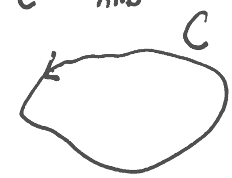
ROUCHE'S THEOREM LET C BE A SIMPLE CLOSED CURVE AND ASSUME THAT f, g ARE ANALYTIC INSIDE AND ON C AND ASSUME THAT



$$|g(z)| < |f(z)| \text{ ON } C.$$

THEN, $f(z)$, $f(z)+g(z)$ HAVE THE SAME NUMBER OF ZEROES INSIDE C (COUNTING MULTIPLICITY).

PROOF BY $|g|<|f|$ ON C, IT FOLLOWS THAT $f(z)$ HAS NO ZEROS ON C. WE FIRST ASSUME THAT IF $f(z_j)=0$ WITH z_j INSIDE C THEN $g(z_j) \neq 0$. DEFINE

$$\mathcal{F}(z) \equiv \frac{f(z)+g(z)}{f(z)}.$$

BY RESIDUE THEOREM,

$$\frac{1}{2\pi i}\int_C \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\,dz = N_0(\mathcal{F}) - N_p(\mathcal{F})$$

WHERE $N_0(\mathcal{F}) = \#$ ZEROES OF $\mathcal{F}(z)$ IN C, i.e. $\#$ ZEROES OF $f+g$ IN C

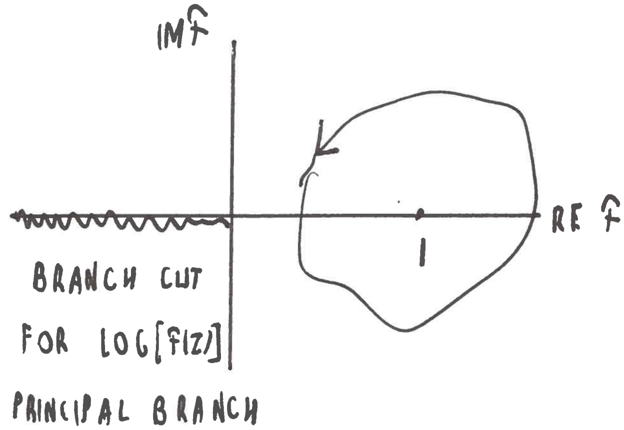
$N_p(\mathcal{F}) = \#$ poles of $\mathcal{F}(z)$ in C, i.e. $\#$ zeroes of f in C.

SINCE ZEROES OF $f+g$ AND OF f ARE TWO DISTINCT SETS SINCE WE ASSUMED $g(z_j) \neq 0$ WHEN $f(z_j)=0$ WE HAVE

$$\frac{1}{2\pi i}\int_C \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\,dz = N_0(f+g) - N_0(f). \qquad (*)$$

NEXT WE CALCULATE LHS OF $(*)$. WE FIRST OBSERVE THAT $|\mathcal{F}(z) - 1| < 1$ ON C.

PROOF $$|\mathcal{F}(z)-1| = \left|\frac{f+g}{f} - 1\right| = \left|\frac{g}{f}\right| < 1 \text{ ON } C.$$



AS SUCH, AS Z TRAVERSES C WE ALWAYS HAVE AN ANALYTIC ANTI-DERIVATIVE $\mathrm{LOG}[f(z)]$ FOR WHICH

$$\frac{d}{dz} \mathrm{LOG}[f(z)] = \frac{f'(z)}{f(z)}.$$

THEREFORE, $|f(z)-1|<1$ WHEN $z \in C$ MEANS WE ARE NEVER ON THE BRANCH CUT OF $\mathrm{LOG}[f(z)]$. HENCE BY FUNDAMENTAL THEOREM OF CALCULUS FOR A SIMPLE CLOSED CURVE C,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i} \mathrm{LOG}[f(z)]\Big|_{z_i}^{z_f} = 0 \quad \text{SINCE } z_i = z_f.$$

IT FOLLOWS FROM (*) THAT $N_0(f+g) - N_0(f) = 0$

I.E. THE # ZEROES OF $f(z)$ AND OF $f(z)+g(z)$ INSIDE C ARE THE SAME.

<u>REMARK</u> DOES THEOREM STILL HOLD IF ZEROES OF $f(z)+g(z)$ AND OF $f(z)$ COINCIDE? YES.

ASSUME z_0 IS A p-FOLD ZERO OF $f+g$ AND A q-FOLD ZERO OF f.

(I) . IF $p>q$, F HAS A $p-q$ FOLD ZERO AT z_0.

(II) . IF $p=q$, F HAS A REMOVABLE SINGULARITY AT z_0

(III) . IF $p<q$ F HAS A $q-p$ FOLD POLE AT z_0.

SO IN EITHER CASE THE CONTRIBUTION OF z_0 TO $\frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}\,dz$ IS $p-q$.

IT FOLLOWS THAT $N_0(f+g) - N_0(f)$ HAS A TERM $p-q$ FOR z_0.

NOTICE THAT CASE (I) CAN NEVER OCCUR SINCE IF $g(z_0)=0$ OF ORDER q WITH $q<p$ F HAS REMOVABLE SINGULARITY.

EXAMPLE 1 LET $p(z) = z^5 + z + 1$

PROVE THAT p HAS 5 ZEROES IN $1/2 < |z| < 2$.

PROOF . LET $C: |z| = 2$, AND LET $f = z^5$, $g = z + 1$.

THEN $|f| = 2^5$ ON C AND $|g| \leq |z| + 1 = 3$ BY Δ-INEQUALITY

SO $|g| < |f|$ ON C,

AND f HAS FIVE ZEROES IN $|z| \leq 2 \longrightarrow$ BY ROUCHE THEN $f + g = p$ HAS FIVE ZEROES INSIDE $|z| \leq 2$.

. NOW LET $C: |z| = 1/2$ AND LET $f = 1$ AND $g = z^5 + z$.

THEN $|f| = 1$ ON C AND $|g| \leq |z|^5 + |z| = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right) = \frac{17}{32}$.

THUS, $|g| < |f|$ ON C.

BUT f HAS NO ZEROES INSIDE C, AND SO $f + g = p$ HAS NO ZEROES INSIDE $|z| \leq 1/2$. NOW COMBINE THE TWO RESULTS.

HENCE $\exists$ 5 ROOTS IN $1/2 \leq |z| \leq 2$.

EXAMPLE 2 PROVE FUNDAMENTAL THEOREM OF ALGEBRA THAT EVERY POLYNOMIAL OF DEGREE n HAS n ROOTS.

SKETCH OF PROOF LET $p(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_0$.

LET $C: |z| = R$ WITH $R >> 1$. DEFINE $f = z^n$ AND $g = a_{n-1} z^{n-1} + \dots + a_0$.

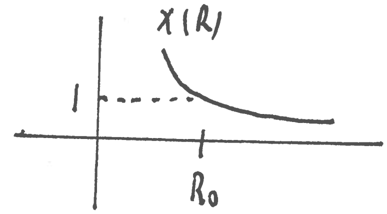
NOW ON C WE HAVE $|f| = R^n$. WE ESTIMATE BY Δ INEQUALITY THAT $|g| \leq |a_{n-1}| R^{n-1} + |a_{n-2}| R^{n-2} + \dots + |a_0|$.

WE MUST CHOOSE R_0 SUCH THAT $|g| < |f|$ ON C.

I.E. THAT $|a_{n-1}| R^{n-1} + |a_{n-2}| R^{n-2} + \dots + |a_0| < R^n$ FOR $R > R_0$.

DEFINE $\chi(R) = \dfrac{|a_{n-1}|}{R} + \dfrac{|a_{n-2}|}{R^2} + \dots + \dfrac{|a_0|}{R^n}$.

AS $R \to \infty$, $\chi(R) \to 0$, χ CONTINUOUS. HENCE FOR R SUFFICIENTLY BIG, $\exists R_0$ FOR WHICH $\chi(R) < 1$ $\forall R > R_0$



THUS $|g| < |f|$ ON C WHEN $R > R_0$.

IT FOLLOWS THAT f, $f+g$ HAVE SAME # ZEROES inside $|z| = R$, FOR ANY $R > R_0$. BUT f HAS n ZEROES, SO $\longrightarrow$ p HAS n ZEROES $\blacksquare$.

<u>EXAMPLE 3</u> DETERMINE THE NUMBER OF ZEROES OF

$$p(z) = z^{28} - \sin(z^3) + 10z \quad \text{IN THE ANNULUS} \quad 1 < |z| < 3.$$

<u>SOLUTION</u> WE FIRST WILL ESTIMATE $|\sin w| = \left| \dfrac{e^{iw} - e^{-iw}}{2i} \right| \leq \dfrac{1}{2}(|e^{iw}| + |e^{-i}$

BY Δ INEQUALITY IN A DISK $|w| \leq \rho$.

WE USE $|e^s| \leq e^{|s|}$ (PROOF IS $|e^{x+iy}| = e^x \leq e^{(x^2+y^2)^{1/2}}$)

AND $|iw| = |w|$ SO $|\sin w| \leq \frac{1}{2}(e^\rho + e^\rho) = e^\rho$, IN $|w| \leq \rho$.

NOTE: IN $|z| \leq 1 \longrightarrow |w| \leq 1$ WITH $w = z^3$ SO $|\sin z^3| \leq e^1$ IN $|z| \leq 1$

IN $|z| \leq 3 \rightarrow |w| \leq 27$ WITH $w = z^3$ SO $|\sin z^3| \leq e^{27}$ IN $|z| \leq$

- THUS IN $|z| \leq 1$ WE DEFINE $f = 10z$ AND $g = z^{28} - \sin(z^3)$.

  BY Δ INEQUALITY $|g| < |z|^{28} + |\sin z^3| \leq 1 + e^1 \cong 3.7$ ON $|z| = 1$, AND $|f| = 10$ ON $|z| = 1$. THUS BY ROUCHE, SINCE f HAS 1 ZERO IN $|z| \leq 1$

  IT FOLLOWS THAT p HAS 1 ZERO IN $|z| \leq 1$.

- NOW ON $|z| = 3$ LET $f = z^{28}$ $g = \sin(z^3) + 10z$. WE HAVE $|f| = 3^{28}$ ON C AND $|g| \leq 30 + e^{27}$ BY Δ-INEQUALITY. SINCE $e = 2.718..$ CLEARLY $|g| < |f|$ ON $C: |z| = 3$. THUS f AND $p = f + g$ HAVE 28 ZEROES IN $|z| < 3$.

$\longrightarrow$ COMBINING THE TWO RESULTS $\longrightarrow \exists$ 27 ZEROES OF p IN $1 < |z| < 3$.

TWO FURTHER PROBLEMS WITH ROOTS OF POLYNOMIALS.

PROPOSITION (*) LET $h(z)$ BE ANALYTIC inside AND ON C (closed curve)

THEN IF $|h-1|<1$ FOR z ON C, THEN h HAS NO zeroes inside C.

PROOF WE CAN SET $f=1$ AND $g=h-1$ IN ROUCHE'S THEOREM.

WE CAN ALSO PROVE BY MAX-MODULUS PRINCIPLE AS FOLLOWS:

LET $f=h-1$, THEN f IS ANALYTIC IN AND ON C AND SO

BY MAX-MODULUS PRINCIPLE FOR ANY z INSIDE C

$$|f| \leq \max_{z \text{ on } C} |f(z)| < 1 \quad \text{(BY ASSUMPTION)}$$

(NOTE: $|f|<1$ ON C $\rightarrow$ $\max_{z \text{ on } C} |f| < 1$ C IS COMPACT

THUS $|f|<1$ FOR EACH z IN C.

HENCE THERE IS NO POINT z_0 IN C WITH $|f(z_0)|=1$

HENCE h HAS NO ZEROES IN C.

EXAMPLE PROVE TWO WAYS THAT $p(z)=6z^3+z^2+z+1$ HAS

ALL OF ITS ZEROES IN $|z|\leq 1$.

PROOF 1 LET $f=6z^3$ AND $g=z^2+z+1$ IN ROUCHE. LET $C:|z|=1$

THEN $|f|=6$ ON C AND $|g|\leq |z|^2+|z|+1=3$ ON C.

SO $|g|<|f|$ ON C $\rightarrow$ SINCE f HAS 3 ZEROES IN C

THEN SO DOES $p=f+g$. (ROUCHE)

PROOF 2 DEFINE $q(w)=w^3p(1/w)=w^3+w^2+w+6$. WE WANT

TO PROVE THAT q HAS NO ZEROES IN $|w|\leq 1 \rightarrow p(z)$ HAS NO

ZEROES IN $|z|\geq 1 \rightarrow$ ALL ZEROES OF p ARE IN $|z|<1$.

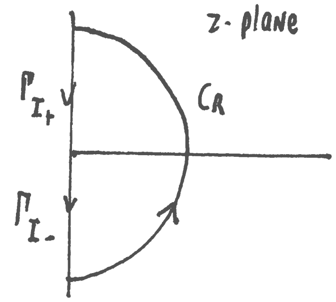
LET $q'=q/6=1+w^3/6+w^2/6+w/6$ AND $|q'-1|\leq \frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{3}<1$

BY Δ-INEQUALITY. THUS PROPOSITION (*) IMPLIES NO ROOTS IN $|w|<1$ $\square$.

EXAMPLE PROVE THAT THE POLYNOMIAL $p(z) = z^4 + z^3 + 5z^2 + 2z + 4$

HAS NO ZEROS IN $\mathrm{RE}\, z > 0$.

SOLUTION LET $C = C_R \cup \Gamma_{I_+} \cup \Gamma_{I_-}$ AS SHOWN



BY THE ARGUMENT PRINCIPLE

$[\arg p]_C = 2\pi N$, $N = \#$ ZEROS OF p INSIDE C
(ASSUME $p \neq 0$ ON C).

NOW LET $R \to \infty$, $\lim_{R\to\infty} \left([\arg p]\big|_{C_R} + [\arg p]\big|_{\Gamma_{I_+}} + [\arg p]\big|_{\Gamma_{I_-}} \right) = 2\pi N$

$N = \#$ zeros of p
in $\mathrm{RE}\, z > 0$.

BUT SINCE p IS POLYNOMIAL OF degree 4, $\lim_{R\to\infty} [\arg p]\big|_{C_R} = 4\pi$.

ALSO SINCE p HAS REAL COEFFICIENTS $[\arg p]\big|_{\Gamma_{I_-}} = [\arg p]\big|_{\Gamma_{I_+}}$. THUS,

$(*)$ $2\pi N = 4\pi + 2[\arg p]\big|_{\Gamma_{I_+}}$ WHERE $\Gamma_{I_+} : 0 < y < \infty$, $z = iy$ downwards.

WE CALCULATE $p(iy) = p_R(y) + i\, p_I(y) = y^4 - 5y^2 + 4 + i(2y - y^3)$

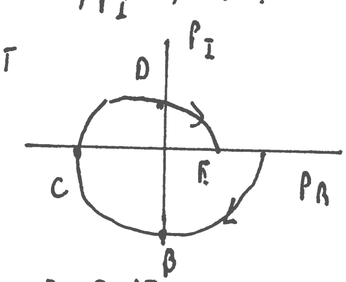
WE HAVE $p_R = 0$ WHEN $y = 1, 4$ AND $p_I = 0$ WHEN $y = 0, \sqrt{2}$.

OUR TABLE IS

	y	p_R	p_I
A	∞	>0	<0
B	4	$=0$	<0
C	$\sqrt{2}$	<0	$=0$
D	1	$=0$	>0
E	0	>0	$=0$

AS $y \to +\infty$ p_R, p_I IN FOURTH QUADRANT WITH
$p_R / p_I \to -\infty$.

THUS WE GET



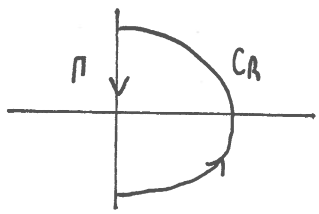
WE CONCLUDE THAT

$[\arg p]\big|_{\Gamma_{I_+}} = -2\pi$ (clockwise)

SO $N = 0$ BY $(*)$.

EXAMPLE LET $p(z) = a - z - e^{-z}$. PROVE THAT $p(z) = 0$ HAS ONE ROOT IN $\mathrm{Re}\, z > 0$ WHEN $a > 1$ REAL.

PROOF WE USE ROUCHE'S THEOREM ON $C = C_R \cup \Gamma$ Γ IMAGINARY AXIS.



LET $f = a - z$

AND $g = -e^{-z}$

IN ROUCHE.

WE HAVE $|g| = e^{-x} \leq 1$ IN $\mathrm{Re}\, z \geq 0$.

NOW ON Γ: $|f| = |a - z| = |a - iy| = \sqrt{a^2 + y^2} \geq a$

THUS IF $a > 1$ WE HAVE $|g| \leq 1 < a \leq |f|$ ON Γ

$\longrightarrow$ $|g| < |f|$ ON Γ.

CLEARLY $|g| < |f|$ ON C_R FOR ANY $R > 0$

(SINCE $|g| \leq 1$ IN $\mathrm{Re}\, z \geq 0$).

THUS IF $a > 1$, $|g| < |f|$ ON C AND SO BY ROUCHE, f, $f + g = p$ HAVE SAME NUMBER OF ZEROS IN C. LET $R \to \infty$ SO f, p HAVE SAME # ZEROS IN $\mathrm{Re}\, z > 0$. BUT $f = a - z$ HAS A ZERO AT $z = a$;

HENCE $p = 0$ HAS A UNIQUE ROOT IN $\mathrm{Re}\, z > 0$.

SINCE ROOTS OF $p = 0$ COME IN COMPLEX conjugate pairs (easily proved), IT follows that this ROOT IS REAL.

EXAMPLE SUPPOSE f IS ANALYTIC IN $|z| \leq 1$ AND $|f| < 1$ FOR $|z| = 1$. FIND THE NUMBER OF SOLUTIONS TO $f(z) = z^n$ IN $|z| < 1$.

PROOF LET $g = -f$, $\hat{F} = z^n$ WITH $g, \hat{F}$ IN ROUCHE.

WE HAVE $|\hat{F}| = 1$ ON $|z| = 1$, $|g| = |f| < 1$ ON $|z| = 1$ $\longrightarrow$ $|g| < |\hat{F}|$ ON $|z| = 1$.

THUS $\hat{F}$ AND $\hat{F} + g = z^n - f$ HAVE SAME NUMBER OF ZEROS IN $|z| < 1$.

$\longrightarrow$ $z^n - f = 0$ HAS n ZEROS IN $|z| < 1$.