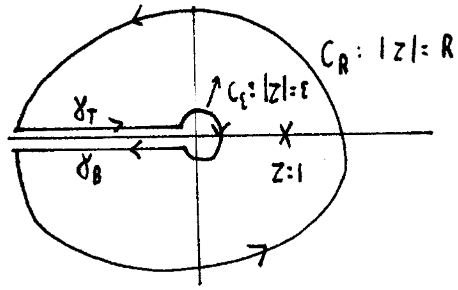
INTEGRALS OF MULTI-VALUED FUNCTIONS

PROBLEM 1 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \qquad 0<\alpha<1.$

TAKE THE INTEGRAL $\int_C \frac{z^{\alpha-1}}{1-z}\,dz$ WHERE C IS THE CONTOUR SHOWN BELOW. TAKE PRINCIPAL BRANCH OF $z^{\alpha-1}$. THEN,



$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0 R\,\frac{R^{\alpha-1}}{R} = a_0 R^{\alpha-1} \to 0 \text{ as } R\to\infty$$
SINCE $\alpha<1$

$$\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0\,\varepsilon\,\frac{\varepsilon^{\alpha-1}}{1} = a_0\varepsilon^{\alpha} \to 0 \text{ as } \varepsilon\to 0$$
SINCE $\alpha>0$.

THUS $$\lim_{R\to\infty}\int_{C_R}\frac{z^{\alpha-1}}{1-z}\,dz + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = 2\pi i\,\mathrm{RES}\left[\frac{z^{\alpha-1}}{1-z};1\right]$$

SO (*) $$\int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = -2\pi i\,(1)^{\alpha-1} = -2\pi i.$$

ON γ_T: $\quad z = \rho e^{i\pi} \qquad 0<\rho<\infty.$

SO $1-z = 1+\rho$ (this is why we took $1-z$ rather than $1+z$)

$dz = e^{i\pi}\,d\rho.$

SO $$\int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz = \int_\infty^0 \frac{e^{i\pi}\rho^{\alpha-1}}{1+\rho}\,e^{i\pi(\alpha-1)}\,d\rho = e^{i\pi(\alpha-1)}\int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$$

ON γ_B: $\quad z=\rho e^{-i\pi} \qquad 0<\rho<\infty \qquad 1-z = 1+\rho \qquad dz = e^{-i\pi}\,d\rho$

$$\int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = \int_0^\infty \frac{e^{-i\pi}\rho^{\alpha-1}e^{-i\pi(\alpha-1)}}{1+\rho}\,d\rho = -e^{-i\pi(\alpha-1)}\int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$$

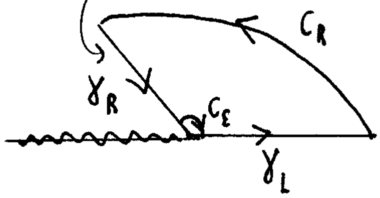
substitute these integrals into (*). Then,

$$\left[e^{i\pi(\alpha-1)} - e^{-i\pi(\alpha-1)}\right]\int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = -2\pi i. \qquad \int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{\left(e^{i\pi(\alpha-1)}-e^{-i\pi(\alpha-1)}\right)}$$

THUS $$\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{2i\sin[\pi(\alpha-1)]} = \frac{\pi}{\sin[\pi(1-\alpha)]}$$

<u>PROBLEM 2</u> $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx \qquad 0<\alpha<3. \qquad z = \rho e^{2\pi i/3}.$

TAKE THE CONTOUR $\displaystyle\int_C \frac{z^{\alpha-1}}{1+z^3}\,dz$



TAKE BRANCH CUT ON negative Real axis.

THEN $$\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le a_0 \underbrace{\varepsilon}_{\text{length of curve}} \varepsilon^{\alpha-1} = a_0\varepsilon^{\alpha} \to 0 \text{ as } \varepsilon\to 0 \text{ since } \alpha>0.$$

$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le a_0 R \frac{R^{\alpha-1}}{R^3} = a_0 R^{\alpha-3} \to 0 \text{ AS } R\to\infty \text{ since } \alpha<3.$$

NOW inside C: $1+z^3=0$ WHEN $z = e^{i\pi/3}$.

THUS $$\text{RES}\left[\frac{z^{\alpha-1}}{1+z^3};\, e^{i\pi/3}\right] = \frac{e^{i\pi/3(\alpha-1)}}{3e^{2\pi i/3}} = \frac{1}{3}e^{\alpha\pi i/3}e^{-i\pi} = -\frac{1}{3}e^{\alpha\pi i/3}.$$

<u>NEXT</u> $$\lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_R} + \int_{\gamma_L} = 2\pi i\,\text{RES}\left(\frac{z^{\alpha-1}}{1+z^3};\, e^{\pi i/3}\right) = -\frac{2\pi i}{3}e^{\alpha\pi i/3}$$

letting $R\to\infty$, $\varepsilon\to 0$ $\quad (*) \quad \displaystyle\int_{\gamma_R} + \int_{\gamma_L} = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$

<u>NOW ON γ_R:</u> $\quad z = \rho e^{2\pi i/3} \qquad dz = e^{2\pi i/3}\,d\rho \qquad 1+z^3 = 1+\rho^3.$

so $$\int_{\gamma_R} = \int_\infty^0 \frac{\rho^{\alpha-1}}{1+\rho^3} e^{2\pi i(\alpha-1)/3} e^{2\pi i/3}\,d\rho = -e^{2\pi i\alpha/3}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$$

<u>NOW ON γ_L</u> $\quad z=\rho \qquad dz = d\rho \qquad \displaystyle\int_{\gamma_L}\frac{z^{\alpha-1}}{1+z^3}\,dz = \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$

substituting these integrals INTO $(*)$ we get
$$\left(1-e^{+2\pi i\alpha/3}\right)\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^2}\,d\rho = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$$

so $$\left(e^{-\alpha\pi i/3} - e^{\alpha\pi i/3}\right) I = -\frac{2\pi i}{3} \longrightarrow +2i\sin\left(-\alpha\pi/3\right) I = -2\pi i/3$$

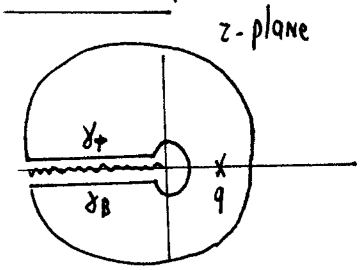
OR $$\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho = \frac{\pi}{3\sin(\pi\alpha/3)}$$

PROBLEM 3 WHAT CONTOURS SHOULD WE TAKE FOR CALCULATING

$$I_1 = \int_0^\infty \frac{x^\alpha}{(x+q)^2}\,dx \qquad I_2 = \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dz \qquad I_3 = \text{P.V.}\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$

$$-1 < \alpha < 1 \qquad\qquad -1 < \alpha < 3 \qquad\qquad -1 < \alpha < 1$$

INTEGRAL I_1



$\int_C \frac{z^\alpha}{(q-z)^2}\,dz$ TAKE z^α with branch cut shown.

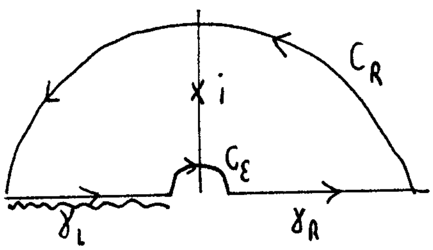
ON γ_T : $z = \rho e^{i\pi}$ so $(q-z)^2 = (q+\rho)^2$

ON γ_B : $z = \rho e^{-i\pi}$ so $(q-z)^2 = (q+\rho)^2$

this contour will work.

INTEGRAL I_2 WE could take precisely the same contour as for I_1 above. This would give poles at $z = \pm i$ when integrating $\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$.

HOWEVER, PERHAPS IT IS EASIER TO TAKE CONTOUR BELOW:



$\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$ with taking principal branch FOR z^α.

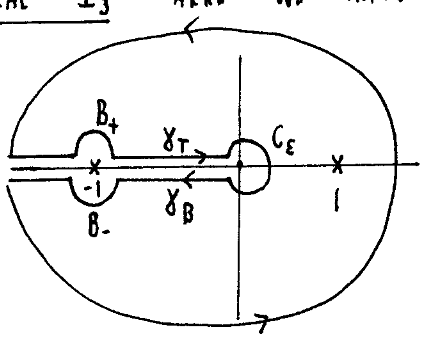
ON γ_L : $z = \rho e^{i\pi}$ so $1+z^2 = 1+\rho^2$.

ON γ_R : $z = \rho$ so $1+z^2 = \rho^2$. $\quad i^\alpha = e^{\frac{\pi i}{2}\alpha}$.

NOW $\text{RES}\left[\frac{z^\alpha}{(z^2+1)^2} ; i\right] = \lim_{z\to i} \frac{d}{dz}\left[\frac{(z+i)^2 z^\alpha}{(z+i)^2(z-i)^2}\right]$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^{-2} z^\alpha\right] = \alpha z^{\alpha-1}(z-i)^{-2}\Big|_i - 2(z-i)^{-3} z^\alpha\Big|_i$$

INTEGRAL I_3 HERE WE HAVE NO CHOICE BUT TO TAKE CONTOUR



$\int_C \frac{z^\alpha}{z^2-1}\,dz$ pole at ± 1 along Real axis

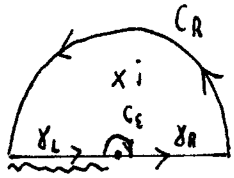
NOTE $\int_{B_\pm} \not\to 0$ AS $\varepsilon \to 0$.

ON B_+ : $z = -1 + \varepsilon e^{i\varphi}$ $\quad \pi < \varphi < 0$

$$\lim_{\varepsilon\to 0} \int_{B_+} \frac{z^\alpha}{z^2-1}\,dz \to \int_\pi^0 \frac{e^{i\pi\alpha}}{\varepsilon e^{i\varphi}(-2)}\,\varepsilon i e^{i\varphi}\,d\varphi \to \frac{i\pi}{2} e^{i\pi\alpha}.$$

PROBLEM 4 $\quad I = \int_0^\infty \frac{\ln x}{x^2+1}\,dx.$

CONSIDER THE CONTOUR



$$\int_C \frac{\operatorname{LOG} z}{z^2+1}\,dz.$$

HERE $\operatorname{LOG}(z)$ is principal branch.

$$\left|\int_{C_R} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0 R\,\frac{\ln R}{R^2} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{C_\varepsilon} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0\,\varepsilon\,\frac{\ln \varepsilon}{1} \to 0 \text{ AS } \varepsilon\to 0.$$

NOW $\displaystyle \lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i\,\operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right]$

LET $R\to\infty$, $\varepsilon\to 0$ TO GET (*) $\displaystyle \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i\,\operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right] = 2\pi i\,\frac{\operatorname{LOG} i}{2i} = \pi\left(i\pi\right.$

$= i\pi^2/2.$

NOW ON γ_L: $\quad z = p e^{i\pi}.\quad dz = e^{i\pi}\,dp.\quad z^2+1 = p^2+1.$

$\operatorname{LOG} z = \ln p + i\pi$

SO $\displaystyle \int_{\gamma_L} = \int_\infty^0 \left(\frac{\ln p + i\pi}{p^2+1}\right) e^{i\pi}\,dp = \int_0^\infty \frac{\ln p + i\pi}{p^2+1}\,dp.$

NOW ON γ_R: $\quad z = p \quad$ SO $\quad dz = dp \quad z^2+1 = p^2+1$

$\operatorname{LOG} z = \ln p$

SO $\displaystyle \int_{\gamma_R} = \int_0^\infty \frac{\ln p}{p^2+1}\,dp.$

substitute $\int_{\gamma_L}$ AND $\int_{\gamma_R}$ INTO (*) TO get $\displaystyle \int_0^\infty \frac{\ln p}{p^2+1}\,dp + \int_0^\infty \frac{\ln p}{p^2+1}\,dp + i\pi\int_0^\infty \frac{1}{1+p^2}\,dp$

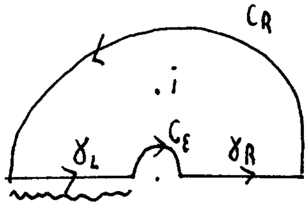
$= i\pi^2/2.$

EQUATING REAL AND IMAGINARY PARTS:

$$\int_0^\infty \frac{\ln p}{p^2+1}\,dp = 0 \quad \text{AND} \quad \int_0^\infty \frac{1}{1+p^2}\,dp = \pi/2.$$

PROBLEM 5 $\quad I = \displaystyle\int_0^\infty \frac{(\ln r)^2}{1+r^2}\,dr.$ $\qquad \text{LOG } z$ IS PRINCIPAL VALUE.

TAKE $\displaystyle\int_C \frac{(\operatorname{LOG} z)^2}{1+z^2}\,dz$ ON



NOW $\operatorname{RES}\left[\dfrac{(\operatorname{LOG} z)^2}{1+z^2};\ i\right] = \dfrac{(\operatorname{LOG} i)^2}{2i} = \dfrac{(i\pi/2)^2}{2i} = -\dfrac{\pi^2}{8i} = \dfrac{i\pi^2}{8}$

THUS AGAIN $\displaystyle\int_{C_R} \to 0$ AND $\displaystyle\int_{C_\varepsilon} \to 0$. SO

$$(*) \quad \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}(\ ;\ i) = 2\pi i\,(i\pi^2/8) = -\pi^3/4.$$

NOW ON γ_L: $\quad z = \rho e^{i\pi} \quad 0<\rho<\infty \quad 1+z^2 = 1+\rho^2 \quad dz = e^{i\pi}\,d\rho$

$$(\operatorname{Log} z)^2 = (\ln\rho + i\pi)^2 = (\ln\rho)^2 + 2i\pi\ln\rho - \pi^2.$$

$$\text{THUS} \int_{\gamma_L} = \int_\infty^0 \left(\frac{(\ln\rho)^2 + 2i\pi\ln\rho - \pi^2}{1+\rho^2}\right) e^{i\pi}\,d\rho = \int_0^\infty \frac{(\ln\rho)^2 + 2i\pi\ln\rho - \pi^2}{1+\rho^2}\,d\rho$$

NOW ON γ_R: $\quad z = \rho$

$$\int_{\gamma_R} = \int_0^\infty \frac{(\ln\rho)^2}{1+\rho^2}.$$

substitute $\displaystyle\int_{\gamma_L}, \int_{\gamma_R}$ INTO $(*)$ to get:

$$2\int_0^\infty \frac{(\ln\rho)^2}{1+\rho^2}\,d\rho + 2i\pi\int_0^\infty \frac{\ln\rho}{1+\rho^2}\,d\rho - \pi^2\int_0^\infty \frac{1}{1+\rho^2}\,d\rho = -\pi^3/4.$$

EQUATING REAL AND IMAGINARY PARTS:

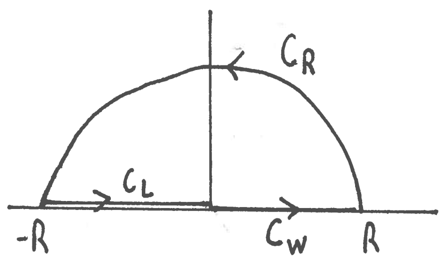
$$\int_0^\infty \frac{\ln\rho}{1+\rho^2}\,d\rho = 0 \quad \text{AND} \quad \int_0^\infty \frac{(\ln\rho)^2}{1+\rho^2}\,d\rho = \frac{\pi^2}{2}\int_0^\infty \frac{1}{1+\rho^2}\,d\rho - \frac{\pi^3}{8}$$

$$= \frac{\pi^2}{2}\tan^{-1}\rho\,\Big|_0^\infty - \frac{\pi^3}{8} = \frac{\pi^2}{2}\left(\frac{\pi}{2}\right) - \frac{\pi^3}{8}$$

$$\text{so} \quad \int_0^\infty \frac{(\ln\rho)^2}{1+\rho^2}\,d\rho = \pi^3/8.$$

<u>EXAMPLE 6</u> CALCULATE $I_1 = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx.$

<u>SOLUTION</u> DEFINE $\sqrt{z}$ AS PRINCIPAL BRANCH AND CONSIDER CONTOUR SHOWN WHERE C_L IS ON TOP OF BRANCH CUT.



THEN LET $C = C_L \cup C_W \cup C_R$

WITH $C_R : |z| = R,\ \mathrm{Im}\, z \geq 0.$

NOW POLES ARE AT $z^2 + z + 1 = 0$

SO $(z + 1/2)^2 = -3/4 \rightarrow z_\pm = -1/2 \pm i\sqrt{3}/2$

NOTICE ONLY z_+ IN UPPER 1/2 PLANE. $z_+ = e^{2\pi i/3}.$

THUS $$\lim_{R\to\infty}\left(\int_{C_L} + \int_{C_W} + \int_{C_R}\right)\frac{z^{1/4}}{z^2+z+1}\,dz = 2\pi i\,\mathrm{RES}\left(\frac{z^{1/4}}{z^2+z+1};\, e^{2\pi i/3}\right) \quad (*)$$

NOW $$\left|\int_{C_R}\frac{\sqrt[4]{z}}{z^2+z+1}\,dz\right| \leq \frac{R^{1/4}}{|R^2-(R+1)|}\,\pi R = O\left(\frac{R^{5/4}}{R^2}\right) \rightarrow 0 \text{ AS } R\to\infty.$$

<u>NOW ON C_L:</u> $z = xe^{i\pi} \longrightarrow dz = e^{i\pi}\,dx \qquad \frac{z^{1/4}}{z^2+z+1} = \frac{e^{i\pi/4}x^{1/4}}{x^2-x+1}.$

THUS $$\int_{C_L} = \int_\infty^0 \frac{e^{i\pi}e^{i\pi/4}x^{1/4}\,dx}{x^2-x+1} = \int_0^\infty \frac{e^{i\pi/4}x^{1/4}}{x^2-x+1}\,dx.$$

<u>NOW ON C_W:</u> $z = x \quad dz = dx \longrightarrow \int_{C_W} = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx.$

THUS $(*)$ GIVES $$\int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx + e^{i\pi/4}\int_0^\infty \frac{x^{1/4}}{x^2-x+1}\,dx = 2\pi i\left(\frac{e^{\pi i/6}}{2e^{2\pi i/3}+1}\right) = \frac{2\pi i\, e^{\pi i/6}}{2i\sqrt{3}/2}.$$

DEFINE $I_1 = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx,\ I_2 = \int_0^\infty \frac{x^{1/4}}{x^2-x+1}\,dx$. WE GET 1-COMPLEX EQUATION FOR 2 UNKNOWNS I_1 AND $I_2 \longrightarrow I_1 + e^{i\pi/4} I_2 = \frac{2\pi}{\sqrt{3}}\, e^{\pi i/6}.$

WE WANT I_1: MULTIPLY BOTH SIDES BY $e^{-i\pi/4} \rightarrow e^{-i\pi/4} I_1 + I_2 = \frac{2\pi}{\sqrt{3}}\, e^{-i\pi/12}$

NOW TAKE IMAGINARY PARTS: $\mathrm{IM}(e^{-i\pi/4} I_1) = \frac{2\pi}{\sqrt{3}}\,\mathrm{IM}(e^{-i\pi/12})$

THUS $I_1 = (2\pi/\sqrt{3})\,[\,\mathrm{SIN}(\pi/12)/\ \ldots\,]$

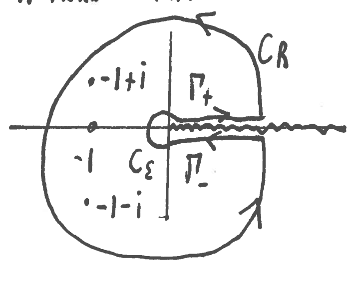
EXAMPLE 7 CALCULATE $I = \int_0^{\infty} \frac{1}{(x+1)(x^2+2x+2)}\, dx$.

SOLUTION WE CANNOT USE SYMMETRY HERE TO GET $\int_{-\infty}^{\infty}$. THUS, TRY

TO INTEGRATE

$$J = \int_C \frac{\log z}{(z+1)(z^2+2z+2)}$$

WHERE THE CONTOUR IS AS SHOWN



WE CHOOSE THE BRANCH OF $\log z$ WITH BRANCH CUT ON positive Real AXIS

$\log z = \ln|z| + i\varphi \quad 0 \le \varphi <$

POLES ARE AT $z = -1$ AND $z^2 + 2z + 1 = -1 \rightarrow (z+1)^2 = -1 \rightarrow z_{\pm} = -1 \pm i$

CLEARLY $\left|\int_{C_R}\right| \rightarrow 0$ AS $R \rightarrow \infty$ AND $\left|\int_{C_\varepsilon}\right| \rightarrow 0$ AS $\varepsilon \rightarrow 0$.

THUS
$$\lim_{R\to\infty}\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right) = 2\pi i\left[\operatorname{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; -1\right) + \operatorname{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; z_+\right) + \operatorname{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; z_-\right)\right]$$

NOW ON Γ_+: $z = x \longrightarrow \int_{\Gamma_+} = \int_0^{\infty} \frac{\ln x}{(x+1)(x^2+2x+2)}\, dx$

NOW ON Γ_-: $z = x e^{2\pi i} \longrightarrow \int_{\Gamma_-} = \int_{\infty}^{0} \frac{(\ln x + i 2\pi)}{(x+1)(x^2+2x+2)}\, dx = -\int_0^{\infty} \frac{(\ln x + 2\pi i)}{(x+1)(x^2+2x+2)}\, dx$.

THUS ADDING $\int_{\Gamma_+} + \int_{\Gamma_-} \longrightarrow -2\pi i \int_0^{\infty} \frac{1}{(x+1)(x^2+2x+2)}\, dx = 2\pi i\left[\operatorname{RES}(\,;-1) + \operatorname{RES}(\,;z_+) + \operatorname{RES}(\,;z_-)\right]$. (*)

$\operatorname{RES}(\,;-1) = \frac{\log(-1)}{1} = i\pi$. $\operatorname{RES}(\,;z_+) = \frac{\log(z_+)}{(z_+ + 1)(2z_+ + 2)} = \frac{\log(\sqrt{2}\, e^{3\pi i/4})}{i(2i)}$

THUS $\operatorname{RES}(\,;z_+) = -\frac{1}{2}\left(\ln(\sqrt{2}) + 3\pi i/4\right)$. SIMILARLY $\operatorname{RES}(\,;z_-) = \frac{\log(\sqrt{2}\, e^{5\pi i/4})}{(z_- + 1)(2z_- + 1)}$

SO $\operatorname{RES}(\,;z_-) = -\frac{1}{2}\left(\ln\sqrt{2} + 5\pi i/4\right)$

THUS (*) BECOMES $-2\pi i\, I = 2\pi i\left(i\pi - \frac{1}{2}\ln\sqrt{2} - \frac{3\pi i}{8} - \frac{1}{2}\ln\sqrt{2} - \frac{5\pi i}{8}\right) \rightarrow \underline{\underline{I = -\ln(\sqrt{2})}}$