FOURIER TRANSFORMS

A COMPLEX FOURIER SERIES IS

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx .$$

PROOF

$$\int_{-L}^{L} e^{-im\pi x/L} f(x)\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{i(n-m)\pi x/L}\,dx = 2L\,c_m \quad \text{SO THAT}$$

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-im\pi x/L}\,dx.$$

TO DERIVE FOURIER TRANSFORM,

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f(s) e^{-in\pi s/L}\,ds\right) e^{in\pi x/L}$$

LET $k_n = n\pi/L$ TO GET $(*)$ $f(x) = \dfrac{1}{2\pi}\displaystyle\sum_{n=-\infty}^{\infty} \frac{\pi}{L}\hat{F}(k_n) e^{ik_n x}$,

WHERE $\hat{F}(k_n) = \displaystyle\int_{-L}^{L} f(s) e^{-ik_n s}\,ds$. NOW $k_{n+1} - k_n = \pi/L = \Delta k$

NOW THE SUM IN $(*)$ IS THE DISCRETIZATION IN THE K-AXIS FROM $-\infty$ TO ∞. SO

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k\,\hat{F}(k_n) e^{ik_n x} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(k) e^{ikx}\,dk \quad \text{AS} \quad L\to\infty, \Delta k\to 0.$$

THUS WE GET THE TRANSFORM PAIR

$$(+)\ \left\{\begin{aligned} \hat{F}(k) &= \int_{-\infty}^{\infty} f(s) e^{-iks}\,ds = \mathcal{F}(f(x)) \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(k) e^{ikx}\,dk = \mathcal{F}^{-1}[\hat{F}(k)] \end{aligned}\right.$$

FOR $(+)$ TO BE VALID WE NEED

(i) f IS PIECEWISE-SMOOTH

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty$ SQUARE INTEGRABLE FUNCTIONS.

$\hat{F}(k)$

EXAMPLE 1 FIND F.T. OF $f(x) = \begin{cases} 1 & \text{IF } |x| < T \\ 0 & \text{IF } |x| > T \end{cases}$

$$\hat{F}(k) = \int_{-T}^{T} e^{-ikx}\,dx = \frac{1}{-ik} e^{-ikx}\Big|_{-T}^{T} = -\frac{i}{k}\left(e^{ikT} - e^{-ikT}\right) = \frac{2}{k}\sin(kT)$$

EXAMPLE 2 FIND F.T. OF $f(x) = \frac{1}{1+x^2}$.

$$\hat{F}(k) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx$$

· IF $k > 0$ ENCLOSE IN LOWER $1/2$ PLANE,

$\int_{C_R} \to 0$ AS $R \to \infty$

$$\text{SO} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx + \lim_{R\to\infty} \int_{C_R} \frac{e^{-ikz}}{1+z^2}\, dz$$

$$= -2\pi i \,\mathrm{RES}\left[\frac{1}{1+x^2} e^{-ikx}; -i\right] = \pi e^{-k}$$

· IF $k < 0$ ENCLOSE IN UPPER $1/2$ PLANE:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx = 2\pi i \,\mathrm{RES}\left[\frac{e^{-ikx}}{1+x^2}; i\right] = \pi e^{k}$$

THUS $\quad \hat{F}(k) = \pi e^{-|k|}$

PROBLEM 3 FIND F.T. OF $e^{-|x|}$.

$$\hat{F}(k) = \int_0^{\infty} e^{-x} e^{-ikx}\, dx + \int_{-\infty}^{0} e^{x-ikx}\, dx = -\frac{e^{-(x+ikx)}}{1+ik}\Big|_0^{\infty} + \frac{e^{x(1-ik)}}{1-ik}\Big|_{-\infty}^{0}$$

$$\hat{F}(k) = \frac{1}{1+ik} + \frac{1}{1-ik} = \frac{2}{1+k^2}. \longrightarrow \hat{F}(k) = \frac{2}{1+k^2}.$$

EXAMPLE 4 LET $f(x) = N e^{-x^2/2\sigma^2}$.

$$\text{THEN} \quad \hat{F}(k) = N \int_{-\infty}^{\infty} e^{-ikx - x^2/2\sigma^2}\, dx = N \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 + 2\sigma^2 ikx - \sigma^4 k^2)}\, e^{-\sigma^2 k^2/2}\, dx$$

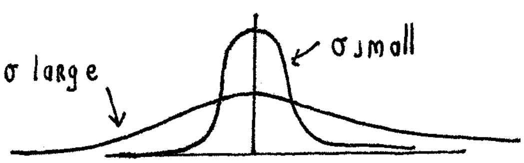
$$\text{THUS,} \quad \hat{F}(k) = N e^{-\sigma^2 k^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x + \sigma^2 ik)^2}\, dx$$

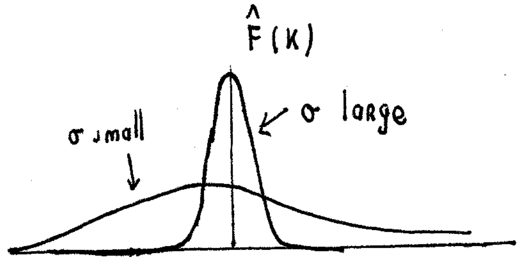
NOW BY INTEGRATING OVER A BOX IT EASILY FOLLOWS THAT $\int_{-\infty}^{\infty} e^{-(x+i\mu)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$.

$$\text{THUS} \quad \hat{F}(k) = N e^{-\sigma^2 k^2/2} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx.$$

RECALL THAT $\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = 1$. THUS, $\hat{F}(k) = \sqrt{2\pi}\, \sigma N e^{-\sigma^2 k^2/2}$

SO $\quad f(x) = N e^{-x^2/2\sigma^2}$





LET $N = \dfrac{1}{\sqrt{2\pi}\,\sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} \qquad \hat{F}(k) = e^{-\sigma^2 k^2/2}$$

NOW WE LET $\sigma \to 0$ THEN $\int_{-\infty}^{\infty} f(x)\,dx = 1$ AND SO

$$f(x) \to \delta(x) \quad \text{AND} \quad \hat{F}(k) = 1$$

SO $\mathcal{F}[\delta(x)] = 1 = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\,dx$,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx.$$

DELTA-SEQUENCE $\quad g_\sigma(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$

NOW $\lim_{\sigma\to 0} g_\sigma(x) = \begin{cases} \infty & \text{IF } x = 0 \\ 0 & \text{IF } x \neq 0 \end{cases}$ AND $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ FOR ALL σ.

THUS $g_\sigma(x) \longrightarrow \delta(x)$ AS $\sigma \to 0$.

FOURIER TRANSFORM PROPERTIES

$$\mathcal{F}(f(x)) = \hat{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx.$$

(i) $\mathcal{F}(f(x+a)) = \int_{-\infty}^{\infty} f(x+a) e^{-ikx}\,dx = e^{ika}\,\mathcal{F}(f(x))$

$$\mathcal{F}(f(x+a)) = e^{ika}\,\mathcal{F}(f(x)).$$

EX: $\mathcal{F}\left[\dfrac{1}{(x+1)^2+1}\right] = e^{ik}\,\mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \pi e^{ik} e^{-|k|}$ SINCE $\mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \pi e^{-|k|}$.

(ii) $\mathcal{F}[f(x/b)] = \int_{-\infty}^{\infty} f(x/b) e^{-ikx}\,dx = b\int_{-\infty}^{\infty} f(s) e^{-i(kb)s}\,ds = b\hat{F}(bk)$.

$b > 0$ HENCE, $\mathcal{F}[f(x/b)] = b\hat{F}(bk)$.

EX: NOW $\mathcal{F}\left[\dfrac{1}{\omega^2+x^2}\right] = \dfrac{1}{\omega^2}\mathcal{F}\left[\dfrac{1}{1+x^2/\omega^2}\right] = \dfrac{1}{\omega^2}\,\omega\,\hat{F}(\omega k) = \dfrac{1}{\omega}\,\pi e^{-\omega|k|}$.

$$\mathcal{F}\left[\frac{1}{\omega^2+x^2}\right] = \frac{\pi}{\omega} e^{-\omega|k|}.$$

(iii) $\mathcal{F}[f'] = \int_{-\infty}^{\infty} f' e^{-ikx}\,dx = f e^{-ikx}\Big|_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx.$

THUS $\mathcal{F}[f^{(n)}(x)] = (ik)\,\mathcal{F}[f^{(n-1)}(x)].$

HENCE IF $f, f', \ldots, f^{(n-1)} \to 0$ AS $|x| \to \infty$

THEN $\mathcal{F}[f^{(n)}(x)] = (ik)^n\,\mathcal{F}[f(x)].$

(iv) CONVOLUTION $(f*g)(x) = \int_{-\infty}^{\infty} f(x-x')g(x')\,dx' = \int_{-\infty}^{\infty} f(x')g(x-x')\,dx'.$

$$\mathcal{F}[f*g] = \int_{-\infty}^{\infty}(f*g)e^{-ikx}\,dx = \int_{-\infty}^{\infty} dx\left(\int_{-\infty}^{\infty} f(x-x')g(x')\,dx'\right)e^{-ikx}$$
$$= \int_{-\infty}^{\infty} g(x')e^{-ikx'}\left(\int_{-\infty}^{\infty} f(s)e^{-iks}\,ds\right)dx' = \left(\int_{-\infty}^{\infty} g(x')e^{-ikx'}\,dx'\right)\left(\int_{-\infty}^{\infty} f(s)e^{-iks}\,ds\right).$$

$$\mathcal{F}[f*g] = \hat{G}(k)\,\hat{F}(k)$$

EX NOW $\hat{G}(k) = \pi e^{-|k|}$, $\hat{F}(k) = \frac{2}{1+k^2}$. SO TO INVERT

$$\hat{H}(k) = \frac{2\pi\, e^{-|k|}}{1+k^2}$$

THEN $g(x) = \frac{1}{1+x^2}$, $f(x) = e^{-|x|}$ SO THAT

WE GET $h(x) = \int_{-\infty}^{\infty} f(x-x')g(x')\,dx'.$

(v) PARSEVAL'S IDENTITY

$$\int_{-\infty}^{\infty} |f(x')|^2\,dx' = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(k)|^2\,dk.$$

$\int_{-\infty}^{\infty} f(x-x')g(x')\,dx' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(k)\hat{G}(k)e^{ikx}\,dx.$

(AND SET $x=0$)

(vi) COSINE TRANSFORM ($f(x)$ EVEN)

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \int_{-\infty}^{0} f(x)e^{-ikx}\,dx + \int_{0}^{\infty} f(x)e^{-ikx}\,dx.$$

LET $x \mapsto -x$.

THEN, $\hat{F}(k) = \int_{0}^{\infty} f(-x)e^{ikx}\,dx + \int_{0}^{\infty} f(x)e^{-ikx}\,dx.$

NOW IF $f(x) = f(-x)$ SO THAT f IS EVEN, THEN

$$\hat{F}(k) = 2\int_{0}^{\infty} f(x)\cos(kx)\,dx \longrightarrow \hat{F}(k) = \hat{F}(-k).$$

SIMILARLY $f(x) = \frac{1}{\pi}\int_{0}^{\infty}\hat{F}(k)\cos(kx)\,dk.$

(V) SINE TRANSFORM $f(x)$ ODD

NOW IF $f(x) = -f(-x)$ THEN

$$(*)\left\{\begin{aligned} \hat{F}(k) &= 2\int_0^\infty f(x)\sin(kx)\,dx \\ f(x) &= \frac{1}{\pi}\int_0^\infty \hat{F}(k)\sin(kx)\,dk \end{aligned}\right.$$

APPLICATIONS

EXAMPLE 1 $\quad u'' - \omega^2 u = -f(x) \quad -\infty < x < \infty$

$$u(\pm\infty) = 0 \qquad \omega > 0$$

$$\mathcal{F}(u'') - \omega^2\,\mathcal{F}(u) = -\mathcal{F}(f)$$

$$(-k^2 - \omega^2)\,\mathcal{F}(u) = -\mathcal{F}(f) \quad \longrightarrow \quad U(k) = \frac{\hat{F}(k)}{k^2+\omega^2} \qquad \hat{F}(k) = \mathcal{F}(f).$$

NOW USE CONVOLUTION:

$$\mathcal{F}^{-1}\left[\frac{1}{k^2+\omega^2}\right] = \frac{1}{\omega^2}\,\mathcal{F}^{-1}\left[\frac{1}{1+k^2/\omega^2}\right] = \frac{1}{\omega}\,\frac{\omega}{2\omega}\,e^{-|x|\omega} = \frac{1}{2\omega}\,e^{-|x|\omega}$$

THUS $\quad u(x) = \dfrac{1}{2\omega}\displaystyle\int_{-\infty}^{\infty} e^{-\omega|x-\xi|} f(\xi)\,d\xi.$

EXAMPLE 2 $\quad u_t = D\,u_{xx} \quad -\infty < x < \infty,\ t > 0$

$$u(x,0) = f(x)$$

NOW TAKE F.T. IN x: DEFINE

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)\,e^{-ikx}\,dk,$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)\,e^{ikx}\,dk$$

TAKE F.T.

$$\left.\begin{aligned} U_t &= -Dk^2 U \\ U(k,0) &= \hat{F}(k) \end{aligned}\right\} \longrightarrow U(k,t) = \hat{F}(k)\,e^{-Dk^2 t}.$$

NOW $\hat{F}(k) = \mathcal{F}(f(x))$. THEN $\quad \mathcal{F}^{-1}\left[e^{-k^2\sigma^2/2}\right] = \dfrac{1}{\sqrt{2\pi}\,\sigma}\,e^{-x^2/2\sigma^2}$ WITH $\dfrac{\sigma^2}{2} = Dt.$

$$\mathcal{F}^{-1}\left[e^{-k^2 Dt}\right] = \frac{1}{2\sqrt{\pi Dt}}\,e^{-x^2/4Dt}. \quad \text{THEN,}$$

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4Dt} f(\xi)\,d\xi.$$

EXAMPLE 3 $\quad U_{xx} + U_{yy} = 0 \quad -\infty < x < \infty,\ y > 0$

$$U(x,0) = h(x).$$

THEN, $$\bar{U}(k,y) = \int_{-\infty}^{\infty} U(x,y)\, e^{-ikx}\, dx.$$

THIS GIVES, $$\bar{U}_{yy} - k^2 \bar{U} = 0 \qquad \bar{U} \to 0 \text{ AS } y \to +\infty.$$
$$\bar{U}(k,0) = \hat{H}(k)$$

NOW WE GET $$\bar{U}(k,y) = \hat{H}(k)\, e^{-|k|y}.$$

USE THE CONVOLUTION, $$\mathcal{F}^{-1}\left[e^{-|k|y} \right] = \frac{y}{\pi(x^2+y^2)}$$

NOW $$U(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{[(x-\xi)^2+y^2]}\, h(\xi)\, d\xi.$$

NOW LET, $$\lim_{y\to 0} \frac{y}{\pi[(x-\xi)^2+y^2]} = \delta(x-\xi) \longrightarrow U(x,0) = h(x).$$

EXAMPLE 4 $\quad U_t = U_{xx} \quad 0 < x < \infty,\ t > 0$

$$U(x,0) = 0, \quad U(0,t) = h(t) \qquad U, U_x \to 0 \text{ as } x \to \infty$$

SINCE $U(\ ,t)$ IS GIVEN WE TRY A SINE TRANSFORM:

$$\bar{U}(k,t) = 2\int_0^{\infty} U(x,t)\sin(kx)\, dx \qquad U(x,t) = \frac{1}{\pi}\int_0^{\infty} \bar{U}(k,t)\sin(kx)\, dk.$$

SO $$2\int_0^{\infty} U_t \sin(kx)\, dx = 2\int_0^{\infty} U_{xx}\sin(kx)\, dx = 2\left[U_x \sin(kx)\Big|_0^{\infty} - k\int_0^{\infty} U_x \cos kx\, dx \right]$$

BUT $\sin kx = 0$ AT $x = 0$ AND $U_x \to 0$ AT ∞. INTEGRATE AGAIN

$$\frac{d}{dt}\, 2\int_0^{\infty} U \sin(kx)\, dx = -2k\left[U\cos(kx)\Big|_0^{\infty} + k\int_0^{\infty} U\sin(kx)\, dx \right]$$
$$= 2k\, h(t) - k^2 \left(2\int_0^{\infty} U \sin kx\, dx \right)$$

THUS $$\bar{U}_t = -k^2 \bar{U} + 2k\, h(t) \qquad \bar{U}(k,0) = 0$$

NOW $$\bar{U}_t + k^2 \bar{U} = 2k\, h(t)$$
$$\left(e^{k^2 t} \bar{U} \right)_t = 2k\, h(t)\, e^{k^2 t}.$$

INTEGRATE TO GET $e^{k^2 t}\, \bar{U} = 2k \int_0^t h(\tau) e^{k^2\tau}\, d\tau$,

SO THAT $\bar{U}(k,t) = 2k\, e^{-k^2 t} \int_0^t h(\tau)\, e^{k^2\tau}\, d\tau$. NOW USE THE INVERSE TRANSFORM TO GET AFTER INTERCHANGING ORDERS OF INTEGRATION:

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \sin(kx) \left(\int_0^t k\, h(\tau) e^{-k^2(t-\tau)}\, d\tau \right) dk$$

SO THAT

$$\left.\begin{aligned} u(x,t) &= \frac{2}{\pi} \int_0^t h(\tau)\, J(x, t-\tau)\, d\tau \\ \text{WHERE} \quad J(x, t-\tau) &= \int_0^\infty k \sin(kx)\, e^{-k^2(t-\tau)}\, dk, \end{aligned}\right\} (*)$$

WITH $\tau < t$.

TO EVALUATE THE INTEGRAL OBSERVE THAT $J = -\partial I / \partial x$

WHERE $$I = \int_0^\infty \cos(kx)\, e^{-k^2(t-\tau)}\, dk.$$

TO EVALUATE I WE WRITE $I = \frac{1}{2} \int_{-\infty}^\infty e^{ikx} e^{-k^2(t-\tau)}\, dk$ AND COMPLETE THE SQUARE:

$$I = \frac{1}{2} \int_{-\infty}^\infty e^{-(t-\tau)\left(k^2 - ikx/(t-\tau) - x^2/4(t-\tau)^2\right)} e^{-x^2/4(t-\tau)}\, dk$$

$$= \frac{1}{2} e^{-x^2/4(t-\tau)} \int_{-\infty}^\infty e^{-(t-\tau)\left(k - ix/2(t-\tau)\right)^2}\, dk$$

BY SHIFTING PATH IN k-PLANE $\longrightarrow$ $I = \frac{1}{2} e^{-x^2/4(t-\tau)} \int_{-\infty}^\infty e^{-(t-\tau)k^2}\, dk$.

NOW RECALL $\int_{-\infty}^\infty e^{-k^2/2\sigma^2}\, dk = \sqrt{2\pi}\, \sigma$ SO IF $2\sigma^2 = 1/(t-\tau) \rightarrow \sigma = \frac{1}{\sqrt{2(t-\tau)}}$

WE HAVE $\int_{-\infty}^\infty e^{-(t-\tau)k^2}\, dk = \sqrt{2\pi} / \sqrt{2(t-\tau)} = \sqrt{\pi/t-\tau}$. THIS GIVES

$I = \frac{1}{2} e^{-x^2/4(t-\tau)} \sqrt{\frac{\pi}{t-\tau}}$. NOW $J = -\frac{\partial I}{\partial x} = \frac{x}{4(t-\tau)^{3/2}} \sqrt{\pi}\, e^{-x^2/4(t-\tau)}$

PUTTING THIS INTO (*) GIVES

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{h(\tau)}{(t-\tau)^{3/2}} e^{-x^2/4(t-\tau)}\, d\tau.$$

REMARK TO EVALUATE I WITH OUR GAUSSIAN INTEGRAL WE COULD FIRST WRITE

$$I = \frac{2\pi}{2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} e^{-k^2(t-\tau)}\,dk\right) = \pi\,\hat{f}^{-1}\left[e^{-k^2(t-\tau)}\right].$$

RECALLING THAT $\hat{f}^{-1}\left[e^{-\sigma^2 k^2/2}\right] = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$ WE SET $\frac{\sigma^2}{2} = t-\tau$, TO GET $\sigma = \sqrt{2(t-\tau)}$. THEN

$$I = \pi\left[\frac{1}{\sqrt{2\pi}\sqrt{2(t-\tau)}} e^{-x^2/4(t-\tau)}\right] = \frac{\sqrt{\pi}}{2(t-\tau)^{1/2}} e^{-x^2/4(t-\tau)} \quad \square.$$

EXAMPLE 5 USING A FOURIER SINE TRANSFORM SOLVE

$$u_t = D u_{xx}, \quad 0 < x < \infty, \ t > 0$$

WITH $u(0,t) = 0$, $t > 0$

$u(x,0) = e^{-x}$ $\quad u, u_x \to 0$ AS $x \to +\infty$, FOR t FIXED.

SOLUTION DEFINE $\bar{u}(k,t) = 2\int_0^{\infty} u(x,t)\sin(kx)\,dx$ AND $u(x,t) = \frac{1}{\pi}\int_0^{\infty}\bar{u}(k,t)\sin(kx)\,dk$.

WE HAVE

$$2\int_0^{\infty} u_t \sin(kx)\,dx = 2D\int_0^{\infty} u_{xx}\sin(kx)\,dx = D\,2\left[u_x \sin(kx)\Big|_0^{\infty} - k\int_0^{\infty} u_x \cos(kx)\,dx\right]$$

$$= -2D\int_0^{\infty} k u_x \cos(kx)\,dx = -2D\left[k u \cos(kx)\Big|_0^{\infty} + k^2\int_0^{\infty} u \sin(kx)\,dx\right]$$

$$\frac{d}{dt}\int_0^{\infty} 2u\sin(kx)\,dx = -2D\int_0^{\infty} u\sin(kx)\,dx$$

$$\rightarrow \qquad \frac{d\bar{u}}{dt} = -Dk^2\bar{u}, \quad t > 0.$$ FIND INITIAL CONDITION.

NOW $$\bar{u}(k,0) = 2\int_0^{\infty} e^{-x}\sin(kx)\,dx = 2\,\mathrm{Im}\left[\int_0^{\infty} e^{-x+ikx}\,dx\right]$$

SO $$\bar{u}(k,0) = 2\,\mathrm{Im}\left[\frac{1}{-1+ik} e^{-x+ikx}\Big|_0^{\infty}\right] = 2\,\mathrm{Im}\left[\frac{1}{(1-ik)}\right]$$

SO $$\bar{u}(k,0) = 2\,\mathrm{Im}\left((1+ik)/_{1+k^2}\right) = 2k/_{1+k^2}.$$

SO THE SOLUTION TO $\dfrac{d\bar{U}}{dt} = -DK^2\bar{U}$ WITH $\bar{U}(K,0) = \dfrac{2K}{1+K^2}$

GIVES $$\bar{U}(K,t) = \frac{2K}{1+K^2} e^{-DK^2t}.$$

THEN FROM INVERSE TRANSFORM

$$U(X,t) = \frac{1}{\pi}\int_0^\infty \bar{U}(K,t)\sin(KX)\,dK = \frac{2}{\pi}\int_0^\infty \frac{K}{1+K^2} e^{-DK^2t}\sin(KX)\,dK.$$

<u>EXAMPLE 6</u> USING A FOURIER COSINE SERIES TO SOLVE

$U_t = DU_{XX}$, $0 < X < \infty$, $t > 0$

$U_X = -Q$ ON $X=0$ FOR $t>0$ WITH Q CONSTANT

$U = 0$ AT $t=0$ FOR $X \geq 0$; $U, U_X \to 0$ AS $X \to \infty$, t FIXED.

<u>SOLUTION</u> WE USE FOURIER COSINE TRANSFORM

$$\bar{U}(K,t) = 2\int_0^\infty U(X,t)\cos(KX)\,dX, \quad U(X,t) = \frac{1}{\pi}\int_0^\infty \bar{U}(K,t)\cos(KX)\,dK.$$

WE CALCULATE:

$$2\int_0^\infty U_t\cos(KX)\,dX = 2D\int_0^\infty U_{XX}\cos(KX)\,dX = 2D\left[U_X\cos(KX)\Big|_0^\infty + K\int_0^\infty U_X\sin(KX)\,dX\right]$$

$$\frac{d}{dt}\left(\int_0^\infty 2U\cos(KX)\,dX\right) = 2D\left[-U_X(0,t) + K\left[U\sin(KX)\Big|_0^\infty - K\int_0^\infty U\cos(KX)\,dX\right]\right]$$

$$\to (*) \begin{cases} \dfrac{d\bar{U}}{dt} = 2DQ - DK^2\bar{U}. \\ \bar{U}(K,0) = 0 \quad \text{SINCE} \quad U(X,0) = 0 \end{cases}$$

PARTICULAR SOLUTION IS $\bar{U}_p = \dfrac{2DQ}{DK^2} = \dfrac{2Q}{K^2}$.

THE SOLUTION TO THE ODE IS $\bar{U} = \dfrac{2Q}{K^2} - \dfrac{2Q}{K^2}e^{-DK^2t}$.

THEREFORE, $$U(X,t) = \frac{2Q}{\pi}\int_0^\infty \frac{1}{K^2}\left(1 - e^{-DK^2t}\right)\cos(KX)\,dK.$$

DISPERSION RELATION

(i) $u_t = D u_{xx} \quad -\infty < x < \infty,\ t > 0; \quad u(x,0) = f(x)$

let $u = e^{ikx + \sigma t}$

substitute to get $\sigma = -Dk^2. \longrightarrow u = e^{ikx - Dk^2 t}$

NOW MULTIPLY BY $A(k)$ AND integrate $-\infty$ TO ∞ IN $k \longrightarrow$ superposition.

THUS GIVES $$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - Dk^2 t}\, dk$$

NOW $u(x,0) = f(x) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \qquad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\, dx'$

THUS $$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left(\int_{-\infty}^{\infty} e^{ik(x-x') - Dk^2 t}\, dk\right) dx'$$

(ii) $u_t - c\,u_x = u_{xxx} \quad -\infty < x < \infty,\ t > 0$

$u(x,0) = f(x)$

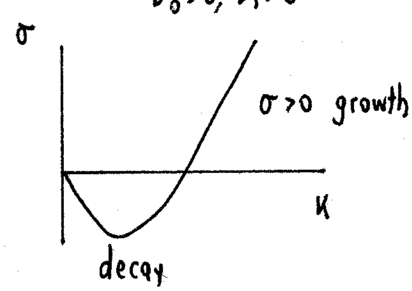
let $u(x,t) = e^{ikx + \sigma t} \longrightarrow$ SUBSTITUTE TO GET $\sigma - cik = (ik)^3$.

THUS $\sigma = cik - ik^3. \longrightarrow$ wave propagation since σ is imaginary

THEN $$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ik(x+c) - ik^3 t}\, dk$$

NOW $u(x,0) = f(x)$ so $f(x) \Longrightarrow \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$

$D_0 > 0,\ D_1 > 0$



(iii) $u_t = D_0 u_{xxxx} + D_1 u_{xx} \quad -\infty < x < \infty$

let $u = e^{ikx + \sigma t} \longrightarrow \sigma = D_0 k^4 - D_1 k^2$

so IF k small, large wavelength $\longrightarrow$ decay.

IF k large, small wavelength $\longrightarrow$ growth.

(iv) $u_{tt} - c^2 u_{xx} = 0 \quad$ NOW $u = e^{ikx + \sigma t} \longrightarrow \sigma^2 = -c^2 k^2. \quad \sigma = \pm ick.$

so $u(x,t) = \int_{-\infty}^{\infty} A_1(k)\, e^{ik(x+ct)}\, dk + \int_{-\infty}^{\infty} A_2(k)\, e^{ik(x-ct)}\, dk$ wave propagation.

MULTIPLE TRANSFORMS

$$\hat{F}(K_1, K_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{-iK_1 x - iK_2 y}\, dx\, dy$$

$$F(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{F}(K_1, K_2)\, e^{iK_1 x + iK_2 y}\, dK_1\, dK_2 .$$

EXAMPLE $\qquad U_{xx} + U_{yy} - m^2 U = f(x,y)$

$$U \rightarrow 0 \text{ AS } x^2 + y^2 \rightarrow \infty$$

NOW TAKE F.T.

$$\left[+(iK_1)^2 + (iK_2)^2\right]\bar{U} - m^2 \bar{U} = \hat{F}(K_1, K_2)$$

THUS $\qquad \bar{U}(K_1, K_2) = -\dfrac{\hat{F}(K_1, K_2)}{K_1^2 + K_2^2 + m^2}$

NOW $\qquad U(x,y) = -\dfrac{1}{4\pi^2}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{F}(K_1, K_2)}{K_1^2 + K_2^2 + m^2}\, e^{iK_1 x + iK_2 y}\, dK_1\, dK_2$

BUT NOW $\qquad \hat{F}(K_1, K_2) = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x', y')\, e^{-iK_1 x' - iK_2 y'}\, dx'\, dy'.$

THEN $\qquad U(x,y) = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, G(x - x', y - y')\, dx'\, dy'$

$$G(x,y) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{iK_1 x + iK_2 y}}{K_1^2 + K_2^2 + m^2}\, dK_1\, dK_2$$

NOW $\qquad G(x,y) = -\dfrac{1}{4\pi}\displaystyle\int_{-\infty}^{\infty} \frac{e^{iK_2 y - \sqrt{K_2^2 + m^2}\,|x|}}{\sqrt{K_2^2 + m^2}}\, dK_2$

SINCE $\qquad \displaystyle\int_{-\infty}^{\infty} \frac{e^{iK_1 x + iK_2 y}}{K_1^2 + (K_2^2 + m^2)}\, dK_1 = \frac{e^{iK_2 y - \sqrt{K_2^2 + m^2}\,|x|}}{2i\,(K_2^2 + m^2)^{1/2}}$ BY RESIDUE CALCULUS.

WELL-POSEDNESS

CONSIDER $\quad U_{xx} - \frac{1}{c^2} U_{tt} = F(x,t) = e^{i\omega t} f(x). \quad -\infty < x < \infty, \; -\infty < t < \infty.$

LET $\quad U = e^{i\omega t}\,\Phi(x)$. THEN,

$$\Phi_{xx} + \frac{\omega^2}{c^2}\Phi = f(x)$$

NOW TAKE F.T. $\quad \hat{\Phi}(k) = \mathcal{F}[\Phi(x)]$. THEN,

$$-k^2\hat{\Phi} + \frac{\omega^2}{c^2}\hat{\Phi} = \hat{F} \longrightarrow \hat{\Phi} = \frac{-\hat{F}(k)}{k^2 - \omega^2/c^2}$$

THEN $\quad \Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{F}(k)}{\omega^2/c^2 - k^2}\, e^{ikx}\,dk. \qquad$ POLES ALONG REAL AXIS IN k-plane.

NON-UNIQUENESS SINCE WE HAVE POLES ALONG THE REAL AXIS: WHICH CONTOUR DO WE TAKE. PROBLEM IS NEED OUTGOING CONDITION AT ∞.

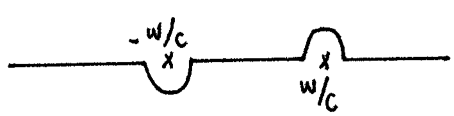
SUPPOSE WE WRITE

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(s)\, e^{-iks}\,ds.$$

THEN, $\quad \Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, H[x-s]\,ds \quad$ WHERE $H(y) \equiv -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iky}}{k^2 - \omega^2/c^2}\,dy.$

HOW DO WE INTERPRET $H(y)$? A PRINCIPAL VALUE INTEGRAL? NO! IT COMES FROM AN EXTRA CONDITION. NOW TO GET OUTGOING WAVE WE WANT $\quad H(y) = C\,e^{-i\omega/c\,|y|}$. THEN WITH $U = e^{i\omega t}\Phi$ WE GET OUTGOING WAVE. TAKE CONTOUR TO CALCULATE $H(y)$ AS SHOWN:

k-plane



$H(y) = 2\pi i \,\text{RES}\left[-\frac{1}{2\pi}\frac{e^{iky}}{k^2 - \omega^2/c^2}; -\omega/c\right] \qquad y > 0$ enclose in upper ½ plane

$H(y) = -2\pi i \,\text{RES}\left[\frac{-1}{2\pi}\frac{e^{iky}}{k^2 - \omega^2/c^2}; +\omega/c\right] \qquad y < 0$ lower ½ plane enclose.

THUS $\quad H(y) = \frac{ic}{2\omega}\, e^{-i\omega/c\,|y|}$. THEN

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\frac{ic}{2\omega}\, e^{-\frac{i\omega}{c}|x-s|}\,ds$$

FINALLY $\quad U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\frac{ic}{2\omega}\, e^{i\omega t - \frac{i\omega}{c}|x-s|}\,ds$

TO GET AN OUTGOING WAVE WE IMPOSED A PARTICULAR PATH AROUND THE POLES AND DID NOT DO A PRINCIPAL VALUE INTEGRAL.

ANOTHER APPROACH IS TO MODIFY THE PROBLEM TO ALLOW A SMALL AMOUNT OF DAMPING

$$u_{xx} - \frac{1}{c^2} u_{tt} - \varepsilon u_t = e^{i\omega t} f(x) \qquad \varepsilon < 0$$

FOR ANY $\varepsilon > 0 \Longrightarrow u \to 0$ AS $|x| \to \infty$. SO WE WILL NOT HAVE POLES ON REAL AXIS IN F.T. PLANE.

LET $\quad u = e^{i\omega t} \Phi(x) \longrightarrow \Phi_{xx} + \frac{\omega^2}{c^2}\Phi - \varepsilon i \omega \Phi = f(x)$

THEN, $\quad -k^2 \hat{\Phi} + \frac{\omega^2}{c^2}\hat{\Phi} - i\varepsilon\omega\hat{\Phi} = \hat{F}(k)$

THEN $\quad \hat{\Phi}(k) = \dfrac{\hat{F}(k)}{-k^2 + \frac{\omega^2}{c^2} - i\varepsilon\omega} . \qquad \Phi(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{\Phi}(k) e^{ikx}\, dk$

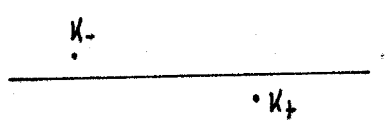
THIS LEADS TO

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\, H[x-\xi]\, d\xi$$

WITH $\quad H(y) = -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{e^{iky}}{k^2 - \omega^2/c^2 + i\varepsilon\omega}\, dk$

poles are at $\quad k^2 = \dfrac{\omega^2}{c^2}\left(1 - \dfrac{i\varepsilon c^2}{\omega}\right) \qquad k_\pm = \pm\dfrac{\omega}{c}\left(1 - \dfrac{i\varepsilon c^2}{2\omega}\right)$

so $\quad k_+ = \dfrac{\omega}{c} - \dfrac{i\varepsilon c}{2} + \ldots \qquad k_- = -\dfrac{\omega}{c} + \dfrac{i\varepsilon c}{2} + \ldots$



RE$(k_+) < 0$ RE$(k_-) > 0$. THUS FOR $\varepsilon > 0$ THE POLES ARE AS SHOWN.

IF WE DO THE RESIDUE CALCULATION WE WILL GET SAME RESULT WHEN $\varepsilon \to 0$ AS ON PREVIOUS PAGE.

PROBLEM (VIBRATING ELASTIC BEAM)

FOR A BEAM THE DEFLECTION $u(x,t)$ SATISFIES

$$\begin{cases} u_{tt} = -a^2 u_{xxxx}, & -\infty < x < \infty,\ t > 0 \\ u \to 0 \text{ AS } |x| \to \infty & (a^2 > 0) \\ u(x,0) = f(x), \quad u_t(x,0) = 0. \end{cases}$$

DETERMINE AN INTEGRAL REPRESENTATION OF THE SOLUTION WHEN $f(x) = e^{-\sigma x^2}$, $\sigma > 0$.

SOLUTION LET $U(k,t) = \mathcal{F}(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dk.$

THEN $$U_{tt} = -a^2 k^4 U$$

$$U(k,0) = F(k), \quad U_t(k,0) = 0.$$

THUS $$U(k,t) = A\cos(atk^2) + B\sin(atk^2).$$

BUT $U_t(k,0) = 0 \to B = 0$; $U(k,0) = F(k) = \mathcal{F}(f(x)) = A.$

THUS, $$U(k,t) = F(k)\cos(atk^2) \qquad F(k) = \mathcal{F}(e^{-\sigma x^2}) = \sqrt{\frac{\pi}{\sigma}}\, e^{-k^2/4\sigma}$$

NOW $$U(k,t) = \sqrt{\frac{\pi}{\sigma}}\, e^{-k^2/4\sigma}\cos(atk^2).$$

INVERTING THE TRANSFORM WE OBTAIN

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk = \frac{1}{2\sqrt{\pi\sigma}}\int_{-\infty}^{\infty} e^{-k^2/4\sigma}\cos(atk^2)\,e^{ikx}\,dk.$$

BY WRITING $e^{ikx} = \cos(kx) + i\sin(kx)$, THE INTEGRAL WITH $\sin(kx)$ VANISHES SINCE IT IS ODD.

HENCE, $$u(x,t) = \frac{1}{2\sqrt{\pi\sigma}}\int_{-\infty}^{\infty} e^{-k^2/4\sigma}\cos(atk^2)\cos(kx)\,dk.$$

PARSEVAL'S RELATION LET $F(x)$ BE COMPLEX-VALUED AND DEFINE $|F|^2 = F(x)\overline{F(x)}$. THEN IF $\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty$ WE HAVE

$$(1) \quad \left\{ \int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(k)|^2 dk. \right. \quad \text{WHERE} \quad \hat{F}(k) = \mathcal{F}(F(x)).$$

PROOF $\int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} F(x)\overline{F}(x)dx.$ BUT $\overline{F(x)} = \overline{\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ikx}dk}$

SO $\int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} \frac{F(x)}{2\pi}\left(\int_{-\infty}^{\infty} \overline{\hat{F}(k)}\, e^{-ikx}dk\right)dx$ SINCE $\overline{e^{ikx}} = e^{-ikx}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{F}(k)}\Big(\underbrace{\int_{-\infty}^{\infty} F(x)e^{-ikx}dx}_{=\hat{F}(k)}\Big)dk$$

(SWAPPING INTEGRATION ORDERS)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{F}(k)}\,\hat{F}(k)\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(k)|^2 dk \qquad \square.$$

NEXT, WE WILL DERIVE AN INEQUALITY THAT IS RELATED TO HEISENBERG'S UNCERTAINTY PRINCIPLE IN QUANTUM MECHANICS.

IN QUANTUM MECHANICS, x IS POSITION AND k IS MOMENTUM. FOR A WAVEFUNCTION $F(x) \in \mathbb{C}$, NORMALIZED BY $\int_{-\infty}^{\infty} |F(x)|^2 dx = 1$, THE EXPECTED VALUE OF THE SQUARE OF THE POSITION IS

$$\bar{x}^2 \equiv \int_{-\infty}^{\infty} x^2 |F(x)|^2 dx \Big/ \int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} x^2 |F(x)|^2 dx.$$

SIMILARLY, THE EXPECTED VALUE OF THE SQUARE OF THE MOMENTUM IS DEFINED BY

$$\bar{k}^2 \equiv \int_{-\infty}^{\infty} k^2 |\hat{F}(k)|^2 dk \Big/ 2\pi$$

SINCE WE KNOW BY PARSEVAL'S RELATION THAT $1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(k)|^2 dk = \int_{-\infty}^{\infty} |F(x)|^2 dx$

THE CLAIM IS THAT WE CANNOT MAKE PRECISE MEASUREMENTS IN SPACE AND MOMENTUM SIMULTANEOUSLY IN THE SENSE THAT WE ALWAYS HAVE

$$\bar{X}\,\bar{K} \geq 1/2 \qquad (2)$$

EQUATION (2) IS CALLED THE HEISENBERG PRINCIPLE WHERE WE SET $\hbar = 1$ (QUANTUM). TO PROVE (*) WE NEED A FEW FACTS.

(I) $\int_{-\infty}^{\infty} |g(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(k)|^2\,dk$ WHERE $\hat{G}(k) = \mathcal{F}[g(x)]$. (PARSEVAL)

(II) $\left|\int_{-\infty}^{\infty} g(x)\,h(x)\,dx\right| \leq \left(\int_{-\infty}^{\infty} |g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty} |h(x)|^2\,dx\right)^{1/2}$ SCHWARTZ INEQUALITY

(III) $|z + \bar{z}| \leq 2|z|$ BY Δ-INEQUALITY. (IV) $\left|\int_{-\infty}^{\infty} f\,dx\right| \leq \int_{-\infty}^{\infty} |f|\,dx$.

WE NOW PROVE (2). WE INTEGRATE BY PARTS ON NORMALIZATION CONDITION TO GET

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = -\int_{-\infty}^{\infty} x\frac{d}{dx}\left(|\psi|^2\right)dx, \text{ WHERE WE SET } \psi(x) = f(x).$$

NOW $|\psi|^2 = \psi\bar{\psi}$ SO BY PRODUCT RULE

$$\frac{d}{dx}|\psi|^2 = \psi\bar{\psi}' + \psi'\bar{\psi} \longrightarrow \left|\frac{d}{dx}|\psi^2|\right| \leq |\psi\bar{\psi}' + \psi'\bar{\psi}| \leq |\psi||\psi'| + |\psi'||\psi|$$

SO $$1 \leq \int_{-\infty}^{\infty} \left|x\frac{d}{dx}(\psi^2)\right|dx \leq 2\int_{-\infty}^{\infty} |x|\,|\psi(x)|\,|\psi'(x)|\,dx$$

SO $$1 \leq 2\int_{-\infty}^{\infty} |\psi'|\,|x\psi|\,dx \leq 2\left(\int_{-\infty}^{\infty} |x\psi|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty} |\psi'|^2\,dx\right)^{1/2}$$

BY SCHWARTZ INEQUALITY. BUT $\int_{-\infty}^{\infty} |x\psi|^2\,dx = \bar{X}^2$ AND SINCE $\mathcal{F}(\psi') = ik\hat{F}(k)$ WHERE $\hat{F}(k) = \mathcal{F}(\psi)$ WE HAVE

$$1 \leq 2\left(\bar{X}^2\right)^{1/2}\left(\int_{-\infty}^{\infty} |ik\hat{F}(k)|^2\,dk\right)^{1/2} = 2\,\bar{X}\,\bar{K}.$$

HENCE $$\bar{X}\,\bar{K} \geq 1/2. \quad \square$$

POISSON SUMMATION FORMULA

LET $\hat{F}(\nu) = \int_0^\infty f(x) e^{-i\nu x}\,dx$. THEN

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{F}(2n\pi) \quad \square$$

PROOF LET $G(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. WE CLAIM THAT THE MINIMAL PERIOD OF $G(x)$ IS 1. TO PROVE THIS $G(x+1) = \sum_{n=-\infty}^{\infty} f(x+1+n)$. LET $n' = n+1$ SO THAT $G(x+1) = \sum_{n'=-\infty}^{\infty} f(x+n') = G(x)$. NOW SINCE $G(x)$ HAS PERIOD ONE, WE HAVE THAT

$$G(x) = \sum_{m=-\infty}^{\infty} C_m e^{2m\pi i x} \quad \text{so} \quad C_m = \int_0^1 G(t) e^{-2m\pi i t}\,dt.$$

THIS GIVES,

$$G(x) = \sum_{m=-\infty}^{\infty} \left(\int_0^1 G(t) e^{-i\nu_m t}\,dt \right) e^{2m\pi i x} \qquad \nu_m = 2m\pi.$$

$$= \sum_{m=-\infty}^{\infty} \left(\int_0^1 \sum_{n=-\infty}^{\infty} f(t+n) e^{-i\nu_m t}\,dt \right) e^{2m\pi i x}$$

NOW $I = \sum_{n=-\infty}^{\infty} \int_0^1 f(t+n) e^{-i\nu_m t}\,dt = \sum_{n=-\infty}^{\infty} \left(\int_n^{n+1} f(s) e^{-i\nu_m s}\,ds \right)$ SINCE $e^{-i\nu_m n} = 1$.

let $s = t+n$ to get $I = \hat{F}(\nu_m)$.

THUS,

$$G(x) = \sum_{m=-\infty}^{\infty} \hat{F}(\nu_m) e^{2m\pi i x}, \quad \text{WITH } \nu_m = 2m\pi.$$

THIS YIELDS THAT

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{m=-\infty}^{\infty} \hat{F}(\nu_m) e^{2m\pi i x}.$$

FINALLY, SET $x = 0$ TO OBTAIN

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \hat{F}(2m\pi) \quad \square.$$

<u>EXAMPLE</u> USE POISSON SUMMATION FORMULA TO PROVE THAT

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2+n^2} = \frac{\pi}{\lambda} \coth(\pi\lambda)$$

<u>PROOF</u> DEFINE $f(x) = \frac{1}{\lambda^2+x^2}$ THEN $\hat{F}(k) = \hat{f}(f(x)) = \frac{\pi}{\lambda} e^{-|k|\lambda}$.

BY POISSON SUMMATION FORMULA,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{F}(2n\pi) = \frac{\pi}{\lambda} \sum_{n=-\infty}^{\infty} e^{-2\pi\lambda|n|}$$

$$= \frac{\pi}{\lambda}\left[1 + 2\sum_{n=1}^{\infty} e^{-Bn}\right] \qquad B = 2\pi\lambda$$

$$= \frac{\pi}{\lambda}\left[1 + \frac{2e^{-B}}{1-e^{-B}}\right] \qquad \text{SINCE } \sum_{n=1}^{\infty}(e^{-B})^n = \frac{e^{-B}}{1-e^{-B}}.$$

$$= \frac{\pi}{\lambda}\left[\frac{1+e^{-B}}{1-e^{-B}}\right] = \frac{\pi}{\lambda}\left[\frac{e^{B/2}+e^{-B/2}}{e^{B/2}-e^{-B/2}}\right] = \frac{\pi}{\lambda}\coth(B/2)$$

WE LET $B = 2\pi\lambda$ TO CONCLUDE

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2+n^2} = \frac{\pi}{\lambda}\coth(\pi\lambda).$$

<u>REMARK</u> WE CAN WRITE

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2+n^2} = \frac{1}{\lambda^2} + 2\sum_{n=1}^{\infty}\frac{1}{\lambda^2+n^2}.$$

NOW $\coth(z) \sim 1/z + z/3 \cdots$ SO THAT

$$2\sum_{n=1}^{\infty}\frac{1}{\lambda^2+n^2} = -\frac{1}{\lambda^2} + \frac{\pi}{\lambda}\coth(\pi\lambda)$$

NOW let $\lambda \to 0$, $\quad 2\sum_{n=1}^{\infty} 1/n^2 = \lim_{\lambda\to 0}\left[-\frac{1}{\lambda^2} + \frac{\pi}{\lambda}\left(\frac{1}{\pi\lambda} + \frac{\pi\lambda}{3} + \cdots\right)\right] = \frac{\pi^2}{3}$

THUS $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ D.