CONFORMAL MAPPING AND FLUID FLOW

CONSIDER A STEADY, TWO-DIMENSIONAL FLOW OF AN INCOMPRESSIBLE, INVISCID FLUID. LET $\underline{V} = (V_1, V_2)$ BE THE VELOCITY VECTOR FOR THE FLUID. WE WILL ASSUME

(i) $\nabla \cdot \underline{V} = 0 \longrightarrow$ INCOMPRESSIBLE FLUID

(ii) $\nabla \times \underline{V} = 0 \longrightarrow$ IRROTATIONAL FLUID (NO VORTICITY)

SINCE $\nabla \times \underline{V} = 0$ THERE IS A FUNCTION $\Phi(x,y)$ CALLED THE <u>VELOCITY POTENTIAL</u> FOR WHICH

$$\underline{V} = \nabla \Phi \longrightarrow V_1 = \partial\Phi/\partial x \qquad V_2 = \partial\Phi/\partial y .$$

FROM THE MASS-CONSERVATION CONDITION (i) WE HAVE

$$(1) \qquad \nabla^2 \Phi = 0$$

NOW INTRODUCE A <u>COMPLEX VELOCITY POTENTIAL</u> $\Omega(z)$ GIVEN BY

$$(2) \qquad \Omega(z) = \Phi + i\Psi$$

IF Ψ IS THE HARMONIC CONJUGATE TO Φ, THEN $\Omega(z)$ IS ANALYTIC.

NOW $\Omega'(z) = \Phi_x + i\Psi_x$. BUT $\Psi_x = -\Phi_y$ BY CAUCHY RIEMANN EQUATION.

HENCE $\Omega'(z) = \Phi_x - i\Phi_y$.

THEREFORE THE X AND Y COMPONENTS OF $\underline{V}$, LABELLED BY V_1 AND V_2 SATISFY

$$\Omega'(z) = V_1 - iV_2 .$$

THUS $V_1 + iV_2 = \overline{\Omega'(z)}$

THE <u>SPEED</u> OF THE FLOW IS $|\underline{V}| = (V_1^2 + V_2^2)^{1/2} = |\overline{\Omega'(z)}|$.

A <u>STAGNATION POINT</u> OF THE FLOW IS A VALUE OF (x, y) FOR WHICH $\underline{V} = 0$. IN OTHER WORDS, SUCH A POINT SATISFIES $|\overline{\Omega'(z_0)}| = 0$

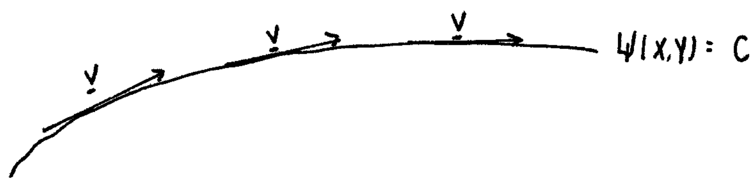
NOW WE INTERPET Ψ:

FIRST NOTICE THAT $\nabla\Phi \cdot \nabla\Psi = 0$ AS FOLLOWS FROM THE CAUCHY RIEMANN EQUATIONS. BUT $\underline{v} = \nabla\Phi$ AND HENCE

$$\underline{v} \cdot \nabla\Psi = 0$$

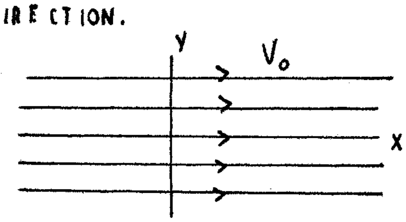
CONSIDER A LEVEL LINE $\Psi(x,y) = C$. THEN $\nabla\Psi$ IS $\perp$ TO THIS LEVEL LINE AND $\underline{v} \cdot \nabla\Psi = 0$. THUS $\underline{v}$ IS PARALLEL TO A TANGENT VECTOR ON THE LEVEL LINE $\Psi(x,y) = C$.

THEREFORE, WE CALL LEVEL LINES Ψ = CONSTANT STREAMLINES SINCE THEY REPRESENT THE DIRECTION OF THE PATHS OF THE FLUID PARTICLES TRANSPORTED BY THE FLOW. Ψ is called the <u>streamfunction</u>.



<u>EXAMPLE</u> FIND THE COMPLEX VELOCITY POTENTIAL CORRESPONDING TO A UNIFORM FLUID FLOW IN THE POSITIVE X-DIRECTION.

WE WANT $\underline{v} = (V_0, 0)$.



V_0 constant

THUS $\partial\Phi/\partial x = V_0$ $\quad \partial\Phi/\partial y = 0$

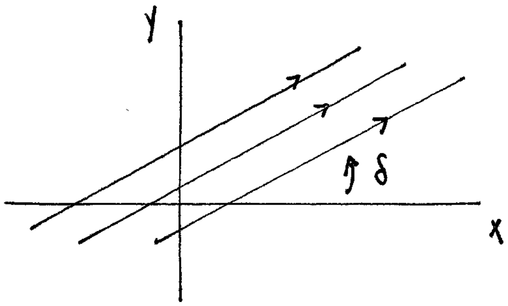
$\longrightarrow$ $\Phi = V_0 x$ (we can neglect the additive constant).

NOW $\Psi_y = \Phi_x$ $\quad \Psi_x = -\Phi_y$ $\longrightarrow$ $\Psi = V_0 y$. streamfunction

HENCE $\Omega(z) = \Phi + i\Psi = V_0(x + iy)$

$\Omega(z) = V_0 z$ is complex velocity potential FOR UNIFORM FLOW IN positive-x-direction.

EXAMPLE Find the velocity potential and complex velocity potential for flow at an angle δ above the horizontal.



want $V_1 = V_0 \cos\delta$

$V_2 = V_0 \sin\delta$

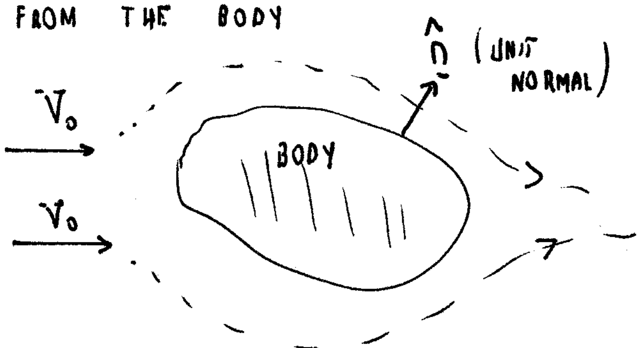
so $\partial\Phi/\partial x = V_0 \cos\delta \qquad \partial\Phi/\partial y = V_0 \sin\delta$

We have $\Omega'(z) = \partial\Phi/\partial x - i\,\partial\Phi/\partial y$

Therefore $\Omega'(z) = V_0(\cos\delta - i\sin\delta) = V_0 e^{-i\delta} \longrightarrow \Omega(z) = V_0 z e^{-i\delta}$ is complex velocity potential

FLOW OVER AN OBJECT

Want to determine streamlines and velocity for flow over a 2-D object. We assume that there is a uniform stream in x-direction far from the body



The complex velocity potential must satisfy

$$\Omega(z) = V_0 z + G(z)$$

where $G(z)$ is bounded as $|z| \to \infty$, or equivalently

$$\lim_{|z|\to\infty} G'(z) = 0 .$$

We also require that the boundary of the body is a streamline of the flow. Hence, no flow penetrates into the body. Writing $\Omega = \Phi + i\Psi$, this implies that (Ψ: streamfunction).

$$\mathrm{Im}(\Omega) = \text{constant on the body} \iff \nabla\Phi\cdot\hat{n} = V\cdot\hat{n} = 0 \text{ on body}$$

EXAMPLE Consider the Joukowski map

$$\Omega(z) = V_0\left(z + a^2/z\right)$$

What type of flow does this correspond to? As $|z| \to \infty$ we have $\Omega(z) \to V_0 z$ which is a uniform stream in x-direction.

NOW CALCULATE Φ AND Ψ.

$$\Omega = \Phi + i\Psi = V_0\left(re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\right)$$

$$\Omega = V_0\left(r + \frac{a^2}{r}\right)\cos\varphi + i\,V_0\left(r - a^2/r\right)\sin\varphi$$

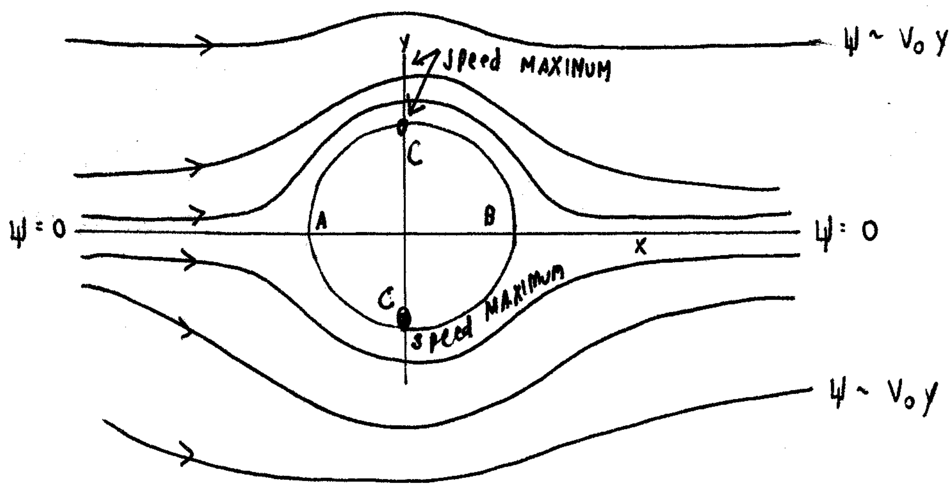
THUS $\quad \Phi = V_0\left(r + a^2/r\right)\cos\varphi \qquad \Psi = V_0\left(r - a^2/r\right)\sin\varphi$

NOTICE THAT $r = a$ IS A STREAMLINE $\Psi = 0$. THUS WE CAN INTERPRET THIS COMPLEX POTENTIAL AS REPRESENTING A UNIFORM FLOW OVER A CIRCULAR CYLINDER OF RADIUS a. THE STREAMLINES $\Psi = C$ SATISFY

$$r^2 - a^2 - \frac{Cr}{V_0\sin\varphi} = 0 \qquad r = \frac{C}{2V_0\sin\varphi} \pm \frac{1}{2}\sqrt{\frac{C^2}{V_0^2\sin^2\varphi} + 4a^2} \qquad C > 0.$$

+ (root is needed)
SINCE WANT $r > a$ OUTSIDE BODY.

PLOTTING, WE GET THE FOLLOWING PICTURE FOR STREAMLINES



NOW CALCULATE THE VELOCITY. RECALL $\underline{V} = (V_1, V_2)$ AND $\overline{\Omega'(z)} = V_1 + iV_2$.

NOW $\quad \Omega'(z) = V_0\left(1 - a^2/z^2\right) = V_0\left(1 - \frac{a^2}{r^2}e^{-2i\varphi}\right)$

$$\overline{\Omega'(z)} = \left(V_0 - \frac{a^2}{r^2}V_0 e^{2i\varphi}\right) = \left(V_0 - V_0\frac{a^2}{r^2}\cos 2\varphi\right) - iV_0\frac{a^2}{r^2}\sin 2\varphi$$

HENCE $\quad V_1 = V_0\left(1 - a^2/r^2\cos 2\varphi\right) \qquad V_2 = -V_0\frac{a^2}{r^2}\sin 2\varphi.$

NOW $\quad |\underline{V}| = \left[1 - \frac{2a^2}{r^2}\cos 2\varphi + \frac{a^4}{r^4}\right]^{1/2} V_0 \quad$ magnitude of velocity.

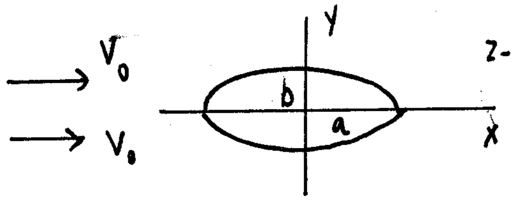
THE STAGNATION POINTS ARE WHERE $\Omega'(z) = 0$. SINCE $\Omega'(z) = \bar{V}_0\left(1 - a^2/z^2\right)$ WE HAVE $\Omega'(z) = 0$ AT $z = \pm a$ (POINTS A, B IN FIGURE). THIS IS WHERE $\underline{v} = \underline{0}$.

NOW THE SPEED OF THE FLOW IS MAXIMIZED AT $\varphi = \pi/2$ AND $\varphi = 3\pi/2$ WITH $r = a$. AT THESE POINTS LABELED BY C

$$|\underline{v}| = \left[\frac{a^4}{a^4} + \frac{2a^2}{a^2} + 1\right]^{1/2} \bar{V}_0 \rightarrow |v| = 2\bar{V}_0.$$

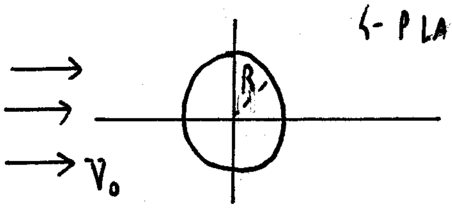
FLOW OVER AN ELLIPSE

FIND THE COMPLEX VELOCITY POTENTIAL FOR FLOW OVER A THIN ELLIPSE WITH A UNIFORM STREAM AT INFINITY AS SHOWN.

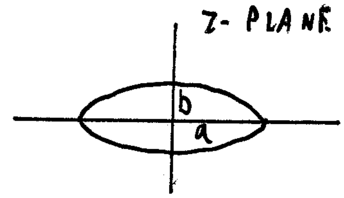


$(a > b > 0)$

WE RECALL THAT $\Omega_0(\zeta) = \bar{V}_0\left(\zeta + R^2/\zeta\right)$ CORRESPONDS TO A UNIFORM FLOW IN POSITIVE $Re(\zeta)$ DIRECTION OVER A CIRCULAR CYLINDER OF RADIUS R IN ζ-PLANE. AS SUCH WE NEED TO FIND MAPPING $\zeta = \zeta(z)$ TO THE ELLIPSE GEOMETRIC IN Z-PLANE. WE WRITE



$$z = \zeta + \tau^2/\zeta$$



OUR GOAL IS TO FIND R AND τ SO THAT DISK $|\zeta| = R$ IS MAPPED TO ELLIPSE AS SHOWN IN Z-PLANE.

PUT $\zeta = Re^{i\varphi}$. THEN $z = Re^{i\varphi} + \dfrac{\tau^2}{R}e^{-i\varphi} = \left(R + \dfrac{\tau^2}{R}\right)\cos\varphi + i\left(R - \dfrac{\tau^2}{R}\right)\sin\varphi$.

THIS GIVES $x = \left(R + \tau^2/R\right)\cos\varphi$, $y = \left(R - \tau^2/R\right)\sin\varphi$. AS SUCH WE SET $a = R + \tau^2/R$ AND $b = R - \tau^2/R$. SOLVING FOR R AND τ WE GET

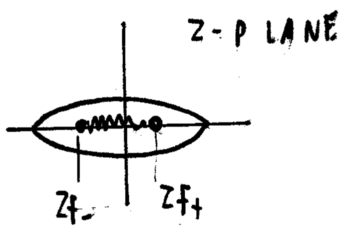
$$R = \frac{a+b}{2}, \quad \tau^2 = (a - R)R = \left(\frac{a-b}{2}\right)\left(\frac{a+b}{2}\right) \rightarrow \tau = \frac{1}{2}\left(a^2 - b^2\right)^{1/2}.$$

THIS GIVES THE MAP $z = \zeta + \tau^2/\zeta$. WE WANT THE INVERSE MAP $\zeta = \zeta(z)$. WE HAVE $\zeta^2 - \zeta z + \tau^2 = 0$, OR $\zeta - z + \tau^2/\zeta = 0$. $(*)$

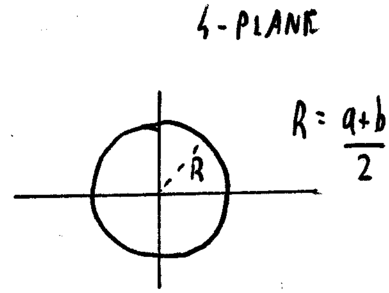
WE GET
$$\zeta = \frac{z \pm (z^2 - 4\tau^2)^{1/2}}{2}.$$
BRANCH POINTS AT
$$z_{F\pm} = \pm 2\tau = \pm (a^2 - b^2)^{1/2}.$$

WE OBSERVE THAT BRANCH POINTS ARE AT THE FOCAL POINTS OF THE ELLIPSE. WE PUT A BRANCH CUT BETWEEN $z_{F\pm} = \pm (a^2 - b^2)^{1/2}$ SO THAT $\zeta(z)$ IS ANALYTIC OUTSIDE ELLIPSE. WITH THIS CHOICE TAKE + SIGN SO THAT AS $|z| \to \infty$, $|\zeta| \to \infty$ AND $\zeta \approx z$ AT INFINITY.

OUR MAPPING IS



$$\zeta = \frac{z + (z^2 - 4\tau^2)^{1/2}}{2}$$



FINALLY, OUR COMPLEX VELOCITY POTENTIAL IS
$$\Omega(z) = \Omega_0[\zeta(z)] = V_0\left[\zeta(z) + R^2/\zeta(z)\right], \quad R = \frac{(a+b)}{2}.$$

NOW WE DETERMINE THE VELOCITY FROM $\Omega'(z) = V_1 - iV_2$ ON THE ELLIPSE. BY CHAIN RULE,
$$\frac{d\Omega}{dz} = \frac{d\Omega_0}{d\zeta}\,\frac{d\zeta}{dz}. \quad \rightarrow \quad \frac{d\Omega_0}{d\zeta} = V_0\left(1 - R^2/\zeta^2\right)$$
$$\frac{d\zeta}{dz}\left(1 - \tau^2/\zeta^2\right) = 1 \quad \text{BY DIFFERENTIATING } (*)$$

THIS GIVES,
$$\frac{d\Omega}{dz} = V_0\left(\frac{\zeta^2 - R^2}{\zeta^2 - \tau^2}\right) \quad \text{WITH} \quad R^2 = (a+b)^2/4, \quad \tau^2 = (a^2 - b^2)/4.$$

TO FIND VELOCITY AT EACH POINT ON ELLIPSE WE CAN SET $\zeta = Re^{i\varphi}$ IN ζ-PLANE TO OBTAIN
$$\frac{d\Omega}{dz} = V_0\left(\frac{e^{2i\varphi} - 1}{e^{2i\varphi} - \tau^2/R^2}\right) \quad \text{WITH} \quad \tau^2/R^2 = \frac{(a-b)(a+b)}{(a+b)(a+b)} = \frac{a-b}{a+b}.$$

OR
$$\tau^2/R^2 = \left(\frac{1 - b/a}{1 + b/a}\right).$$

CORRESPONDING TO A POINT ON BOUNDARY IN ζ-PLANE WE HAVE FROM THE MAP THAT

$$Z = Re^{i\varphi} + \frac{\tau^2}{R} e^{-i\varphi} = R\left[e^{i\varphi} + \frac{\tau^2}{R^2} e^{-i\varphi} \right].$$

THEREFORE, IN TERMS OF THE PARAMETER φ IN $0 \le \varphi \le 2\pi$ WE CAN PARAMETRIZE THE VELOCITY ON ELLIPSE BY

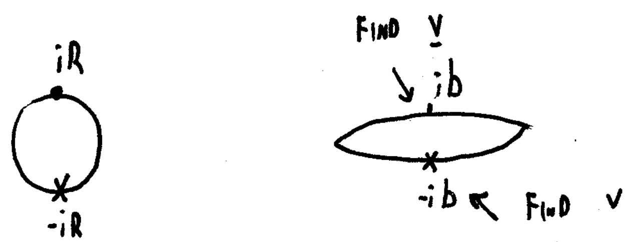
$$\begin{cases} V_1 - iV_2 = \overline{\dfrac{d\Omega}{dz}} = \bar{V}_0 \left(\dfrac{e^{-2i\varphi} - 1}{e^{-2i\varphi} - \tau^2/R^2} \right) \\ x + iy = Z = R\left[e^{i\varphi} + \dfrac{\tau^2}{R^2} e^{-i\varphi} \right] \quad \text{WITH } \dfrac{\tau^2}{R^2} = \left(\dfrac{1 - b/a}{1 + b/a} \right). \end{cases}$$

THIS IMPLICITLY GIVES V_1, V_2 IN TERMS OF EACH POINT ON BOUNDARY OF ELLIPSE.

NOW OBSERVE THAT IF $\zeta = iR$, $\varphi = \pi/2$ SO $x + iy = R\left(i - \frac{i\tau^2}{R^2} \right) = iR\left(1 - \frac{\tau^2}{R^2} \right)$

BUT $R\left(1 - \tau^2/R^2\right) = \frac{(a+b)}{2}\left(1 - \frac{(a-b)}{(a+b)} \right) = b.$ SO $\zeta = iR \to z = ib$ on ellipse

likewise $\zeta = -iR \to z = -ib$ on ellipse



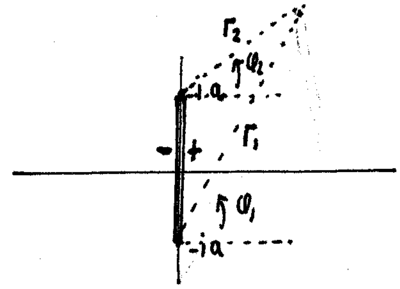
WE CALCULATE AT POINTS $z = \pm ib$ THAT WITH $\varphi = \pi/2$, (OR $\varphi = -\pi/2$)

$$V_1 - iV_2 = \bar{V}_0 \left(\frac{e^{-i\pi} - 1}{e^{-i\pi} - \frac{\tau^2}{R^2}} \right) = 2\bar{V}_0 \left(\frac{1}{\tau^2/R^2 + 1} \right) = 2\bar{V}_0 \left(\frac{1}{\frac{(a-b)}{(a+b)} + 1} \right) = 2\bar{V}_0 \left(\frac{a+b}{2a} \right)$$

THEREFORE $V_2 = 0$, $V_1 = \bar{V}_0 \left(1 + \frac{b}{a} \right)$.

SINCE $b < a$ WE HAVE $V_1 < 2\bar{V}_0$

EXAMPLE (FLOW PAST A PLATE)



CONSIDER THE COMPLEX VELOCITY POTENTIAL

$$\Omega(z) = (z^2 + a^2)^{1/2}.$$

WE CHOOSE THE BRANCH CUT BETWEEN $z = ia$, $z = -ia$.

THE ANGLES ARE DEFINED BY $\frac{-3\pi}{2} < \theta_1 < \frac{\pi}{2}$, $\frac{-3\pi}{2} < \theta_2 < \frac{\pi}{2}$, WHICH GIVES THE BRANCH CUTS AS SHOWN. WE WRITE $\Omega(z) = r_1^{1/2} r_2^{1/2} e^{i(\theta_1+\theta_2)/2}$ WITH $r_1 = |z+ia|$, $r_2 = |z-ia|$.

NOW ON THE + SIDE OF THE PLATE $\theta_1 = \pi/2$, $\theta_2 = -\pi/2$, WHICH GIVES

$$\Omega = (z^2 + a^2)^{1/2} = |z+ia|^{1/2}|z-ia|^{1/2}. \quad \text{THUS } \psi = IM(\Omega) = 0.$$

ON THE - SIDE OF THE PLATE $\theta_1 = -3\pi/2$, $\theta_2 = -\pi/2$

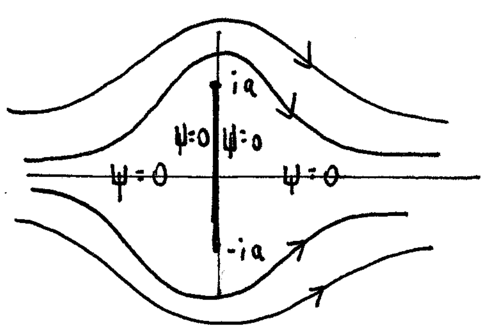
$$\Omega = (z^2+a^2)^{1/2} = |z+ia|^{1/2}|z-ia|^{1/2} e^{-2\pi i/2} = -|z+ia|^{1/2}|z-ia|^{1/2}. \quad \text{THUS } \psi = IM(\Omega) = 0.$$

HENCE, ON EITHER SIDE OF THE PLATE WE HAVE $IM(\Omega) = 0$, SO THAT THE PLATE IS THE STREAMLINE $\psi = 0$. NOW

$$\Omega(z) = |z^2 + a^2|^{1/2} e^{i(\theta_1+\theta_2)/2}$$

SO $\psi = IM(\Omega) = |z^2 + a^2|^{1/2} \sin\left((\theta_1 + \theta_2)/2\right)$ IS THE STREAMFUNCTION

WHERE $-\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$, $-\frac{3\pi}{2} < \theta_2 < \frac{\pi}{2}$. NOTICE THAT $\theta_1 = \theta_2 \to 0 \longrightarrow \psi = 0$, AND THAT $\theta_1 = \theta_2 \to -\pi \longrightarrow \psi = 0$. IF WE PLOT LEVEL CURVES OF ψ WE GET



NOTE

(i) $\Omega(z) = (z^2)^{1/2}\left[1 + \frac{a^2}{z^2}\right]^{1/2} \sim z\left(1 - \frac{a^2}{2z^2} + \dots\right)$ FOR $|z| >> 1$

$\longrightarrow \Omega(z) \approx z$ FOR $|z| \to \infty$ (free stream)

(ii) $\Omega'(z) = z(z^2 + a^2)^{-1/2}$. VELOCITY $\underline{V}$ IS $\underline{V} = \overline{\Omega'(z)}$. NOTE $\underline{V} = 0$ WHEN $z = 0$, AND $\underline{V}$ IS infinite when $z = \pm ia$.

(iii) ON POSITIVE REAL AXIS $\theta_1 = -\theta_2$ SO $\psi = 0$. SAME ON NEGATIVE REAL AXIS.

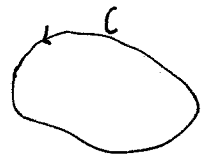
FORCE DUE TO FLUID PRESSURE

THE BERNOULLI'S LAW IS

$$P + \frac{1}{2}\rho\,|\underline{v}|^2 = \alpha \qquad \alpha = \text{CONSTANT along each streamline.}$$

<u>MAIN RESULT</u> LET C BE A STREAMLINE REPRESENTING A BODY. LET $\Omega(z)$ BE THE CORRESPONDING VELOCITY POTENTIAL. LET $\underline{F} = F_1 + iF_2$ BE THE FORCE ON THE BODY. THEN

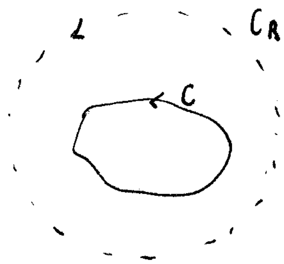
$$\bar{\underline{F}} = \frac{i\rho}{2}\oint_C [\Omega'(z)]^2\,dz.$$



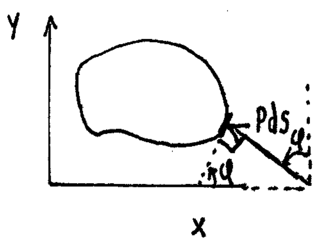
EQUIVALENTLY SINCE $\Omega'(z)$ IS ANALYTIC OUTSIDE C THEN WE CAN USE THE RESIDUE THEOREM TO CALCULATE $\underline{F}$ AS

$$\underline{F} = \frac{i\rho}{2}\lim_{R\to\infty}\oint_{C_R}[\Omega'(z)]^2\,dz$$

WHERE C_R IS A CIRCLE OF RADIUS R ($|z| = R$) THAT ENCLOSES C.



<u>DERIVATION</u> CONSIDER THE FORCE $d\underline{F}$ induced ON A SMALL segment ds OF C AS SHOWN



$$d\underline{F} = dF_1 + i\,dF_2 = -P\sin\varphi\,ds + iP\cos\varphi\,ds$$

THUS $\quad dF = iP\,e^{i\varphi}\,ds.$

NOW $\quad dz = dx + i\,dy = \cos\varphi\,ds + i\sin\varphi\,ds = e^{i\varphi}\,ds. \to dz = e^{i\varphi}ds$

NOW $\quad \Omega'(z) = V_1 - iV_2 \quad$ WITH $\quad (V_1, V_2) = |v|e^{i\varphi} \longrightarrow \Omega'(z) = |v|\,e^{-i\varphi}.$

NOW USE BERNOULLI'S LAW $\quad d\underline{F} = i\left(\alpha - \frac{1}{2}\rho|v|^2\right)e^{i\varphi}\,ds.$

NOW $\quad \int_C \alpha e^{i\varphi}\,ds = \alpha\int_C dz = 0.$ THUS,

$$\underline{F} = -\frac{i\rho}{2}\oint_C |v|^2 e^{i\varphi}\,ds \longrightarrow \bar{\underline{F}} = \frac{i\rho}{2}\oint_C |v|^2 e^{-i\varphi}\,ds = \frac{i\rho}{2}\oint_C |v|^2 e^{-2i\varphi}\,dz.$$

HOWEVER $\quad \Omega'(z) = |v|\,e^{-i\varphi}$, SO THAT

$$\bar{\underline{F}} = \frac{i\rho}{2}\oint_C [\Omega'(z)]^2\,dz.$$

SINKS, SOURCES, AND VORTICES

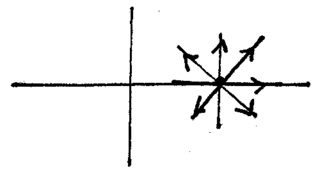
A POINT SOURCE HAS THE FORM

$$\Omega(z) = K \log(z-a), \quad \text{WHERE } K>0 \text{ AND } K \text{ REAL.}$$

LET $\Omega'(z) = \dfrac{K}{z-a}$. WE CALCULATE, WITH $z-a = re^{i\varphi}$

$$\Omega'(z) = \frac{K}{z-a} = \frac{K}{r}e^{-i\varphi} = \frac{K}{r}(\cos\varphi - i\sin\varphi). = V_1 - iV_2.$$

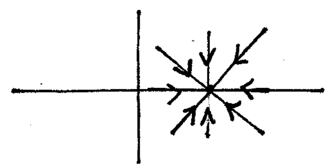
THUS $V_1 = \dfrac{K\cos\varphi}{r}$, $V_2 = \dfrac{K\sin\varphi}{r}$ $\qquad |V| = K/r$



THIS REPRESENTS OUTWARD FLOW FROM A SINGULAR POINT.

THE STREAMLINES CORRESPOND TO $\mathrm{IM}[\Omega(z)] = $ CONSTANT. $\longrightarrow$ streamlines CORRESPOND TO $\varphi = $ CONSTANT.

A POINT SINK LIKEWISE HAS THE FORM $\Omega(z) = -K\log(z-a)$ WITH K REAL AND $K>0$.



A VORTEX CORRESPONDS TO SWIRLING FLOW AROUND SOME POINT.

CONSIDER $\qquad \Omega(z) = iK\log(z-a)$ WITH $K>0$ AND REAL.

WE LET $z = a + re^{i\varphi}$ AND CALCULATE $\Omega'(z) = \dfrac{iK}{z-a} = \dfrac{iK}{r}e^{-i\varphi}$.

WE GET $\qquad \Omega'(z) = \dfrac{K}{r}(\sin\varphi + i\cos\varphi) = V_1 - iV_2.$

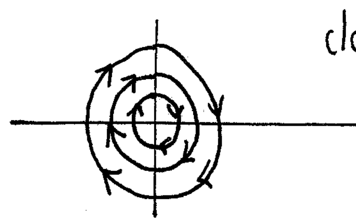
THUS $\quad V_1 = \dfrac{K}{r}\sin\varphi, \quad V_2 = -\dfrac{K}{r}\cos\varphi.$ $\qquad$ NOTICE (WITH $K>0$)

QUADRANT	V_1	V_2
I	>0	<0
II	>0	>0
III	<0	>0
IV	<0	<0

NOW STREAMLINES ARE $\mathrm{IM}(\Omega) = $ CONSTANT.

THIS IMPLIES WITH $\log(z-a) = \log|z-a| + i\varphi$

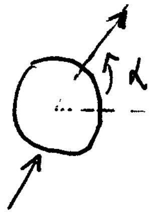
THAT $r = $ CONSTANT. THIS GIVES THE FLOW



clockwise SWIRL when $K>0$.

IF $K<0 \longrightarrow$ counterclockwise SWIRL.

EXAMPLE FOR FLOW OVER A CIRCLE OF RADIUS a AT AN ANGLE OF ATTACK α AS SHOWN WE SIMPLY REPLACE Z IN



$$\Omega = V_0\left(Z + a^2/Z\right)$$

WITH $ze^{-i\alpha}$.

THE COMPLEX VELOCITY POTENTIAL IS THEN

$$\Omega(z) = V_0\left(ze^{-i\alpha} + \frac{a^2}{ze^{-i\alpha}}\right).$$

EXAMPLE CONSIDER FLOW OVER AN ELLIPSE $x^2/a^2 + y^2/b^2 = 1$ WHERE $a > b$. ASSUME THAT FAR FROM THE ELLIPSE THE FLOW HAS SPEED V_0 AND IS INCLINED AT AN ANGLE $\alpha > 0$ WITH RESPECT TO THE POSITIVE X-AXIS.

(i) DERIVE COMPLEX VELOCITY POTENTIAL FOR THE FLOW

(ii) FIND A FORMULA FOR THE SPEED OF THE FLOW AT ANY POINT ON THE ELLIPSE.

(iii) WHERE ARE THE STAGNATION POINTS? AT WHAT POINT ON THE ELLIPSE IS THE SPEED MAXIMUM? WHAT IS THE MAXIMUM SPEED?

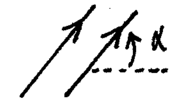
SOLUTION

THE PROCEDURE is:

- RECALL FLOW OVER A CYLINDER OF RADIUS R IN $\tilde{\zeta}$-PLANE WITH UNIFORM STREAM $V_0\tilde{\zeta}$ AT INFINITY IN POSITIVE $\mathrm{RE}(\tilde{\zeta})$ DIRECTION.
- REPLACE $\tilde{\zeta} = \zeta e^{-i\alpha}$ SO THAT FLOW IN ζ-PLANE AT ∞ IS $\nearrow\nearrow$
- USE JOUKOWSKI MAP $Z = \zeta + r^2/\zeta$ TO MAP TO ELLIPSE.

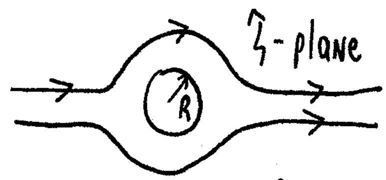
PROBLEM 4

RECALL THAT FLOW AT AN ANGLE α TO HORIZONTAL IS $\Omega = V_0 z e^{-i\alpha}$



NOW THE FLOW OVER A CIRCLE OF RADIUS R IS IN $\tilde{\zeta}$-PLANE

$$\Omega(\zeta) = V_0\left(\tilde{\zeta} + R^2/\tilde{\zeta}\right)$$



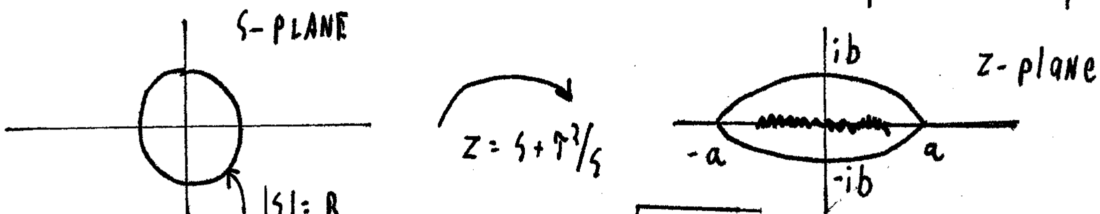
NOW FOR A FLOW AT ANGLE α LET $\hat{\zeta} = \zeta e^{-i\alpha} \longrightarrow \Omega = V_0\left(\zeta e^{-i\alpha} + R^2/\zeta e^{-i\alpha}\right)$

NOW LET $z = \zeta + \tau^2/\zeta$ MAP ζ-PLANE TO z-plane.

THEN $x = a\sin\varphi$, $y = b\sin\varphi$, $\zeta = Re^{i\varphi}$, THEN

$$z = x + iy = \left(R + \tau^2/R\right)\cos\varphi + i\left(R - \tau^2/R\right)\sin\varphi \qquad a = R + \tau^2/R\,,\quad b = R - \tau^2/R$$

THEN $R = (a+b)/2$ AND $2\tau^2/R = a - b \rightarrow \tau^2 = \dfrac{(a-b)(a+b)}{4} = \dfrac{a^2-b^2}{4} \rightarrow \tau = \dfrac{(a^2-b^2)^{1/2}}{2}$



$$\zeta = \frac{z + \sqrt{z^2 - (a^2-b^2)}}{2}$$

WHICH BRANCH CUTS INSIDE BODY BETWEEN $-\sqrt{a^2-b^2} < z < \sqrt{a^2-b^2}$

(i) THEN $\Omega(z) = V_0\left[\zeta e^{-i\alpha} + \dfrac{R^2}{e^{-i\alpha}\zeta}\right]$ $\qquad \zeta = \dfrac{z + \sqrt{z^2-(a^2-b^2)}}{2}$

(ii) GET A FORMULA FOR $|\Omega'(z)|$

NOW $\dfrac{d\Omega}{dz} = \dfrac{d\Omega}{d\zeta}\dfrac{d\zeta}{dz}$ AND $\dfrac{d\Omega}{d\zeta} = V_0\left(e^{-i\alpha} - \dfrac{R^2 e^{i\alpha}}{\zeta^2}\right)$

NOW $2\zeta\, d\zeta/dz - d\zeta/dz\, z - \zeta = 0$ SO $d\zeta/dz = \left(1 - \tau^2/\zeta^2\right)^{-1}$.

HENCE $\dfrac{d\Omega}{dz} = \dfrac{V_0}{\zeta^2}\left(\dfrac{\zeta^2 e^{-i\alpha} - R^2 e^{i\alpha}}{1 - \tau^2/\zeta^2}\right) = V_0\left(\dfrac{\zeta^2 e^{-i\alpha} - R^2 e^{i\alpha}}{\zeta^2 - \tau^2}\right)$

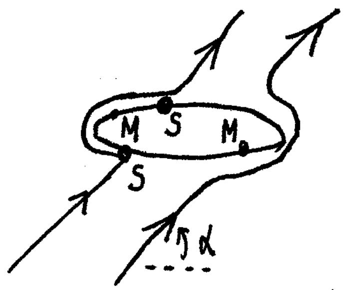
NOW LET $\zeta = Re^{i\varphi}$ SO THAT $\dfrac{d\Omega}{dz} = V_0 e^{-i\alpha}\left(\dfrac{R^2 e^{2i\varphi} - R^2 e^{2i\alpha}}{R^2 e^{2i\varphi} - \tau^2}\right)$

WE THEN RECALL $\dfrac{\tau^2}{R^2} = \dfrac{a-b}{a+b} \equiv \mathcal{B} \longrightarrow \dfrac{d\Omega}{dz} = V_0 e^{-i\alpha}\left(\dfrac{e^{2i\varphi} - e^{2i\alpha}}{e^{2i\varphi} - \mathcal{B}}\right)$

HENCE $\left|\dfrac{d\Omega}{dz}\right| = V = V_0 \left| \dfrac{e^{2i\varphi} - e^{2i\alpha}}{e^{2i\varphi} - B} \right|$ gives a FORMULA FOR speed at any point on ellipse (here φ parametrizes points on ellipse)

(iii) stagnation points where $\left| d\Omega/dz \right| = 0$ so $\varphi = \alpha$ AND $\varphi = \alpha + \pi$ (S = stagnation points)

NOW THE speed should be MAXIMUM when $\varphi = \alpha + \pi/2$, $\varphi = \alpha + 3\pi/2$ (M = MAX speed points)



NOW when $\varphi = \alpha + \pi/2$, OR $\varphi = 3\pi/2 + \alpha$

we get $\left|\dfrac{d\Omega}{dz}\right| = V_0 \dfrac{|-2e^{2i\alpha}|}{|e^{2i\alpha} + B|}$

(NOTICE: $e^{2i(\alpha + \pi/2)} = e^{i\pi} e^{2i\alpha} = -e^{2i\alpha}$.

HENCE $\left|\dfrac{d\Omega}{dz}\right| = V_{MAX} = \dfrac{2V_0}{|e^{2i\alpha} + B|} = \dfrac{2V_0}{\left[(\cos 2\alpha + B)^2 + \sin^2 2\alpha\right]^{1/2}}$

where $B = \dfrac{a-b}{a+b} > 0$.

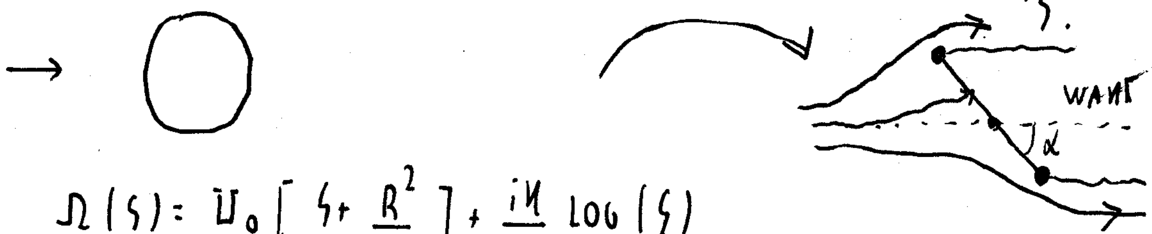
NOTICE IF $\alpha = 0$ THEN $1 + B = \dfrac{2a}{a+b}$

so $\left|\dfrac{d\Omega}{dz}\right| = V_{MAX} = \dfrac{2V_0}{\frac{2a}{(a+b)}} = \left(\dfrac{a+b}{a}\right) V_0$.

IF $a = b \rightarrow V_{MAX} = 2V_0$.

EXAMPLE FIND COMPLEX VELOCITY POTENTIAL FOR UNIFORM FLOW AT INFINITY IN POSITIVE X-DIRECTION WITH COUNTERCLOCKWISE SWIRL OF STRENGTH $\kappa/2\pi$ OVER A PLATE OF LENGTH $2R$ INCLINED AT AN ANGLE α AS SHOWN.

ζ-PLANE $\qquad\qquad$ $Z = \zeta + R^2 e^{-2i\alpha}/\zeta$.



$$\Omega(\zeta) = U_0\left[\zeta + \frac{R^2}{\zeta}\right] + \frac{i\kappa}{2\pi}\log(\zeta).$$

THE BOUNDARY $\zeta = Re^{i\varphi}$ BECOMES

$$Z = Re^{i\varphi} + Re^{-i\varphi}e^{-2i\alpha}$$

$$= Re^{-i\alpha}\left[e^{i(\varphi+\alpha)} + e^{-i(\varphi+\alpha)}\right]$$

$$Z = 2Re^{-i\alpha}\cos(\varphi+\alpha)$$

SINCE $-1 \leq \cos(\varphi+\alpha) \leq 1$ THEN THIS GIVES PLATE

POINT WHERE $\varphi + \alpha = 2\pi$, i.e. $\varphi = -\alpha + 2\pi \rightarrow Z = 2R$

TRAILING EDGE.

• **KUTTA CONDITION** IMPOSE THAT VELOCITY IS ZERO AT TRAILING EDGE

$$\Omega'(\zeta) = U_0\left[1 - R^2/\zeta^2\right] + \frac{i\kappa}{2\pi\zeta} = 0 \quad \text{AT} \quad \zeta = Re^{(2\pi-\alpha)i} = Re^{-i\alpha}$$

$$\rightarrow \quad U_0\left(1 - e^{2i\alpha}\right) + \frac{i\kappa}{2\pi R}e^{i\alpha} = 0 \qquad U_0\left(e^{\pi i\alpha} - e^{-\pi i\alpha}\right) = \frac{i\kappa}{2\pi R}$$

SO $\kappa = 4\pi R U_0 \sin(\pi\alpha)$ IS KUTTA CONDITION.

REMARK CONSIDER A VERTICAL PLATE WITH NO CIRCULATION.

THEN $\alpha = \pi/2$ AND $\Omega(\zeta) = U_0\left[\zeta + R^2/\zeta\right]$ WITH $z = \zeta - R^2/\zeta$.

DEFINE $W = \zeta + R^2/\zeta$. THEN $R^2/\zeta = \zeta - z$ SO $W = 2\zeta - z$.

NOW $\zeta^2 - \zeta z - R^2 = 0 \rightarrow \zeta = \dfrac{z + \sqrt{z^2 + 4R^2}}{2}$.

SO $2\zeta - z = \sqrt{z^2 + 4R^2}$.

WE CONCLUDE THAT

$$\Omega = U_0\sqrt{z^2 + 4R^2}$$

IS THE POTENTIAL FOR A VERTICAL PLATE.