)\right]$

THIS GIVES $\displaystyle y(t) = \frac{\hat{F}_0(-1)e^{-t}}{(1-e^{2})} + \frac{\hat{F}_0(0)}{2} + \sum_{K=1}^{\infty} \frac{1}{(1+K^2\pi^2)} \text{RE}\left[\hat{F}_0(iK\pi)e^{iK\pi t}(1-iK\pi)\right].$

NOW WE CALCULATE:

$$\hat{F}_0(s) = 2\int_1^2 e^{-st}dt = -\frac{2}{s}e^{-st}\Big|_1^2 = \frac{2}{s}\left(e^{-s} - e^{-2s}\right)$$

$$\hat{F}_0(0) = 2$$
$$\hat{F}_0(-1) = 2(e^2 - e^1)$$
$$\hat{F}_0(iK\pi) = \frac{2}{iK\pi}\left(e^{-iK\pi} - 1\right)$$
$$\hat{F}_0(iK\pi) = \frac{-2i}{K\pi}\left[(-1)^K - 1\right]$$

THIS YIELDS

$$y(t) = \underbrace{\frac{2(e^2 - e^1)e^{-t}}{(1-e^{+2})}}_{\text{transient response}} + \underbrace{1 + p(t)}_{\text{steady-state response}}$$

WHERE $\quad p(t) = \sum_{K=1}^{\infty} \frac{2(1-(-1)^K)}{(1+K^2\pi^2)} \mathrm{RE}\left[e^{iK\pi t}\left(1+\frac{i}{K\pi}\right)\right]$

$$p(t) = \sum_{K=1}^{\infty} \frac{2(1-(-1)^K)}{(1+K^2\pi^2)}\left[\cos(K\pi t) - \frac{1}{K\pi}\sin(K\pi t)\right]$$

THIS CAN BE WRITTEN AS $\quad R\cos w = 1, \quad R\sin w = 1/K\pi \quad \tan w = 1/K\pi \quad R = \sqrt{1+1/K^2\pi^2}$

$$p(t) = \sum_{K=1}^{\infty} \frac{2(1-(-1)^K)}{K\pi\sqrt{1+K^2\pi^2}} \cos(K\pi t + w) \qquad \tan w = 1/K\pi$$

FINALLY,

$$y(t) = \frac{2(e^2-e^1)}{(1-e^2)} e^{-t} + 1 + \sum_{K=1}^{\infty} \frac{4}{(2K-1)\pi\sqrt{1+(2K-1)^2\pi^2}} \cos\left((2K-1)\pi t + w\right)$$

$$\tan w = \frac{1}{(2K-1)\pi}.$$

ANOTHER WAY

WE WRITE $$Y(s) = \left(\frac{\hat{F}_0(s)}{(s+1)(1-e^{-2s})} - \frac{\hat{F}_0(-1)}{(s+1)(1-e^{2})} \right) + \frac{\hat{F}_0(-1)}{(s+1)(1-e^{2})}$$

WHERE $$\hat{F}_0(s) = \frac{2}{s}\left(e^{-s} - e^{-2s}\right).$$

WE OBTAIN THAT

$$Y(s) = \frac{P_0(s)}{(1-e^{-2s})} + A(s) \qquad A(s) = \frac{-2(e-e^2)}{(1-e^2)(s+1)} = \frac{-2e}{(1+e)(s+1)}$$

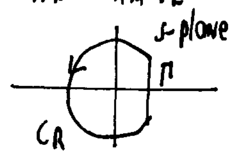
WITH $$P_0(s) = \frac{\hat{F}_0(s)(1-e^2) - \hat{F}_0(-1)(1-e^{-2s})}{(s+1)(1-e^2)}$$ $P_0(s)$ Analytic $\forall s$.

NOW LET $p_0(t) = \mathcal{L}^{-1}[P_0(s)]$. THEN $p_0(t) = 0$ IF $t < 0$, $p_0(t) = 0$ FOR $t > 2$.

NOW $y_a(t) = \mathcal{L}^{-1}[A(s)] = -\frac{2e}{(1+e)} e^{-t}$ is the transient response.

then $y_p(t) = \sum_{n=0}^{\infty} p_0(t-2n)$ WHERE $p_0(t) = \mathcal{L}^{-1}[P_0(s)]$.

NOW TO SEE THAT $p_0(t) = 0$ FOR $t > 2$ WE WRITE $p_0(t) = \frac{1}{2\pi i}\int_{\Gamma} P_0(s) e^{st}\, ds$.
SINCE $P_0(s)$ is ANALYTIC AND $P_0(s) = O(e^{-2s})$ FOR $RE(s) < 0$ WITH $|RE\, s| \to \infty$ WE HAVE THAT $\int_{C_R} \to 0$ AS $R \to \infty$ ONLY WHEN $t > 2 \longrightarrow p_0(t) \equiv 0$ FOR $t > 2$



WE USE $\hat{F}_0(s)$ AND $\hat{F}_0(-1)$ AND WRITE $P_0(s)$ AS

$$P_0(s) = \frac{2(1-e^2)}{s(s+1)}\frac{e^{-s}}{(1-e^2)} + \frac{2(e-e^2)}{(1-e^2)(s+1)} + (\dots)e^{-2s} = \frac{2(1-e^2)}{(1-e^2)}\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-s} + \frac{2e}{(e+1)}\frac{1}{s+1} + (\dots)e^{-2s}.$$

WE DON'T WORRY ABOUT $(\dots)e^{-2s}$ SINCE $p_0(t) = 0$ FOR $t > 2$.

WE INVERT WHAT REMAINS RECALLING $\mathcal{L}(u_c(t) f(t-c)) = F(s)e^{-cs}$ TO OBTAIN

$$p_0(t) = 2u_1(t)\left(1 - e^{-(t-1)}\right) + \frac{2e}{e+1}e^{-t}.$$ THEN $$y_p(t) = \sum_{n=0}^{\infty} p_0(t-2n)$$

AND $$y(t) = \sum_{n=0}^{\infty} p_0(t-2n) + -2e\, e^{-t}/(1+e).$$

PROBLEM 6 $\quad u_t = u_{xx} - u, \quad 0 < x < \infty, \; t > 0$

$u(x,0) = 0, \quad u_x(0,t) = \cos(wt)$

$u, u_x \to 0$ AT ∞.

TAKE LAPLACE TRANSFORMS:

WE DEFINE $\quad U(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt = \mathcal{L}(u) \Rightarrow \mathcal{L}(u_t) = sU - u(x,0)$

THEN $\quad sU = U_{xx} - 1 \quad \to \quad U_{xx} - (s+1)U = 0 \quad$ SINCE $u(x,0) = 0$.

WE HAVE TO SOLVE: $\quad U_{xx} - (s+1)U = 0 \qquad 0 < x < \infty$

$$U_x(0,s) = \frac{s}{s^2+w^2} = \mathcal{L}(\cos(wt))$$

$$U \to 0 \quad \text{as} \quad x \to \infty.$$

THE SOLUTION IS $\quad U = Ae^{-\sqrt{s+1}\,x} + Be^{\sqrt{s+1}\,x}$

CHOOSING THE BRANCH FOR WHICH $\mathrm{RE}(\sqrt{s+1}) > 0$, THEN WE NEED $B = 0$ FOR BOUNDEDNESS AS $x \to +\infty$. $\to U = Ae^{-\sqrt{s+1}\,x}$.

NOW $\quad U_x = -A\sqrt{s+1}$ ON $x = 0 \quad \to \quad -A\sqrt{s+1} = \dfrac{s}{s^2+w^2}$.

HENCE $\quad U(x,s) = \left(-\dfrac{s}{s^2+w^2}\right)\dfrac{e^{-\sqrt{s+1}\,x}}{\sqrt{s+1}}$.

WE INVERT BY CONVOLUTION THEOREM. $\quad \mathcal{L}^{-1}\left(\dfrac{s}{s^2+w^2}\right) = \cos(wt)$

SINCE $\quad \mathcal{L}^{-1}\left(\dfrac{e^{-\sqrt{s}\,x}}{\sqrt{s}}\right) = \dfrac{1}{\sqrt{\pi t}}e^{-x^2/4t} \Rightarrow \mathcal{L}^{-1}\left(\dfrac{e^{-\sqrt{s+1}\,x}}{\sqrt{s+1}}\right) = \dfrac{e^{-t}}{\sqrt{\pi t}}e^{-x^2/4t}$.

BY THE CONVOLUTION THEOREM $\quad u(x,t) = \displaystyle\int_0^t -\cos(w(t-\tau))\frac{e^{-\tau}}{\sqrt{\pi\tau}}e^{-x^2/4\tau}\,d\tau$

OR EQUIVALENTLY,

$$u(x,t) = \int_0^t -\cos(w\tau)\frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}}e^{-x^2/4(t-\tau)}\,d\tau$$