MATH 301: SAMPLE MIDTERM QUESTIONS (M. WARD)

PROBLEM 1: Solve Laplace's equation $T_{xx} + T_{yy} = 0$ in the intersection of the regions $\text{Re}(z) > 1$ and $|z| < 2$. We are given that $T = 1$ on $\text{Re}(z) = 1$ and $T = 3$ on $|z| = 2$.

PROBLEM 2: Find the image of the unit disk $|z| \leq 1$ under the mapping $w = f(z) = \frac{i}{2}\text{Log}\left(\frac{i+z}{i-z}\right)$. (Here Log denotes the principal branch of the logarithim function). (Hint: do it in steps)

PROBLEM 3: Using conformal mapping find the solution $T(x, y)$ to Laplace's equation outside the two circles $C_1 : |z - 2i| = 1$ and $C_2 : |z + 4i| = 2$. You are given that $T = 1$ on C_1 and $T = 2$ on C_2.

PROBLEM 4: Consider incompressible, inviscid, fluid flow past the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b$. Assume that the flow far from the ellipse has speed V_0 and is inclined at an angle $\alpha > 0$ with respect to the positive x-axis.

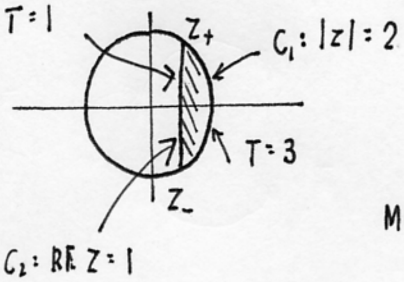
i) Derive the complex velocity potential for the flow

ii) Derive a formula for the speed of the flow at any point on the ellipse.

iii) Where are the stagnation points? At what point on the ellipse is the speed maximum? What is the maximum speed?

M301 SAMPLE MIDTERM

PROBLEM 1 WE SOLVE $T_{xx}+T_{yy}=0$ IN REGION SHOWN



FIND INTERSECTION POINTS OF $|z|=2$ AND $RE(Z)=1$.

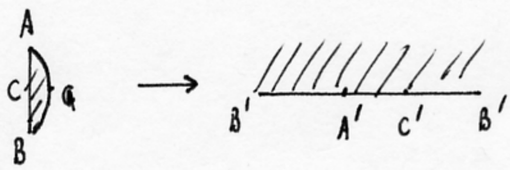
WE HAVE $\left.\begin{matrix} x^2+y^2=4 \\ x=1 \end{matrix}\right\} \Rightarrow y=\pm\sqrt{3}$ SO $Z_+ = 1+i\sqrt{3} = 2e^{\pi i/3}$

$Z_- = 1-i\sqrt{3} = 2e^{-\pi i/3}$

MAP $\begin{matrix} Z=Z_+ \longrightarrow W=0 \\ Z=Z_- \longrightarrow W=\infty \end{matrix}$ SO IMAGE OF C_1 AND C_2 ARE LINES THROUGH THE ORIGIN.

WE THUS HAVE $W = B\left(\dfrac{Z-Z_+}{Z-Z_-}\right)$ CHOOSE $Z=1$ TO MAP TO $W=1$

$\Rightarrow\ 1 = B\left(\dfrac{-i\sqrt{3}}{i\sqrt{3}}\right) = -B$ SO $B=-1$.



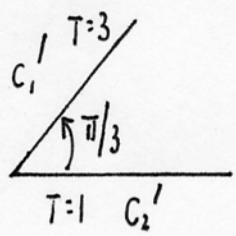
THIS CHOICE $Z=1 \rightarrow W=1$ PRESERVES ORIENTATION AND HENCE $W = \dfrac{Z_+-Z}{Z-Z_-}$

NOW THE ANGLE AT Z_+ IS BY CONFORMALITY GIVEN BY $\cos\varphi = \dfrac{(y,-x)}{2}\cdot(0,-1) =$

SO $\cos\varphi = \dfrac{(y,-x)}{2}\cdot(0,-1) = \dfrac{x}{2} = \dfrac{1}{2}$ AT $Z_+ \longrightarrow \varphi = \pi/3$.

EQUIVALENTLY LET $Z=2$ SO $W = \dfrac{1+i\sqrt{3}-2}{2-(1-i\sqrt{3})} = \dfrac{-1+i\sqrt{3}}{1+i\sqrt{3}} = \dfrac{2e^{2\pi i/3}}{2e^{\pi i/3}} = e^{\pi i/3}$.

HENCE THE IMAGE OF THE DOMAIN IS AS FOLLOWS:



SO $T = A + B\varphi$. BY INSPECTION $T = 1 + \dfrac{6}{\pi}\varphi$

WHERE $\varphi = \tan^{-1}(V/U)$

$$T = 1 + \frac{6}{\pi}\tan^{-1}\left(\frac{V}{U}\right)$$

(NOTICE $\bar{Z}_- = Z_+$)

NOW $W = \dfrac{Z_+-Z}{Z-Z_-} = \dfrac{(Z_+-Z)(\bar{Z}-\bar{Z}_-)}{|Z-Z_-|^2} = \dfrac{-Z_+\bar{Z}_- - |Z|^2 + Z_+\bar{Z} + Z\bar{Z}_-}{|Z-Z_-|^2} = \dfrac{-Z_+^2 - |Z|^2 + Z_+(Z+\bar{Z})}{|Z-Z_-|^2}$

THIS YIELDS, $W = \dfrac{-4e^{2\pi i/3} - |Z|^2 + 2xZ_+}{|Z-Z_-|^2} \Rightarrow \dfrac{V}{U} = \dfrac{-2\sqrt{3}+2x\sqrt{3}}{2x-(x^2+y^2)+2} = \dfrac{2\sqrt{3}(x-1)}{2(1+x)-(x^2+y^2)}$

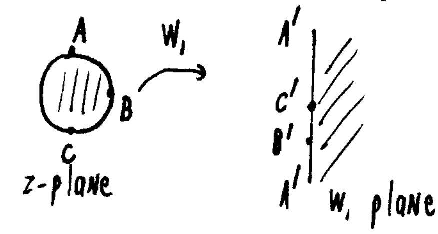
THIS GIVES $T(x,y) = 1 + \dfrac{6}{\pi}\tan^{-1}\left(\dfrac{2\sqrt{3}(x-1)}{2(1+x)-(x^2+y^2)}\right)$.

PROBLEM 2 FIND THE IMAGE OF $|z| \leq 1$ UNDER $W = \frac{i}{2} \operatorname{LOG}\left(\frac{i+z}{i-z}\right)$.

WE LET $W_1 = \frac{i+z}{i-z}$ $\qquad z = i \longrightarrow W_1 = \infty$ (circle maps to line)

$\qquad z = -i \longrightarrow W = 0$ (circle goes through line through origin).

NOW $z = 1 \longrightarrow W = \frac{1+i}{-1+i} = \frac{\sqrt{2}\, e^{\pi i/4}}{\sqrt{2}\, e^{3\pi i/4}} = e^{-i\pi/2} = -i$



PRESERVATION OF ORIENTATION IMPLIES THAT

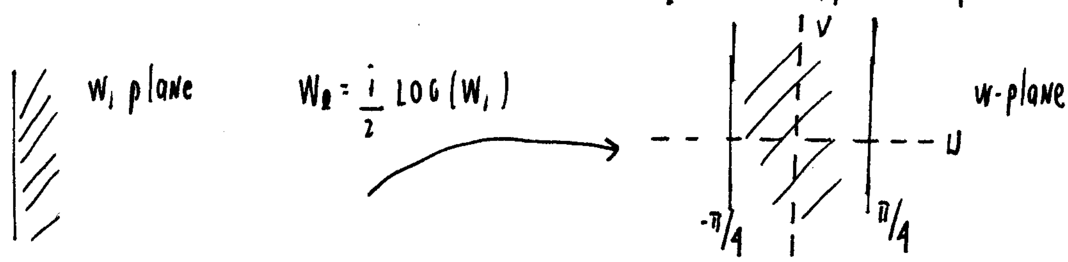
$|z| \leq 1$ MAPS TO $\operatorname{RE} W_1 \geq 0$.

NOW $W = \frac{i}{2} \operatorname{LOG}(W_1)$ $\qquad |\operatorname{Arg}(W_1)| < \pi/2$ (same as $\operatorname{Re} W_1 \geq 0$).

let $W_1 = r e^{i\varphi}$, $-\pi/2 < \varphi < \pi/2$ $\qquad W = U + iV = \frac{i}{2}\left[\ln r + i\varphi\right] = -\frac{\varphi}{2} + \frac{i}{2} \ln r$.

HENCE FOR $0 < r < \infty$ WE HAVE $-\infty < V < \infty$ AND $|\varphi| < \pi/2 \rightarrow -\pi/4 < U < \pi/4$.

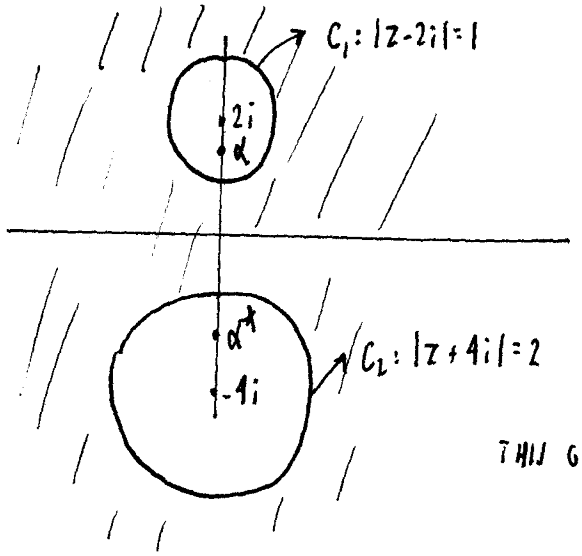
THIS MEANS



THE IMAGE OF $|z| \leq 1$ UNDER $W = \frac{i}{2} \operatorname{LOG}\left(\frac{i+z}{i-z}\right)$ IS SIMPLY $|\operatorname{RE} W| \leq \pi/4$

PROBLEM 3 FIND SOLUTION TO $T_{xx} + T_{yy} = 0$ OUTSIDE THE TWO CIRCLES

$C_1 : |z-2i| = 1$ AND $C_2 : |z+4i| = 2$. HERE $T = 1$ ON C_1 AND $T = 2$ ON C_2:



choose α inside C_1:

$$\alpha^* = 2i + \frac{1}{\bar{\alpha} + 2i} \qquad (\alpha^*, \alpha \text{ symmetric WRT } C_1)$$

$$\alpha = -4i + \frac{4}{\bar{\alpha}^* - 4i} \qquad (\alpha^*, \alpha \text{ symmetric WRT } C_2)$$

THIS GIVES
$$\left.\begin{aligned} (\alpha^* - 2i)(\bar{\alpha} + 2i) &= 1 \\ (\alpha + 4i)(\bar{\alpha}^* - 4i) &= 4 \end{aligned}\right\} \quad (1)$$

NOW WE SET $\alpha = iY_1$ AND $\alpha^* = iY_2$ SO THAT (1) BECOMES
$$(Y_2 - 2)(Y_1 - 2) = 1$$
$$(Y_1 + 4)(Y_2 + 4) = 4$$

WE GET $Y_2 = -4 + \dfrac{4}{Y_1 + 4}$ AND SO $\left(-6 + \dfrac{4}{Y_1 + 4}\right)(Y_1 - 2) = 1 \longrightarrow (-6Y_1 - 20)(Y_1 - 2) = Y_1 + 4$

THIS GIVES $2Y_1^2 + 3Y_1 - 12 = 0 \rightarrow \alpha = iY_1$ where $Y_1 = \dfrac{-3 + \sqrt{105}}{4}$

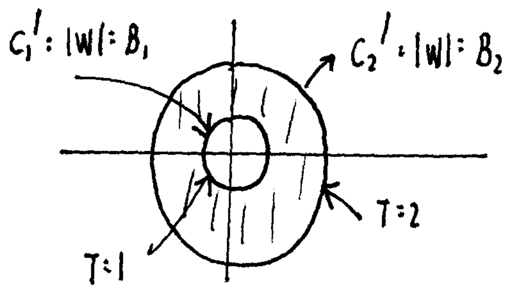
$\alpha^* = iY_2$ where $Y_2 = \dfrac{-3 - \sqrt{105}}{4}$.

WE MAP $z = \alpha \rightarrow w = 0$ AND $z = \alpha^*$ TO $w = \infty \implies w = \beta\left(\dfrac{z - \alpha}{z - \alpha^*}\right)$ WLOG let $\beta = 1$

SO $w = \dfrac{z - \alpha}{z - \alpha^*} \implies$ IMAGE OF C_1 AND C_2 ARE concentric circles centered at $w = 0$

let $z = i$ ON $C_1 \Rightarrow |w| = (Y_1 - 1)/(1 - Y_2) \equiv B_1$: LET $z = -3i$ ON $C_2 \rightarrow |w| = \dfrac{Y_1 + 3}{3 + Y_2} = B_2$

WITH $B_2 > B_1$



NOW $T = A + B \log \rho$ WITH $\rho = |w|$.

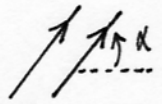
EASILY WE FIT SO THAT

$$T = 1 + \log(\rho/B_1)/\log(B_2/B_1)$$

WITH $\rho = |w| = \left|\dfrac{z - iY_1}{z - iY_2}\right| = \left(\dfrac{x^2 + (y - Y_1)^2}{x^2 + (y - Y_2)^2}\right)^{1/2}$.

PROBLEM 4

RECALL THAT FLOW AT AN ANGLE α TO HORIZONTAL IS $\Omega = V_0 z e^{-i\alpha}$



NOW THE FLOW OVER A CIRCLE OF RADIUS R IS IN $\tilde{\zeta}$-PLANE

$$\Omega(\zeta) = V_0\left(\tilde{\zeta} + R^2/\tilde{\zeta}\right)$$

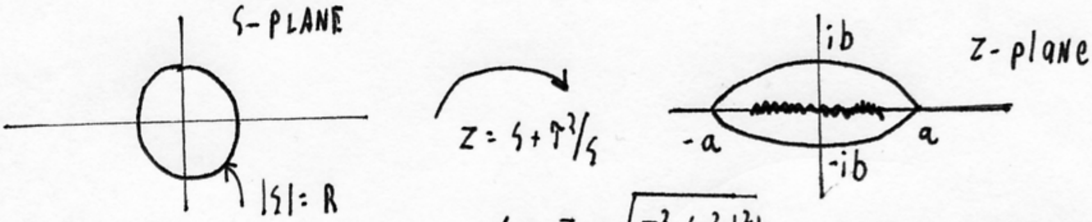
NOW FOR A FLOW AT ANGLE α LET $\hat{\zeta} = \zeta e^{-i\alpha} \rightarrow \Omega = V_0\left(\zeta e^{-i\alpha} + R^2/\zeta e^{-i\alpha}\right)$

NOW LET $z = \zeta + \tau^2/\zeta$ MAP ζ-PLANE TO z-plane.

THEN $x = a\sin q$, $y = b\sin q$, $\zeta = Re^{iq}$, THEN

$$z = x + iy = \left(R + \tau^2/R\right)\cos q + i\left(R - \tau^2/R\right)\sin q \qquad a = R + \tau^2/R, \quad b = R - \tau^2/R$$

THEN $R = (a+b)/2$ AND $2\tau^2/R = a - b \rightarrow \tau^2 = \dfrac{(a-b)(a+b)}{4} = \dfrac{a^2-b^2}{4} \rightarrow \tau = \dfrac{(a^2-b^2)^{1/2}}{2}$



$$\zeta = \frac{z + \sqrt{z^2 - (a^2-b^2)}}{2}$$

WHICH BRANCH CUTS INSIDE BODY BETWEEN $-\sqrt{a^2-b^2} < z < \sqrt{a^2-b^2}$

(i) THEN $\Omega(z) = V_0\left[\zeta e^{-i\alpha} + \dfrac{R^2}{e^{-i\alpha}\zeta}\right] \qquad \zeta = \dfrac{z + \sqrt{z^2-(a^2-b^2)}}{2}$

(ii) GET A FORMULA FOR $|\Omega'(z)|$

NOW $\dfrac{d\Omega}{dz} = \dfrac{d\Omega}{d\zeta}\dfrac{d\zeta}{dz}$ AND $\dfrac{d\Omega}{d\zeta} = V_0\left(e^{-i\alpha} - \dfrac{R^2 e^{i\alpha}}{\zeta^2}\right)$

NOW $2\zeta\, d\zeta/dz - d\zeta/dz\, z - \zeta = 0$ SO $d\zeta/dz = \left(1 - \tau^2/\zeta^2\right)^{-1}$.

HENCE $\dfrac{d\Omega}{dz} = \dfrac{V_0}{\zeta^2}\left(\dfrac{\zeta^2 e^{-i\alpha} - R^2 e^{i\alpha}}{1 - \tau^2/\zeta^2}\right) = V_0\left(\dfrac{\zeta^2 e^{-i\alpha} - R^2 e^{i\alpha}}{\zeta^2 - \tau^2}\right)$

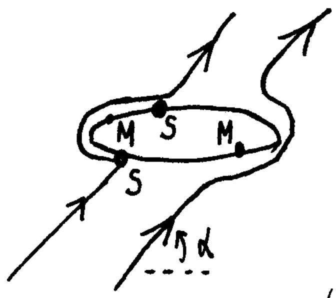
NOW LET $\zeta = Re^{iq}$ SO THAT $\dfrac{d\Omega}{dz} = V_0 e^{-i\alpha}\left(\dfrac{R^2 e^{2iq} - R^2 e^{2i\alpha}}{R^2 e^{2iq} - \tau^2}\right)$

WE THEN RECALL $\dfrac{\tau^2}{R^2} = \dfrac{a-b}{a+b} \equiv \mathcal{B} \longrightarrow \dfrac{d\Omega}{dz} = V_0 e^{-i\alpha}\left(\dfrac{e^{2iq} - e^{2i\alpha}}{e^{2iq} - \mathcal{B}}\right)$

HENCE $\left|\frac{d\Omega}{dz}\right| = V = V_0 \left| \frac{e^{2i\varphi} - e^{2i\alpha}}{e^{2i\varphi} - B} \right|$ gives a FORMULA FOR speed at any point on ellipse (here φ parametrizes point on ellipse)

(iii) stagnation points where $|d\Omega/dz| = 0$ so $\varphi = \alpha$ AND $\varphi = \alpha + \pi$ (S = stagnation points)

NOW THE speed should be MAXIMUM when $\varphi = \alpha + \pi/2$, $\varphi = \alpha + 3\pi/2$ (M = MAX speed points)



NOW when $\varphi = \alpha + \pi/2$, OR $\varphi = 3\pi/2 + \alpha$

we get $\left|\frac{d\Omega}{dz}\right| = V_0 \frac{|-2e^{2i\alpha}|}{|e^{2i\alpha} + B|}$

(NOTICE: $e^{2i(\alpha + \pi/2)} = e^{i\pi} e^{2i\alpha} = -e^{2i\alpha}$.

HENCE $\left|\frac{d\Omega}{dz}\right| = V_{MAX} = \frac{2V_0}{|e^{2i\alpha} + B|} = \frac{2V_0}{\left[(\cos 2\alpha + B)^2 + \sin^2 2\alpha\right]^{1/2}}$

where $B = \frac{a-b}{a+b} > 0$.

NOTICE IF $\alpha = 0$ THEN $1 + B = \frac{2a}{a+b}$

so $\left|\frac{d\Omega}{dz}\right| = V_{MAX} = \frac{2V_0}{\frac{2a}{(a+b)}} = \left(\frac{a+b}{a}\right) V_0$.

IF $a = b \rightarrow V_{MAX} = 2V_0$.