RESIDUE AT INFINITY

SUPPOSE THAT $f(z)$ IS ANALYTIC OUTSIDE $|z|=R$. THEN, IN $|z|>R$ IT HAS A CONVERGENT LAURENT EXPANSION OF THE FORM

$$f(z) \sim a_N z^N + a_{N-1} z^{N-1} + \dots + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots \quad \text{VALID FOR } |z|>R.$$

LET $C_p : |z| = p$ WITH $p > R$ ORIENTED COUNTERCLOCKWISE. SINCE

$$\int_{C_p} z^m \, dz = \begin{cases} 0 & \text{IF } m \neq -1 \\ 2\pi i & \text{IF } m = -1 . \end{cases}$$

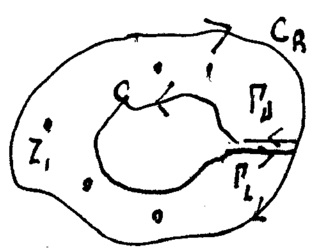
WE HAVE THAT $\displaystyle\int_{C_p} f(z)\,dz = \int_{C_p} \frac{a_{-1}}{z}\,dz = 2\pi i \, a_{-1}.$

WE DEFINE $\displaystyle \text{RES}[f;\infty] = -\frac{1}{2\pi i} \lim_{p\to\infty} \int_{C_p} f(z)\,dz = -a_{-1}.$

THEOREM LET C BE A COUNTERCLOCKWISE-ORIENTED SIMPLE CLOSED CURVE. LET $f(z)$ BE ANALYTIC ON C AND ON THE DOMAIN EXTERIOR TO C EXCEPT AT FINITELY MANY ISOLATED SINGULARITIES AT $z = z_j$ FOR $j = 1, \dots, N$. THEN

$$\int_C f(z)\,dz = -2\pi i \sum_{j=1}^{N} \text{RES}[f; z_j] - 2\pi i \, \text{RES}[f;\infty] . \qquad (+)$$

PROOF



BY RESIDUE THM SINCE CONTOUR IS NEGATIVELY ORIENTED, FOR $R >> 1$,

$$\oint_C f\,dz + \oint_{C_R} f\,dz = -2\pi i \sum_{j=1}^{N} \text{RES}[f; z_j].$$

NOW CHOOSE $C_R : |z| = R$. THEN SINCE $\displaystyle\oint_{C_R} f\,dz = -\oint_{C_R} f\,dz$

WE HAVE

$$\oint_C f\,dz = -2\pi i \sum_{j=1}^{N} \text{RES}[f; z_j] + \lim_{R\to\infty} \oint_{C_R} f\,dz .$$

NOW IF $f(z)$ HAS A CONVERGENT LAURENT EXPANSION FOR $R>>1$

WE HAVE $$\lim_{R\to\infty} \oint_{C_R} f\,dz = 2\pi i\, a_{-1}.$$

IT FOLLOWS THAT $$\oint_C f\,dz = -2\pi i \sum_{j=1}^{N} \mathrm{RES}[f; z_j] + 2\pi i\, a_{-1}. \quad (*)$$

FINALLY, SINCE $a_{-1} = -\mathrm{RES}[f;\infty]$ WE HAVE $(\dagger)$ •

EXAMPLE 1 CALCULATE $I = \oint_C \dfrac{z^4}{z^5+1}\,dz$ $C: |z| = 2$ counterclockwise.

SOLUTION ALL POLES OF $f(z) = \dfrac{z^4}{z^5+1}$ SATISFY $z_j^5 = -1$ SO $|z_j| = 1$

WHICH ARE INSIDE $|z| = 1$. NOW BY $(*)$ WE HAVE $f(z)$ IS

ANALYTIC FOR $|z| \geq 2$ AND $f(z) \cong \dfrac{1}{z} + \ldots$ AS $|z| \to \infty$ SO THAT $a_{-1} = 1$.

FROM $(*)$ WE OBTAIN

$$\oint_C f(z)\,dz = 2\pi i (1) = 2\pi i.$$

EXAMPLE 2 CALCULATE $\mathrm{P.V.} \displaystyle\int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx.$

SOLUTION THE IMPROPER INTEGRAL DOES NOT EXIST IN THE USUAL SENSE BUT THE SYMMETRIC LIMIT $\mathrm{P.V.} \displaystyle\int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx$ DOES

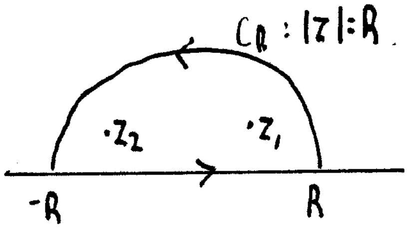
EXIST SINCE $$\lim_{R\to\infty} \int_{-R}^{R} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx < \infty.$$

WE DEFINE $f(z) = \dfrac{5z^3}{1+z+z^2+z^3+z^4}$. USING $1+z+z^2+z^3+z^4 = \dfrac{z^5-1}{z-1}$

SO THAT $f(z) = 5z^3(z-1)/(z^5-1)$.

CLEARLY $Z=1$ IS A REMOVABLE SINGULARITY AND WE HAVE SIMPLE POLES AT $Z_1=e^{2\pi i/5}$, $Z_2=e^{4\pi i/5}$, $Z_3=e^{6\pi i/5}$, $Z_4=e^{8\pi i/5}$.

WE CONSIDER THE CONTOUR SHOWN: ONLY Z_1, Z_2 IN UPPER 1/2 PLANE.



BY RESIDUE THEOREM

$$\left.\begin{aligned} &P.V. \int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx + \lim_{R\to\infty} \int_{C_R} \frac{5z^3(z-1)}{z^5-1}\,dz \\ &= 2\pi i\left(\text{RES}\left[f; e^{2\pi i/5}\right] + \text{RES}\left[F; e^{4\pi i/5}\right]\right). \end{aligned}\right\}(*)$$

NOW $\text{RES}[f; z_j] = \dfrac{5z_j^3(z_j-1)}{5z_j^4} = \dfrac{1}{z_j}(z_j-1) = 1-\dfrac{1}{z_j}$.

NOW FOR $|z|>>1$, $f(z) = \dfrac{5z^3(z-1)}{z^5-1} \simeq \dfrac{5}{z} + \dots$ AS $|z|\to\infty$.

SO $$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{5}{Re^{i\varphi}}\, iRe^{i\varphi}\,d\varphi = 5\pi i.$$

WE HAVE BY $(*)$

$$P.V. \int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx = -5\pi i + 2\pi i\left[\left(1-\frac{1}{e^{2\pi i/5}}\right) + \left(1-\frac{1}{e^{4\pi i/5}}\right)\right]$$

$$= -\pi i + 2\pi i\left[-e^{-2\pi i/5} - e^{-4\pi i/5}\right]$$

SO $$P.V. \int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx = -\pi i - 2\pi i\left[\cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)\right]$$

$$= -2\pi\left[\sin(2\pi/5) + \sin\left(\frac{4\pi}{5}\right)\right] - \pi i - 2\pi i\left[\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\right]$$

BUT $\cos(2\pi/5) + \cos(4\pi/5) = -1/2$ AND SO

$$I = P.V. \int_{-\infty}^{\infty} \frac{5x^3}{1+x+x^2+x^3+x^4}\,dx = -2\pi\left[\sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right)\right].$$

REMARK TO PROVE $\mathrm{RE}(z_1 + z_2) = -1/2$ WITH $z_1 = e^{2\pi i/5}$, $z_2 = e^{4\pi i/5}$

WE NOTE THAT $1 + z_1 + z_2 + z_3 + z_4 = 0$. BUT $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$

SO THAT $1 + (z_1 + z_2) + \overline{(z_1 + z_2)} = 0 \rightarrow \mathrm{RE}(z_1 + z_2) = -1/2$. $\square$.

EXAMPLE CONSTRUCT A BRANCH OF $f(z) = \sqrt{z^2 - 1}$ ANALYTIC IN $|z| > 1$

AND CALCULATE THE RESIDUE AT INFINITY FOR $f(z)$.

SOLUTION THERE ARE TWO APPROACHES.

METHOD 1 (LOG METHOD) WRITE $f(z) = z\left[1 - 1/z^2\right]^{1/2} = z\, e^{\frac{1}{2}\log(1 - 1/z^2)}$.

WHICH IS ANALYTIC IN $|z| > 1$. RECALL $\dfrac{1}{1 - w} = 1 + w + w^2 + \cdots$ FOR $|w| < 1$.

INTEGRATING $-\log(1 - w) = w + w^2/2 + w^3/3 + \cdots$ FOR $|w| < 1$ WHERE INTEGRATION

CONSTANT IS ZERO SINCE $\log(1) = 0$. NOW

$$\log\left(1 - \frac{1}{z^2}\right) = -\left[\frac{1}{z^2} + \frac{1}{2z^4} + \cdots\right].$$

NOW $\dfrac{1}{2}\log\left(1 - \dfrac{1}{z^2}\right) \simeq -\dfrac{1}{2z^2} + O\left(\dfrac{1}{|z|^4}\right)$ FOR $|z| >> 1$.

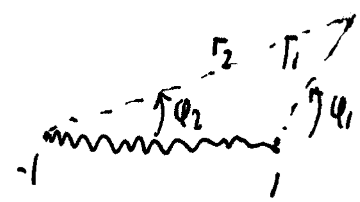
RECALL $e^w \simeq 1 + w + \cdots$ SO WITH $w = \dfrac{1}{2}\log\left(1 - \dfrac{1}{z^2}\right) \rightarrow e^{\frac{1}{2}\log(1 - 1/z^2)} \simeq 1 - \dfrac{1}{2z^2}$.

THUS $f(z) \simeq z\left[1 - \dfrac{1}{2z^2} + \cdots\right]$ AS $|z| \rightarrow \infty$ SO $f(z) \simeq z - \dfrac{1}{2z} + \cdots$ AND

THIS GIVES $a_{-1} = -1/2$.

METHOD 2 BY RANGE OF ANGLES

$$f(z) = r_1^{1/2}\, r_2^{1/2}\, e^{i(\varphi_1 + \varphi_2)/2}$$



WITH $-\pi < \varphi_j \leq \pi$, $j = 1, 2$.

NOW FOR $|z| >> 1$ $f(z) = \pm z\left[1 - 1/z^2\right]^{1/2}$ WHERE WE

MUST DECIDE ON $\pm$ SIGN. SO $f(z) \simeq \pm z\left[1 - 1/2z^2 + \cdots\right]$ BY BINOMIAL THM.

• LET $Z = X$ WITH $X > 0$ AND $X \to \infty$. THEN $\varphi_1 = \varphi_2 = 0$ AND $r_1 = (X-1)$ AND $r_2 = (X+1)$ SO $f(z) = [X^2-1]^{1/2} e^{i0} \hat{=} X$.

WE CONCLUDE THAT FROM $f(z) \hat{=} \pm z\left[1 - 1/2z^2 + \ldots\right]$ WE NEED + SIGN

• ALTERNATIVELY, LET $Z = iy$ WITH $y > 0$ AND $y \to +\infty$. THEN

$\varphi_1 = \varphi_2 = \pi/2$ AND $f(z) \hat{=} y e^{i\pi/2} \hat{=} iy$ THIS IS CONSISTENT WITH CHOOSING + SIGN AND LETTING $Z = iy$ IN $f(z) \hat{=} z\left(1 - 1/2z^2 + \ldots\right)$.

AS SUCH $f(z) = z\left(1 - 1/2z^2 + \ldots\right) = z - 1/2z + \ldots$ SO $a_{-1} = -1/2$ $\square$.

NOW WE APPLY THE RESIDUE OF INFINITY CONCEPT TO FINDING THE ROOTS OF POLYNOMIALS. SUPPOSE THAT $p(z) = z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0$ IS A POLYNOMIAL OF DEGREE N WITH ROOTS AT $Z = Z_j$, $j = 1, \ldots, m$ EACH OF MULTIPLICITY n_j. THEN BY FACTORING WE HAVE

$$p(z) = \prod_{j=1}^{m} (z - z_j)^{n_j}$$

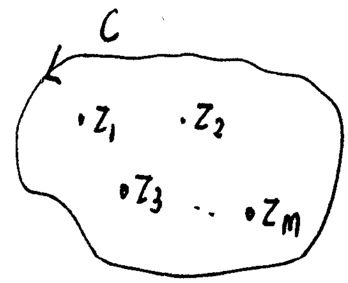
WHERE n_j IS A POSITIVE INTEGER WITH $n_1 + \ldots + n_m = N$

AND $\prod_{j=1}^{m} X_j = X_1 X_2 \cdots X_m$. WE THEN CALCULATE BY PRODUCT RULE THAT

$$\frac{p'(z)}{p(z)} = \frac{\sum_{j=1}^{m} n_j (z - z_j)^{n_j - 1} \prod_{k \neq j}^{m} (z - z_k)^{n_k}}{\prod_{k=1}^{m} (z - z_k)^{n_k}} = \sum_{j=1}^{m} \frac{n_j}{z - z_j}.$$

AS SUCH IF WE TAKE A CONTOUR C THAT ENCLOSES ALL THE ROOTS Z_j WE HAVE BY RESIDUE THEOREM

$$\frac{1}{2\pi i} \int_C \frac{p'(z)}{p(z)}\, dz = \sum_{j=1}^{m} n_j = N.$$



MORE GENERALLY IF C ONLY ENCLOSES THE POLES $z_1, \dots, z_{m'}$ WITH $m' < m$ WE HAVE

$$\frac{1}{2\pi i}\int_C \frac{p'(z)}{p(z)}\,dz = \sum_{j=1}^{m'} n_j = \#\text{ zeroes OF } p(z) \text{ inside } C$$

counting multiplicity.

FOR SIMPLE ROOTS OF $p(z)=0$ WE HAVE $n_j = 1$, $\forall j = 1, \dots, N$ WITH $m = N$.

WE WOULD NOW LIKE TO LINK THE COEFFICIENTS OF THE POLYNOMIAL TO SOME PROPERTIES OF THE ROOTS OF THE POLYNOMIAL.

<u>RESULT 1</u> $\displaystyle\sum_{j=1}^{m} n_j z_j = -a_{N-1}.$

<u>PROOF</u> FROM THE RESIDUE THEOREM IF WE TAKE ALL ROOTS z_j OF $p(z)=0$ inside C, THEN

$$\frac{1}{2\pi i}\int_C z\frac{p'(z)}{p(z)}\,dz = \frac{1}{2\pi i}\sum_{j=1}^{m}\int_C \frac{z_j n_j}{z-z_j}\,dz = \sum_{j=1}^{m} z_j n_j .$$

NOW DEFORM C TO $C_R : |z| = R$ COUNTERCLOCKWISE AND LET $R \to \infty$. SINCE THERE ARE NO SINGULARITIES BETWEEN C AND C_R WE HAVE

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R}\frac{z\,p'(z)}{p(z)}\,dz = \sum_{j=1}^{m} z_j n_j .$$

NOW FOR $|z| >> 1$,

$$\frac{z\,p'(z)}{p(z)} = \frac{z\left[Nz^{N-1} + (N-1)a_{N-1}z^{N-2} + \cdots\right]}{z^N + a_{N-1}z^{N-1}} \simeq \frac{Nz^N\left[1 + \frac{(N-1)}{N}\frac{a_{N-1}}{z}\right]}{z^N\left[1 + a_{N-1}/z + \cdots\right]}$$

SO

$$\frac{z\,p'(z)}{p(z)} \simeq N\left[1 + \frac{(N-1)}{N}\frac{a_{N-1}}{z}\right]\left[1 - \frac{a_{N-1}}{z} + \cdots\right] \simeq N\left[1 + \frac{a_{N-1}}{z}\left(\frac{(N-1)}{N} - 1\right)\right]$$

$$\simeq N\left[1 + \frac{a_{N-1}}{z}\left[-\frac{1}{N}\right] + \cdots\right] = N - \frac{a_{N-1}}{z} + \cdots \qquad \text{SO } a_{-1} = -a_{N-1}.$$

THEREFORE, $\lim_{R\to\infty} \frac{1}{2\pi i}\int_{C_R} \frac{z\,p'(z)}{p(z)}\,dz = -a_{N-1}$.

WE CONCLUDE THAT $a_{N-1} = -\sum_{j=1}^{m} z_j n_j$. □

RESULT 2 $\sum_{j=1}^{m} z_j^2 n_j = a_{n-1}^2 - 2a_{n-2}$ FOR $n \geq 2$.

PROOF BY RESIDUE THEOREM $\frac{1}{2\pi i}\int_C \frac{z^2 p'(z)}{p(z)}\,dz = \sum_{j=1}^{m} z_j^2 n_j$,

WHEN C ENCLOSES ALL THE ZEROS. NOW USE RESIDUE AT INFINITY BY DEFORMING TO $C_R: |z| = R$ WITH $R >> 1$ SO THAT

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R} \frac{z^2 p'(z)}{p(z)}\,dz = \sum_{j=1}^{m} z_j^2 n_j. \qquad (*)$$

WE CALCULATE $\frac{z^2 p'(z)}{p(z)} \simeq \frac{z^2[Nz^{N-1} + (N-1)a_{N-1}z^{N-2} + (N-2)a_{N-2}z^{N-3} + \ldots]}{z^N + a_{N-1}z^{N-1} + a_{N-2}z^{N-2}}$

$$\frac{z^2 p'(z)}{p(z)} \simeq \frac{z^{N+1}}{z^N}\frac{[N + (N-1)a_{N-1}/z + (N-2)a_{N-2}/z^2]}{[1 + a_{N-1}/z + a_{N-2}/z^2]} \simeq z\frac{\left[N + (N-1)\frac{a_{N-1}}{z} + \frac{(N-2)a_{N-2}}{z^2} + \ldots\right]}{[1 + a_{N-1}/z + a_{N-2}/z^2]}.$$

NOW FOR $|h| << 1$, $\frac{1}{1+h} \simeq 1 - h + h^2 + \ldots$ SO THAT

$$\frac{z^2 p'(z)}{p(z)} \simeq z\left[N + \frac{(N-1)a_{N-1}}{z} + \frac{(N-2)a_{N-2}}{z^2}\right]\left[1 - \frac{a_{N-1}}{z} + \frac{1}{z^2}\left(a_{N-1}^2 - a_{N-2}\right)\right] + \ldots$$

$$\simeq zN - a_{N-1} + \frac{1}{z}\left[N a_{N-1}^2 - (N-1)a_{N-1}^2 - N a_{N-2} + (N-2)a_{N-2} + \ldots\right]$$

$$\simeq zN - a_{N-1} + \frac{1}{z}\left[a_{N-1}^2 - 2a_{N-2}\right] + \ldots$$

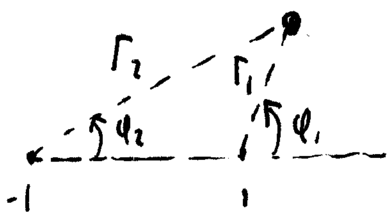
WE CONCLUDE THAT $a_{-1} = a_{N-1}^2 - 2a_{N-2} \longrightarrow$ FROM $(*)$ $\sum_{j=1}^{m} z_j^2 n_j = a_{N-1}^2 - 2a_{N-2}$.

□

EXAMPLE 1 CALCULATE $I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx.$

SOLUTION DEFINE $f(z) = \frac{\sqrt{z^2-1}}{z^2+1}$. WANT A BRANCH CUT ALONG $-1 \le \mathrm{Re}\, z \le 1$, WITH $\mathrm{Im}\, z = 0$.

WE HAVE POLES AT $z = \pm i$.



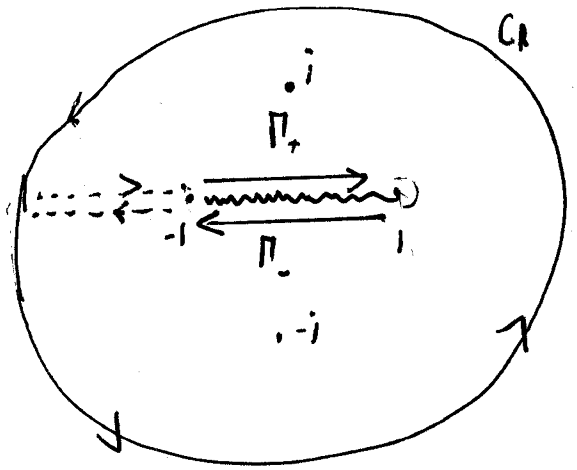
BY RANGE OF ANGLES:

$$\sqrt{z^2-1} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

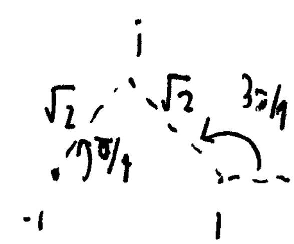
$$-\pi < \varphi_j \le \pi, \; j = 1, 2.$$

OR EQUIVALENTLY $\sqrt{z^2-1} = z\left[1 - 1/z^2\right]^{1/2} = z\, e^{\frac{1}{2}\log(1 - 1/z^2)}$

NOW CONSIDER DOGBONE CONTOUR



$$\lim_{R\to\infty}\left(\left(\int_{\Gamma_+} + \int_{\Gamma_-} + \int_{C_R}\right) f(z)\,dz\right) = 2\pi i\left[\mathrm{RES}(f; i) + \mathrm{RES}(f; -i)\right].$$



NOW RES $(f; i)$ $\varphi_1 = 3\pi/4$, $\varphi_2 = \pi/4$, $r_1 = r_2 = \sqrt{2}$

SO $$\mathrm{RES}[f; i] = \frac{(2^{1/2}\, 2^{1/2})^{1/2}\, e^{i\pi/2}}{2i} = \frac{1}{\sqrt{2}}.$$

RES $(f; -i)$ $\varphi_1 = -3\pi/4$, $\varphi_2 = -\pi/4$, $r_1 = r_2 = \sqrt{2}$.

$$\mathrm{RES}[f; -i] = \frac{(2^{1/2} \cdot 2^{1/2})^{1/2}\, e^{-i\pi/2}}{-2i} = \frac{1}{\sqrt{2}}.$$

SO $$\lim_{R\to\infty}\left(\left(\int_{\Gamma_+} + \int_{\Gamma_-} + \int_{C_R}\right) f(z)\,dz\right) = 2\pi i\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 2\pi i\sqrt{2}.$$

NOW RESIDUE AT $Z = \infty$

WE WORK WITH $f(z) = \dfrac{z e^{\frac{1}{2}\log(1 - 1/z^2)}}{z^2+1}$.

NOW $\dfrac{1}{1-w} = 1 + w + w^2 + \ldots$ FOR $|w| << 1$.

$\longrightarrow \; -\log(1-w) = w + w^2/2 + w^3/3 + \ldots$ SINCE $\log(1) = 0$.

SO WITH $w = 1/z^2 \longrightarrow \log(1 - 1/z^2) \cong -1/z^2$ FOR $|z| >> 1$.

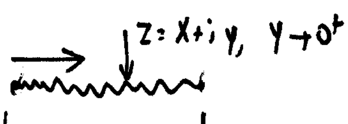
NOW $e^w \cong 1 + w \longrightarrow e^{1/2\log(1-1/z^2)} \cong e^{-1/2z^2} \cong 1 + O(1/|z|^2)$ FOR $|z| >> 1$.

THIS IMPLIES $f(z) \cong \dfrac{z}{z^2+1}\left(1 + O(1/z^2)\right) \cong \dfrac{1}{z} + O(1/|z|^2)$ FOR $|z| >> 1$.

WE THEN CALCULATE $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 2\pi i(1) = 2\pi i$.

THIS YIELDS $\displaystyle\int_{\Gamma_+} + \int_{\Gamma_-} = 2\pi i\sqrt{2} - 2\pi i = 2\pi i(\sqrt{2} - 1)$. (*)

NOW ON Γ_+



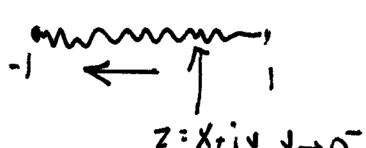
$\varphi_1 = \pi$, $\varphi_2 = 0$. LET $z = x$, $dz = dx$. $r_2 r_1 = (1-x)(1+x)$

$(z^2-1) = \sqrt{1-x^2}\, e^{i\pi/2} = i\sqrt{1-x^2}$.

SO $f(z) = \dfrac{i\sqrt{1-x^2}}{1+x^2}$.

WE GET $\displaystyle\int_{\Gamma_+} = \int_{-1}^{1} \frac{i\sqrt{1-x^2}}{1+x^2}\,dx = iI$, $\quad I = \displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$

NOW ON Γ_-



$\varphi_1 = -\pi$, $\varphi_2 = 0$, $dz = dx$, $r_1 = 1-x$, $r_2 = 1+x$.

$\displaystyle\int_{\Gamma_B} = \int_{1}^{-1} \frac{\sqrt{(1-x)(x+1)}}{x^2+1}\, e^{-i\pi/2}\,dx = i\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$.

SINCE $\displaystyle\int_{\Gamma_+} = i\,I$ AND $\displaystyle\int_{\Gamma_-} = i\,I$ WE PUT INTO (*)

TO OBTAIN THAT $2i\,I = 2\pi i\left(\sqrt{2}-1\right)$.

$$\rightarrow \quad I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \left(\sqrt{2}-1\right)\pi. \quad \square$$

REMARK TO FIND RESIDUE AT ∞ WE CAN ALSO WORK WITH

$$f(z) = \frac{\sqrt{z^2-1}}{z^2+1}.$$

FOR $|z| >> 1$, $\sqrt{z^2-1} \simeq \pm z$. (*)

WHICH SIGN IS CONSISTENT WITH BRANCH CHOSEN?

. ON POSITIVE REAL AXIS $z = x >> 1$. THERE $\varphi_1 = \varphi_2 = 0$ SO THAT

$\sqrt{z^2-1} \simeq x\, e^{i0} = x$. THIS MEANS WE NEED + SIGN IN (*)

HENCE $$f(z) = \frac{z\left[1-1/z^2\right]^{1/2}}{z^2+1} \simeq \frac{z}{z^2+1}\left(1-\frac{1}{2z^2}\right) = \frac{1}{z} + O\left(\frac{1}{|z|^2}\right)$$

FOR $|z| >> 1$.

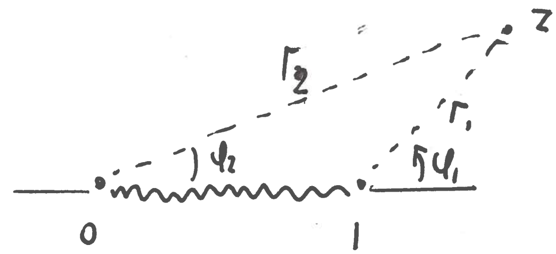
<u>EXAMPLE</u> CALCULATE $I = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\,dx$ FOR $0<\alpha<1$

<u>SOLUTION</u> DEFINE $f(z) = z^{\alpha-1}(z-1)^{-\alpha}$. WE WANT A BRANCH CUT ALONG $0 \le \operatorname{Re} z \le 1$ WITH $\operatorname{Im} z = 0$

$r_1 = |z-1|$, $r_2 = |z|$

(1) SO $f(z) = r_1^{-\alpha} r_2^{\alpha-1} e^{i[-\varphi_1\alpha + (\alpha-1)\varphi_2]}$

WHERE $-\pi < \varphi_j \le \pi$, $j=1,2$.

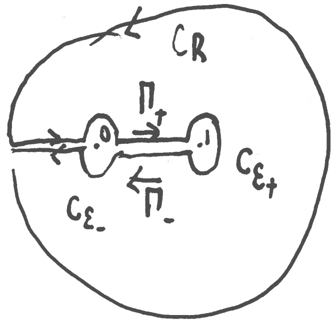


EQUIVALENTLY, CHOOSING A BRANCH OF log WE HAVE

$$f(z) = z^{\alpha-1} z^{-\alpha}\left(1-1/z\right)^{-\alpha} = \frac{1}{z}\, e^{-\alpha \operatorname{LOG}(1-1/z)} \qquad (2)$$

THIS HAS A BRANCH CUT ON $0 \le \operatorname{Re} z \le 1$.

NOW CONSIDER DOG-BONE CONTOUR:



- NOW ON C_{ε_+}: $z-1 = \varepsilon e^{i\varphi}$

SO $\left|\int_{C_{\varepsilon_+}} f(z)\,dz\right| \le O(\varepsilon^{-\alpha})\, 2\pi\varepsilon = O(\varepsilon^{1-\alpha}) \to 0$

AS $\varepsilon \to 0^+$ SINCE $\alpha<1$

- NOW ON C_{ε_-}: $z = \varepsilon e^{i\varphi}$

$\left|\int_{C_{\varepsilon_-}} f(z)\,dz\right| \le O(\varepsilon^{\alpha-1})\, 2\pi\varepsilon = O(\varepsilon^{\alpha}) \to 0$

AS $\varepsilon \to 0^+$ SINCE $\alpha > 0$.

AS A RESULT BY THE RESIDUE THEOREM

$$\left(\int_{\Pi_+} + \int_{\Pi_-}\right) f(z)\,dz = -\lim_{R\to\infty}\int_{C_R} f(z)\,dz. \qquad (*)$$

NOW BY (2) WE HAVE FOR $|z| >> 1$ THAT $\operatorname{LOG}(1-1/z) = O(1/z)$

SO THAT $f(z) \approx \frac{1}{z} + O\left(\frac{1}{z^2}\right)$. $\rightarrow a_{-1} = 1 \rightarrow \int_{C_R} f(z)\,dz = 2\pi i$.

WE CONCLUDE THAT

$$\left(\oint_{\Gamma_+} + \oint_{\Gamma_-} \right) f(z)\, dz = -2\pi i \qquad (**)$$

<u>NOW ON Γ_+</u>

$z = x + iy,\ y \to 0^+$

$0 \quad x \quad 1$

$$\varphi_1 = \pi, \quad \varphi_2 = 0, \quad r_1 = 1 - x, \quad r_2 = x, \quad dz = dx$$

$$\text{so } f(z) = (1-x)^{-\alpha} x^{\alpha-1} e^{-i\pi\alpha}.$$

$$\Rightarrow \oint_{\Gamma_+} f(z)\, dz = \int_0^1 (1-x)^{-\alpha} x^{\alpha-1} e^{-i\pi\alpha}\, dx.$$

<u>NOW ON Γ_-</u>

$0 \quad 1$

$z = x + iy,\ y \to 0^-$

$$\varphi_1 = -\pi, \quad \varphi_2 = 0, \quad r_1 = 1 - x, \quad r_2 = x, \quad dz = dx$$

$$\text{so } f(z) = (1-x)^{-\alpha} x^{\alpha-1} e^{i\pi\alpha}.$$

$$\Rightarrow \oint_{\Gamma_-} f(z)\, dz = \int_1^0 (1-x)^{-\alpha} x^{\alpha-1} e^{i\pi\alpha}\, dx = -\int_0^1 (1-x)^{-\alpha} x^{\alpha-1} e^{i\pi\alpha}\, dx$$

SO FROM (**) WE HAVE

$$\left(e^{-i\pi\alpha} - e^{i\pi\alpha}\right) \underbrace{\int_0^1 (1-x)^{-\alpha} x^{\alpha-1}\, dx}_{\equiv I} = -2\pi i$$

$$\text{OR} \qquad \left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right) I = 2\pi i$$

$$2i \sin(\pi\alpha)\, I = 2\pi i$$

$$\text{OR} \qquad I = \frac{\pi}{\sin(\pi\alpha)} = \int_0^1 (1-x)^{-\alpha} x^{\alpha-1}\, dx$$

$$0 < \alpha < 1.$$