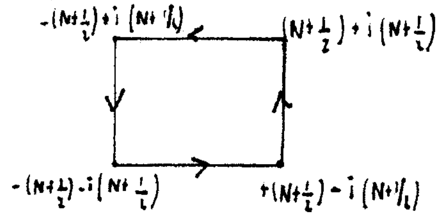
SUMMATION OF SERIES (WATSON TRANSFORM)

CONSIDER $\displaystyle\int_{C_N} \pi \cot(\pi z)\, f(z)\, dz$ WHERE C_N IS SQUARE WITH VERTICES AT $\pm(N+1/2) \pm i(N+1/2)$



NOW THE INTEGRAND HAS RESIDUES AT SINGULARITIES OF $\cot(\pi z)$ AND SINGULARITIES OF $f(z)$.

$\cot \pi z = \infty \longrightarrow \sin(\pi z) = 0 \quad z_K = \pm K \quad K = 0, 1, 2, \ldots$ ARE SIMPLE POLES.

NOW ASSUME $f(z_K) \neq 0$ AND THAT f HAVE SINGULARITIES AT ζ_j FOR $j = 1, 2, \ldots, S$, WHERE $\cot(\pi \zeta_j) \neq 0$. THEN

$$\lim_{N\to\infty} \int_{C_N} \pi \cot(\pi z)\, f(z)\, dz = 2\pi i \sum_{K=-\infty}^{\infty} \mathrm{RES}\left[\pi \cot(\pi z) f ; K\right] + 2\pi i \sum_{j=1}^{S} \mathrm{RES}\left[\pi \cot(\pi z) f ; \zeta_j\right].$$

NOW $\mathrm{RES}\left[f \pi \cot(\pi z) ; K\right] = f(K)\, \pi \dfrac{\cos(\pi K)}{\pi \cos(\pi K)} = f(K).$

$$\longrightarrow (*) \lim_{N\to\infty} \int_{C_N} \pi \cot(\pi z)\, f(z)\, dz = 2\pi i \sum_{K=-\infty}^{\infty} f(K) + 2\pi i \sum_{j=1}^{S} \mathrm{RES}\left[\pi \cot(\pi z) f ; \zeta_j\right].$$

NOW IT CAN BE SHOWN THAT USING THE Δ-INEQUALITY

$$|\cot \pi z| = \left|\frac{\cos \pi z}{\sin \pi z}\right| = \left|\frac{e^{\pi z i} + e^{-\pi z i}}{e^{\pi z i} - e^{-\pi z i}}\right| \le \frac{|e^{i\pi z}| + |e^{-\pi z i}|}{\left||e^{\pi z i}| - |e^{-\pi z i}|\right|} = \frac{e^{-\pi \mathrm{Im}(z)} + e^{\pi \mathrm{Im}(z)}}{\left|e^{-\pi \mathrm{Im}(z)} - e^{\pi \mathrm{Im} z}\right|}$$

NOW AS $z \to \infty i$ IN UPPER 1/2 PLANE $|\cot \pi z|$ is bounded

$z \to -\infty i$ IN LOWER 1/2 PLANE $|\cot \pi z|$ is bounded

$z \to \infty$ IN right 1/2 PLANE $|\cot \pi z|$ is bounded

$z \to -\infty$ IN LEFT 1/2 PLANE $|\cot \pi z|$ is bounded.

THUS $\displaystyle\left|\int_{C_N} f(z) \cot(\pi z)\, \pi\, dz\right| \le K \left|\int_{C_N} f(z)\, dz\right| \to 0$ AS $N \to \infty$ PROVIDED $|f(z)|\,|z| \to 0$ AS $|z| \to \infty$

$\le K \max_{C_N} |f| \text{ length}(C_N)$

SUBSTITUTE IN (*) TO GET $$0 = 2\pi i \sum_{k=-\infty}^{\infty} f(k) + 2\pi i \sum_{j=1}^{s} \mathrm{Res}\left[\pi \cot(\pi z) f ; \zeta_j\right].$$

$$(\dagger) \qquad \sum_{k=-\infty}^{\infty} f(k) = -\sum_{j=1}^{s} \mathrm{Res}\left[f(z)\pi \cot(\pi z) ; \zeta_j\right]$$

IN $(\dagger)$ we have to ensure that
- $|z f(z)| \to 0$ as $|z| \to \infty$ is needed
- ζ_j is a pole of $f(z)$ AND f is analytic at all other points besides $\zeta_1, \ldots, \zeta_s$.

<u>EXAMPLE</u> SHOW $$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a) \qquad a > 0.$$

let $f(z) = \dfrac{1}{z^2+a^2}$. IT HAS SIMPLE POLES AT $z = \pm ia$. THEN

$$\mathrm{Res}\left[\pi f(z) \cot(\pi z); ia\right] = \frac{\pi \cot(\pi i a)}{2ia} = \frac{-\pi}{2a} \coth(\pi a)$$

$$\mathrm{Res}\left[\pi f(z) \cot(\pi z); -ia\right] = \frac{\pi \cot(-\pi i a)}{-2ia} = \frac{-\pi}{2a} \coth(\pi a)$$

$$\cot(\pi i a) = -i \coth(\pi a).$$

THEREFORE $$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\left[\frac{-\pi}{2a}\coth(\pi a) - \frac{\pi}{2a}\coth(\pi a)\right] = \frac{\pi}{a}\coth(\pi a)$$

$$\longrightarrow \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$$

NOTICE $$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2}$$

$$\longrightarrow 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth(\pi a) \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}.$$

let $a \to 0$. $\tanh z = z - z^3/3 + \ldots \quad z \to 0 \qquad \coth z = \dfrac{1}{(z - z^3/3 + \ldots)} = \dfrac{1}{z(1 - z^2/3 + \ldots)} = \dfrac{1}{z}(1 + z^2/3 + \ldots)$

$\coth(\pi a) = \dfrac{1}{\pi a}\left(1 + \pi^2 a^2/3 + \ldots\right) \quad a \to 0.$ HENCE $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \to \frac{\pi}{2a}\,\frac{1}{\pi a}\left(1 + \frac{\pi^2 a^2}{3} + \ldots\right) - \frac{1}{2a^2} \quad a \to$

$$\longrightarrow \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6.$$

MITTAG-LEFFLER GENERALIZED PARTIAL FRACTIONS

WE NOW DERIVE SOME RESULTS INVOLVING TRIGONOMETRIC FUNCTIONS THAT CAN BE VIEWED AS "PARTIAL FRACTION DECOMPOSITIONS" EXCEPT NOW THERE ARE AN INFINITE NUMBER OF TERMS. TWO OF THESE RESULTS ARE

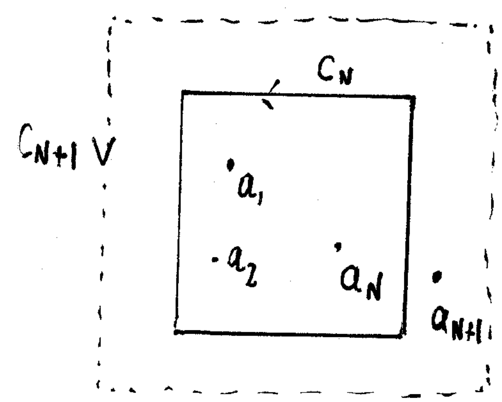
$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi} \right)$$

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi} \right)$$

TO DERIVE THESE RESULTS WE USE THE FOLLOWING MAIN RESULT.

THEOREM (MITTAG-LEFFLER) SUPPOSE THAT $f(z)$ IS A MEROMORPHIC FUNCTION; (i.e. ANALYTIC EXCEPT FOR POLE SINGULARITIES) AND HAS ONLY SIMPLE POLES AT $z_j = a_j$ ORDERED IN SUCH A WAY THAT $0 < |a_1| < |a_2| < \ldots$ SUPPOSE THAT $\text{RES}[f; a_j] = b_j$. MOREOVER, SUPPOSE THAT THE SIMPLE CLOSED COUNTERCLOCKWISE CONTOURS C_N ARE CHOSEN TO CONTAIN $0, a_1, \ldots, a_N$ BUT NO OTHER POLES, AND THAT $C_{N+1} \supset C_N$ FOR ALL $N \geq 1$. LET d_N = MIN DISTANCE FROM C_N TO $z=0$ WITH $d_N \to \infty$ AS $N \to \infty$, AND THAT $\text{LENGTH}(C_N) = O(d_N)$. IF $|f(z)| \leq M$ ON C_N, WITH M, INDEPENDENT OF N, WE HAVE

$$(*) \quad f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{a_n} + \frac{1}{z-a_n} \right).$$



REMARK WE CAN CHOOSE C_N TO BE CIRCLES OR RECTANGLES OR MORE GENERALLY.

PROOF WE SIMPLY APPLY THE RESIDUE THEOREM TO

$$I = \frac{1}{2\pi i} \int_{C_N} \frac{f(w)}{w(w-z)}\, dw \quad \text{AND THEN LET } N \to \infty.$$

BY ASSUMPTION, INSIDE C_N, $f(w)$ HAS POLES AT $w = a_j$, $j = 1, \ldots, N$ THAT ARE SIMPLE. MOREOVER, $w = 0$ IS A SIMPLE POLE IN C_N AND FOR N LARGE ENOUGH $w = z$ IS A SIMPLE POLE IN C_N.

$$\text{RES}\left[\frac{f(w)}{w(w-z)} ; a_j\right] = \frac{b_j}{a_j(a_j - z)}$$

$$\text{RES}\left[\frac{f(w)}{w(w-z)} ; 0\right] = \frac{f(0)}{-z}$$

$$\text{RES}\left[\frac{f(w)}{w(w-z)} ; z\right] = \frac{f(z)}{z}$$

WE HAVE

$$(+) \quad \lim_{N\to\infty} \frac{1}{2\pi i} \int_{C_N} \frac{f(w)}{w(w-z)}\, dw = \frac{f(z)}{z} - \frac{f(0)}{z} + \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{b_n}{a_n(a_n - z)}\right).$$

NOW

$$\left| \frac{1}{2\pi i} \int_{C_N} \frac{f(w)}{w(w-z)}\, dw \right| \leq \frac{1}{2\pi} \max_{C_N} \left| \frac{f(w)}{w(w-z)} \right| \text{LENGTH}(C_N) \leq \frac{M}{O(d_N^2)} O(d_N) \to 0$$

AS $N \to \infty$. SETTING THE LEFT SIDE TO ZERO IN (+) WE SOLVE FOR $f(z)$ TO GET

$$(++) \quad f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z\, b_n}{a_n(z - a_n)}.$$

NOW USING PARTIAL FRACTIONS

$$\frac{A}{z - a_n} + \frac{B}{a_n} = \frac{b_n z}{a_n(z - a_n)} \quad \to \quad A a_n + B(z - a_n) = b_n z$$

WE GET $B = b_n$, $A = a_n$.

PUTTING THIS INTO (++) GIVES

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{a_n} + \frac{1}{z-a_n} \right) \quad \square$$

EXAMPLE IF WE WANT TO WRITE DECOMPOSITION OF $\cot z$ WE MUST MODIFY IT SO AS TO NOT HAVE A POLE AT $z=0$.

SO DEFINE $f(z) = \cot z - \frac{1}{z}$. CLEARLY $|f(z)| \leq M$ (BOUNDED) ON C_N.

- $z=0$ IS NOW A REMOVABLE SINGULARITY (NO POLE).
- POLES OCCUR WHERE $\sin(z)=0 \to z = n\pi$, $n = \pm 1, \pm 2, \ldots$ SO $a_n = n\pi$, $n \neq 0$.
- WRITING $f(z) = \dfrac{z\cot z - 1}{z} = \dfrac{z\cos z - \sin z}{z \sin z}$ WE LABEL $P(z) = \dfrac{z\cos z - \sin z}{z}$

AND $Q(z) = \sin z$, SO $f(z) = \dfrac{P(z)}{Q(z)}$ WITH $P(z_n) = z_n \cos(z_n)/z_n = \cos(z_n)$

$Q(z_n) = 0$, $Q'(z_n) = \frac{d}{dz}\sin z \Big|_{z_n} = \cos(z_n)$

THEREFORE, $\mathrm{RES}[f; z_n] = 1 \quad \forall n \neq 0$.

HENCE SETTING $a_n = n\pi$, $b_n = 1$ WE HAVE

$$(**) \quad f(z) = f(0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{n\pi} + \frac{1}{z-n\pi} \right)$$

TO FIND $f(0)$, WE CALCULATE $f(z) = \dfrac{\cos z}{\sin z} - \dfrac{1}{z} = \dfrac{1 - z^2/2 + \cdots}{z - z^3/6 + \cdots} - \dfrac{1}{z}$

SO $f(z) = \dfrac{1}{z}\left(1 - z^2/6 + \cdots\right)^{-1}\left(1 - z^2/2 + \cdots\right) - \dfrac{1}{z} = O(z)$ AS $z \to 0$.

HENCE $\lim_{z\to 0} f(z) = f(0) = 0$.

NOW WRITING $f(z) = \cot z - 1/z$ WE HAVE FROM (**)

$$\cot z - \frac{1}{z} = 0 + \sum_{n=1}^{\infty} \left(\frac{1}{n\pi} + \frac{1}{z-n\pi} \right) + \sum_{n=-1}^{-\infty} \left(\frac{1}{n\pi} + \frac{1}{z-n\pi} \right)$$

CHANGING INDICES AND COMBINING

$$\cot z - \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{n\pi} + \frac{1}{z-n\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{-1}{n\pi} + \frac{1}{z+n\pi}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right).$$

THIS YIELDS, $$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2\pi^2}. \quad \square$$

<u>REMARK 1</u> CALCULATE THE SUM $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2-1}$.

SET $z=1$ TO GET $$\cot(1) - 1 = 2\sum_{n=1}^{\infty}\frac{1}{(1-n^2\pi^2)}$$

THIS GIVES $$\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2-1)} = \frac{1-\cot(1)}{2}.$$

<u>REMARK 2</u> A SIMILAR RESULT CAN BE OBTAINED FOR $\dfrac{1}{\sin z}$ BY STARTING WITH $f(z) = \dfrac{1}{\sin z} - \dfrac{1}{z}$ TO REMOVE THE POLE AT $z=0$.