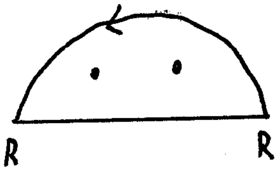
EXAMPLE

$$I = \int_{-\infty}^{\infty} \frac{1-x}{1-x^5}\,dx = \int_{-\infty}^{\infty} \frac{dx}{1+x+x^2+x^3+x^4}.$$



- $z=1$ REMOVABLE.
- $z^5 = 1 \rightarrow z = e^{2\pi i/5\, n}$ $\quad n = 0, \underset{\text{UPPER}}{1, 2}, 3, 4$

LET $\quad z_1 = e^{2\pi i/5}, \quad z_2 = e^{4\pi i/5}$.

NOW $\quad I = 2\pi i\,[\,\mathrm{RES}(f; z_1) + \mathrm{RES}(f; z_2)\,]$ WITH $f(z) = \dfrac{1-z}{1-z^5}$.

THIS GIVES SINCE, z_1, z_2 SIMPLE POLES

$$I = 2\pi i \left[\frac{1-z_1}{-5z_1^4} + \frac{1-z_2}{-5z_2^4} \right] = -\frac{2\pi i}{5} \left[\frac{1-z_1}{z_1^4} + \frac{1-z_2}{z_2^4} \right]$$

USING $z_i^5 = 1$, $i = 1, 2$;

$$I = -\frac{2\pi i}{5} \left[\frac{z_1(1-z_1)}{z_1^5} + \frac{z_2(1-z_2)}{z_2^5} \right] = -\frac{2\pi i}{5} \left[z_1 + z_2 - z_1^2 - z_2^2 \right].$$

BUT NOW $z_2 = z_1^2$ SO THAT

$$I = -\frac{2\pi i}{5} \left[z_1 - z_2^2 \right]$$

BUT $z_2^2 = z_1^4$ SO THAT $\quad I = -\dfrac{2\pi i}{5} \left[z_1 - z_1^4 \right] = -\dfrac{2\pi i}{5} \left[z_1 - \dfrac{z_1^5}{z_1} \right].$

BUT $z_1^5 = 1$ SO THAT

$$I = -\frac{2\pi i}{5} \left[z_1 - \frac{1}{z_1} \right].$$ RECALL $\quad \sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2i}$.

WITH $\varphi = 2\pi/5 \rightarrow \sin(2\pi/5) = \dfrac{e^{2\pi i/5} - e^{-2\pi i/5}}{2i} \rightarrow 2i\sin(2\pi/5) = z_1 - 1/z_1$

SO $$I = -\frac{2\pi i}{5} \left(2i \sin(2\pi/5)\right) = \frac{4\pi}{5} \sin\left(2\pi/5\right).$$

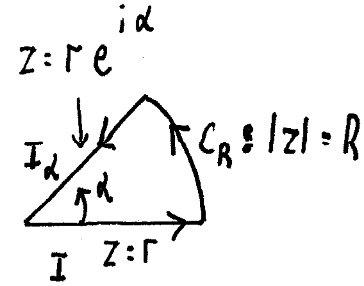
WEDGE CONTOURS

LET $m, n \in \{0, 1, 2, \dots\}$ WITH $n \geq m+2$. EVALUATE

$$I = \int_0^\infty \frac{x^m}{1+x^n}\, dx.$$

SOLUTION

CHOOSE A WEDGE CONTOUR WITH ANGLE α,



AND LABEL $f(z) = \dfrac{z^m}{1+z^n}$

THEN BY RESIDUE THM: $\lim\limits_{R\to\infty} \left(I + I_\alpha + I_{C_R} \right) = 2\pi i \sum\limits_j \mathrm{RES}[f, z_j]$ inside contour.

ON I_α: $z = re^{i\alpha}$ SO THAT $\quad I_\alpha = \displaystyle\int_R^0 e^{i\alpha} \frac{r^m e^{i\alpha m}}{1+r^n e^{i\alpha n}}\, dr$

$dz = e^{i\alpha}\, dr$

CHOOSE α SO THAT I_α IS <u>PROPORTIONAL</u> TO I: THIS GIVES $\alpha n = 2\pi$

OR $\alpha = 2\pi/n$. THEN $\quad I_\alpha = -e^{2\pi i(m+1)/n} \displaystyle\int_0^\infty \frac{r^m}{1+r^n}\, dr$ AS $R \to \infty$.

THE ONLY SINGULARITY INSIDE IS WHERE $z^n = -1 \rightarrow z = z_1 = e^{\pi i/n}$. $\rightarrow$ simple pole

THUS AS $R \to \infty$ WE HAVE

$$I - e^{2\pi i(m+1)/n} I + \lim_{R\to\infty} \int_{C_R} f(z)\, dz = 2\pi i\, \mathrm{RES}[f; z_1] = 2\pi i \frac{z_1^m}{n z_1^{n-1}}.$$

NOW $\lim\limits_{R\to\infty} \left| \displaystyle\int_{C_R} f(z)\, dz \right| \to 0$ SINCE $\left| \displaystyle\int_{C_R} f\, dz \right| \leq \underbrace{\text{LENGTH}(C_R)}_{O(R)}\, O\!\left(R^{n-m}\right) \to 0$

SINCE $n \geq m+2$. NOW THIS GIVES

$$I\left(1 - e^{2\pi i(m+1)/n}\right) = \frac{2\pi i\, z_1^{m+1}}{n z_1^n} = -\frac{2\pi i}{n} z_1^{m+1} = -\frac{2\pi i}{n} e^{\frac{\pi i}{n}(m+1)} \quad \text{SINCE } z_1^n = -1.$$

MULTIPLYING BY $-e^{-\pi i(m+1)/n}$ BOTH SIDES $\rightarrow \quad I\left(e^{\pi i(m+1)/n} - e^{-\pi i(m+1)/n}\right) = \dfrac{2\pi i}{n}$

$$\rightarrow \quad I = \frac{\pi}{n}\left(\frac{2i}{e^{\pi i(m+1)/n} - e^{-\pi i(m+1)/n}}\right) \quad \rightarrow \quad I = \frac{\pi}{n \sin\left(\frac{\pi(m+1)}{n}\right)}.$$