MATH 305 SOLUTIONS HW # 2

SECTION 1.5 # 5a

CALCULATE $(-16)^{1/4}$. IN OTHER WORDS, FIND ALL z FOR WHICH $z^4 = -16 = 16e^{i\pi}$.

LET $z = re^{i\varphi} \rightarrow r^4 e^{4i\varphi} = 16e^{i\pi}$.

THEN $r^4 = 16 \rightarrow r = |z| = 2$.

$$\underset{+2k\pi}{4\varphi = \pi} \rightarrow \varphi = \frac{\pi}{4} + \frac{2k\pi}{4}, \quad k = 0, 1, 2, 3.$$

THUS $z = 2e^{i\pi/4 + ik\pi/2}, \quad k = 0, 1, 2, 3$.

SECTION 1.5 # 5e

CALCULATE $(-1+i)^{1/2}$. IN OTHER WORDS, FIND ALL z FOR WHICH $z^2 = -1+i = \sqrt{2}\, e^{3\pi i/4}$.

WE PUT $z = re^{i\varphi}$, $r = |z|$ TO GET

$$r^2 e^{2i\varphi} = \sqrt{2}\, e^{3\pi i/4}.$$

THEN $r^2 = \sqrt{2} \longrightarrow r = 2^{1/4}$

$$2\varphi = 3\pi/4 + 2k\pi, \quad k = 0, 1.$$

$$\rightarrow \varphi = 3\pi/8 + k\pi.$$

HENCE $z = 2^{1/4} e^{3\pi i/8 + ik\pi} \qquad k = 0, 1.$

$$\longrightarrow z_1 = 2^{1/4} e^{3\pi i/8}, \quad z_2 = -2^{1/4} e^{3\pi i/8}.$$

SECTION 1.5 # 11

SOLVE $(z+1)^5 = z^5$.

NOTICE THAT THIS IS A POLYNOMIAL OF DEGREE 4 SINCE THE z^5 TERM CANCELS. THUS THERE ARE 4 ROOTS.

FOR $z \neq 0$, WE WRITE $\left(\frac{z+1}{z}\right)^5 = w^5 = 1$. $\quad w = \frac{z+1}{z}$

THUS $w_k = e^{2\pi i k/5}$ $\quad k = 0, 1, 2, 3, 4$.

HOWEVER $z \neq 0$ SO WE MUST BE CAREFUL.

WE SOLVE $w_k = \frac{z_k+1}{z_k}$ FOR z TO GET $z_k = \frac{1}{w_k - 1}$

SINCE z IS FINITE, WE MUST EXCLUDE $w_k = 1$, i.e. EXCLUDE $k=0$.

HENCE $z_k = \frac{1}{w_k - 1}$, $w_k = e^{2\pi i k/5}$, $k = 1, 2, 3, 4$.

SECTION 1.5 # 16 SHOW THAT ANY ROOT OF $(z+1)^{100} = (z-1)^{100}$ MUST SATISFY $\mathrm{Re}(z) = 0$.

PROOF $(z+1)^{100} - (z-1)^{100}$ IS A POLYNOMIAL OF DEGREE 99 SINCE z^{100} TERM CANCELS. FOR $z \neq 1$ WE WRITE $\left(\frac{z+1}{z-1}\right)^{100} = w^{100} = 1$ WITH $w = \frac{z+1}{z-1}$.

WE OBTAIN $w_k = e^{2\pi i k/100}$, $k = 0, \ldots, 99$. SOLVE FOR z TO OBTAIN

$$w(z-1) = z+1 \longrightarrow z(w-1) = 1+w \longrightarrow z = \frac{1+w}{w-1}.$$

NOW $z \neq \infty$ SO $w \neq 1$. WE MUST EXCLUDE $k=0$. HENCE

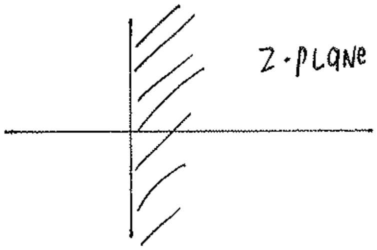
$$z_k = \frac{1+w_k}{w_k - 1}, \quad w_k = e^{2\pi i k/100}, \quad k = 1, \ldots, 99. \qquad \underline{\text{NOTE:}}\ w_k \bar{w}_k = 1.$$

NOW CALCULATE $z_k = \frac{(1+w_k)(\bar{w}_k - 1)}{|w_k - 1|^2} = \frac{w_k\bar{w}_k - 1 - (w_k - \bar{w}_k)}{|w_k - 1|^2} = \frac{-2i\,\mathrm{Im}(w_k)}{|w_k-1|^2}$.

THUS $\mathrm{Re}(z_k) = 0$ $\boxtimes$

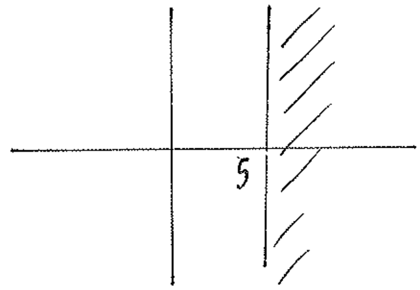
SECTION 2.1 #3

a) $f(z) = z+5$ FOR $\mathrm{RE}(z) > 0$



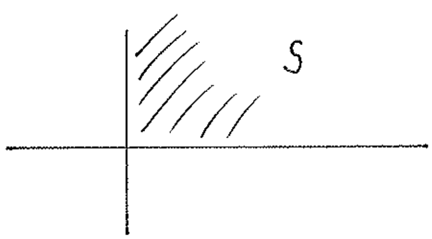
$w = z+5$

translation to right

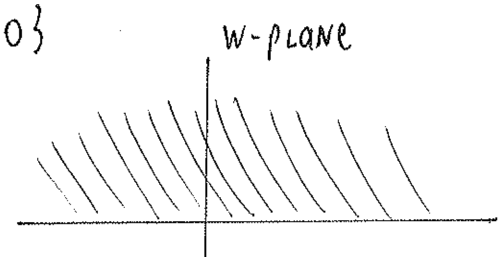


$S' = \{w \mid \mathrm{RE}(w) > 5\}$.

b) $f(z) = z^2$ $S = \{z \mid \mathrm{RE}\,z \geq 0,\ \mathrm{IM}\,z \geq 0\}$



$w = z^2$

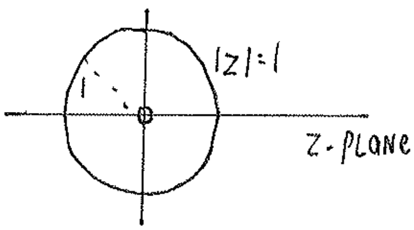


$S' = \{w \mid \mathrm{IM}(w) \geq 0\}$.

LET $z = re^{i\varphi}$ WITH $0 \leq r < \infty$ AND $0 \leq \varphi \leq \pi/2$.

THEN $w = z^2 = r^2 e^{2i\varphi}$ WITH $2\varphi \in [0, \pi] \rightarrow \mathrm{IM}\,W \geq 0$.

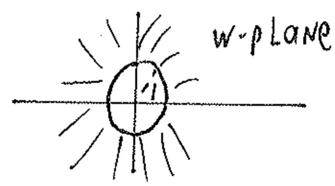
c) $f(z) = 1/z$ FOR $0 < |z| \leq 1$. LET $w = 1/z$ OR $z = 1/w$.



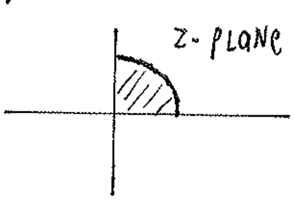
$\rightarrow 0 < |z| \leq 1 \rightarrow 0 < \dfrac{|1|}{|w|} \leq 1$.

THUS $|w| \geq 1$

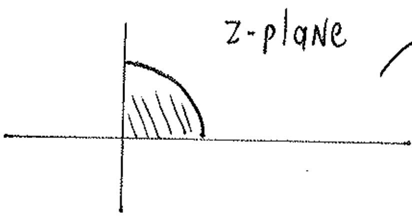
$S' = \{w \mid |w| \geq 1\}$.



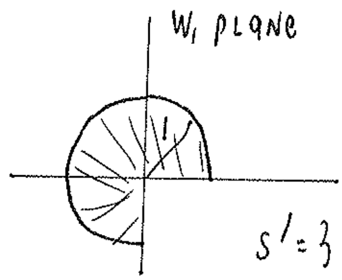
d) $f(z) = -2z^3$ FOR $S = \{z \mid |z| < 1,\ 0 < \mathrm{ARG}\,z < \pi/2\}$.



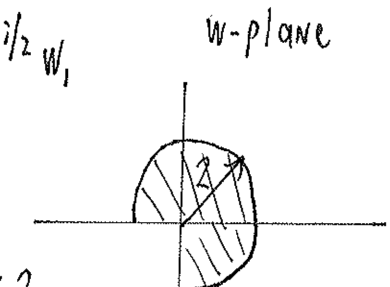
$w = 2e^{3\pi i/2} z^3$. LET $w_1 = z^3$ $w_2 = 2e^{3\pi i/2} w_1$



$w_1 = z^3$



$w = 2e^{3\pi i/2} w_1$



$S' = \{w \mid |w| \leq 2,\ -\pi/2 < \mathrm{ARG}\,w < \pi\}$.

SECTION 2.1 #5

LET $w = f(z) = e^z = e^x(\cos y + i \sin y)$.

(i) THE DOMAIN OF DEFINITION IS THE ENTIRE Z-PLANE $\mathbb{C}$

SINCE $w \neq 0$, THE RANGE IS $\mathbb{C} \setminus \{0\}$, i.e. THE ENTIRE W-PLANE, WITH ORIGIN DELETED.

(ii) SHOW THAT $f(-z) = 1/f(z)$.

PROOF $f(-z) = e^{-x-iy} = e^{-x}[\cos(-y) + i\sin(-y)]$

$$= e^{-x}[\cos y - i \sin y]\frac{[\cos y + i \sin y]}{[\cos y + i \sin y]}$$

$$= \frac{e^{-x}[\cos^2 y + \sin^2 y]}{\cos y + i \sin y} = \frac{1}{e^x(\cos y + i \sin y)}$$

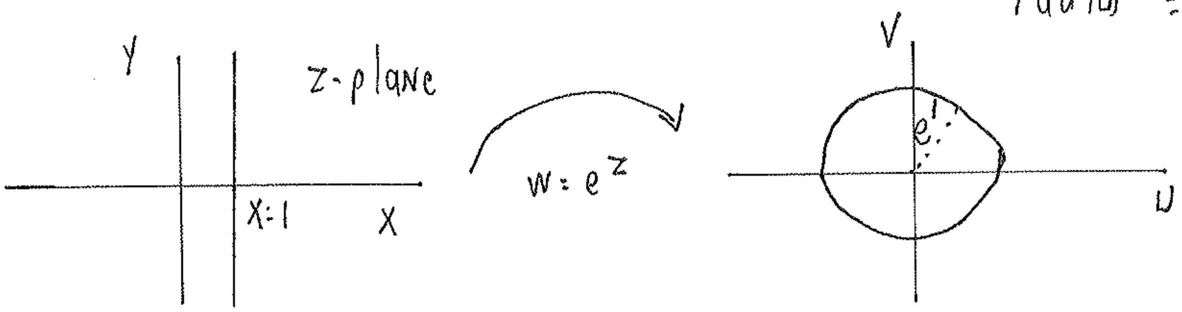
THUS $f(-z) = \dfrac{1}{f(z)}$.

(iii) DESCRIBE IMAGE OF $\mathrm{RE}(z) = 1$.

LET $x = 1$. THEN $w = u + iv = e^1(\cos y + i \sin y)$.

$\longrightarrow$ $u = e^1 \cos y$, $v = e^1 \sin y$

SO $u^2 + v^2 = |e^1|^2 \longrightarrow$ circle center at $(u,v) = (0,0)$ radius $= e^1$.



(iv) DESCRIBE IMAGE OF $\mathrm{IM}\, z = \pi/4$, $-\infty < x < \infty$.

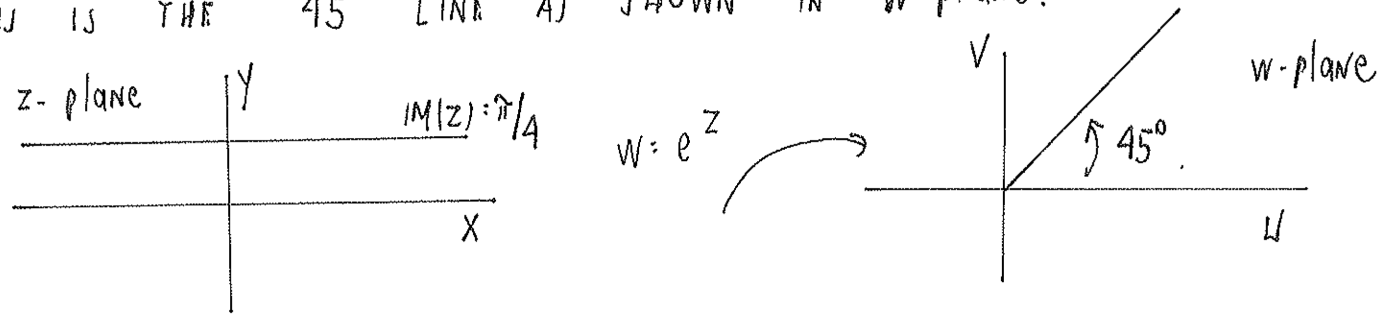
LET $y = \pi/4$, $-\infty < x < \infty$ $\longrightarrow$ $u + iv = e^x[\cos(\pi/4) + i \sin(\pi/4)]$

THUS $\quad u = e^{x}\sqrt{2}/2, \quad v = e^{x}\sqrt{2}/2, \quad -\infty < x < \infty.$

NOTE $(u,v) \to 0$ as $x \to -\infty$

$u = v$

THIS IS THE 45° LINE AS SHOWN IN W-plane.



(V) DESCRIBE THE IMAGE OF $0 \le \mathrm{Im}(z) \le \pi/4$.

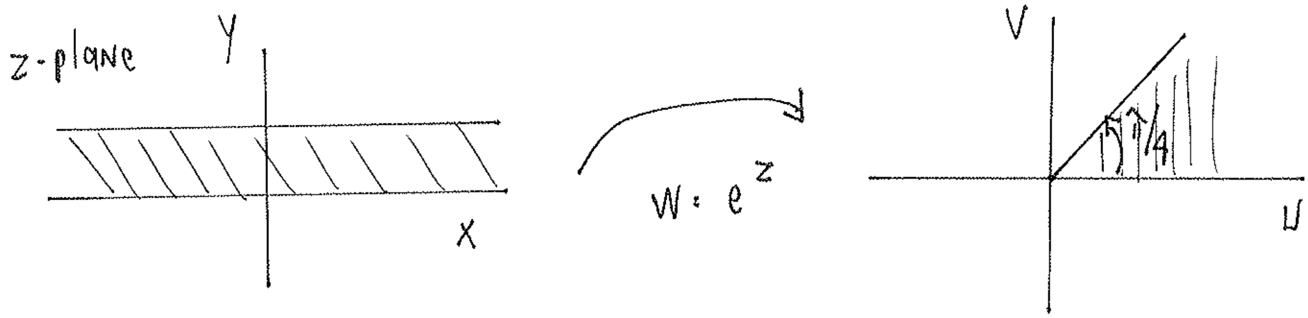
WE LET y SATISFY $0 \le y \le \pi/4$ AND $-\infty < x < \infty$.

THEN $\quad u + iv = e^{x}\cos y + i e^{x}\sin y.$

SO $\quad u = e^{x}\cos y \qquad v = e^{x}\sin y.$

FOR y FIXED $\longrightarrow$ $u/v = \cot y$ OR $v/u = \tan y, \; 0 \le y \le \pi/4$.

THIS IS A FAMILY OF straight LINES with slope IN $0 \le v/u \le 1$.



SECTION 2.1 # 6

THE JOUKOWSKI MAP IS

$$W = J(Z) = \frac{1}{2}\left(Z + \frac{1}{Z}\right)$$

(i) SHOW THAT $J(Z) = J(1/Z)$.

PROOF $J(1/Z) = \frac{1}{2}\left(\frac{1}{Z} + \frac{1}{(1/Z)}\right) = \frac{1}{2}\left(\frac{1}{Z} + Z\right) = J(Z)$ □

(ii) SHOW J MAPS $|Z| = 1$ ONTO $-1 \leq \mathrm{Re}(W) \leq 1$

LET $Z = e^{i\varphi}$ PARAMETRIZE CIRCLE WITH $-\pi < \varphi \leq \pi$.

THEN $J(e^{i\varphi}) = \frac{1}{2}\left(e^{i\varphi} + \frac{1}{e^{i\varphi}}\right) = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) = \frac{1}{2}\, 2\cos\varphi$.

SO $J(e^{i\varphi}) = \cos\varphi$.

SINCE $\varphi \in (-\pi, \pi]$ THEN $W = J(e^{i\varphi})$ IS REAL-VALUED AND $-1 \leq \mathrm{RE}(W) \leq 1$.

(iii) SHOW J MAPS $|Z| = r$, $r > 0$ WITH $r \neq 1$ ONTO AN ELLIPSE.

WE WRITE $Z = re^{i\varphi}$ WITH $-\pi < \varphi \leq \pi$.

THEN, $W = U + iV = J(re^{i\varphi}) = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$

$$U + iV = \frac{1}{2}\left[r\cos\varphi + ir\sin\varphi + \frac{1}{r}(\cos\varphi - i\sin\varphi)\right]$$

$$U + iV = \frac{1}{2}\cos\varphi\left(r + 1/r\right) + i\left[\frac{1}{2}\sin\varphi\left(r - 1/r\right)\right].$$

SO $(*)$ $U = \frac{1}{2}\cos\varphi\left(r + 1/r\right)$, $V = \frac{1}{2}\sin\varphi\left(r - 1/r\right)$

SINCE $-\pi < \varphi \leq \pi \longrightarrow$ $-\frac{1}{2}(r + 1/r) \leq U \leq \frac{1}{2}(r + 1/r)$

$-\frac{1}{2}|r - 1/r| \leq V \leq \frac{1}{2}|r - 1/r|$

SQUARING BOTH SIDES IN $(*)$ AND USING $\sin^2\varphi + \cos^2\varphi = 1$ WE GET

$$U^2 \Big/ \left[\frac{1}{2}(r + 1/r)\right]^2 + V^2 \Big/ \left[\frac{1}{2}(r - 1/r)\right]^2 = 1 \quad \underline{\text{ELLIPSE}}$$

□

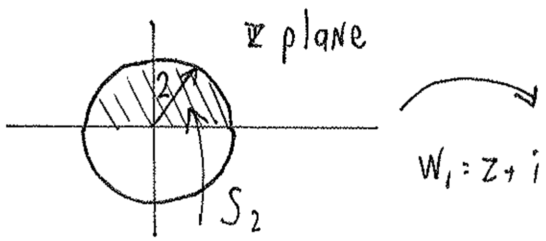
SECTION 2.1 10a)

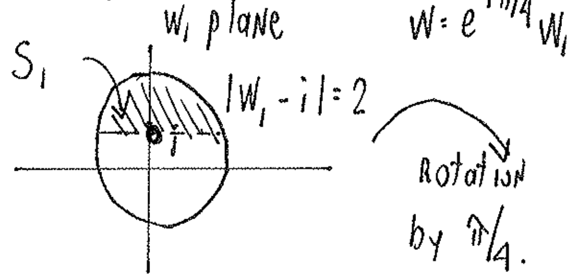
WE CALCULATE $W = G[F(Z)]$ WHERE $F(Z) = Z + i$ AND $G(Z) = e^{i\pi/4} Z$.

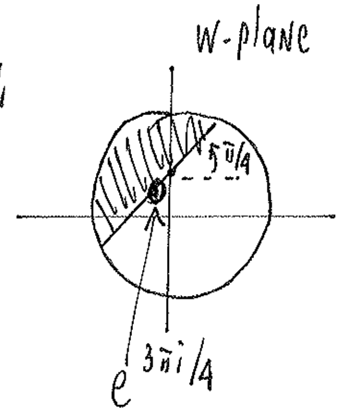
NOW $S = \{ z \mid |z| \le 2, \ IM(z) \ge 0 \}$.

WE DEFINE $W_1 = Z + i$ AND $W = e^{i\pi/4} W_1$. THIS IS EQUIVALENT TO $W = G[F(Z)]$.

WE DRAW A SEQUENCE OF PICTURES







NOW UNDER $W_1 = Z + i$ WE GET

$$S_1 = \{ w_1 \mid |W_1 - i| \le 2 \quad IM(W_1 - i) \ge 0 \}$$

SO $$S_1 = \{ w_1 \mid |W_1 - i| \le 2, \quad IM(W_1) \ge 1 \}$$

$$S' = \{ w \mid |e^{-i\pi/4} w - i| \le 2, \ IM(e^{-i\pi/4} W) \ge 1 \}$$

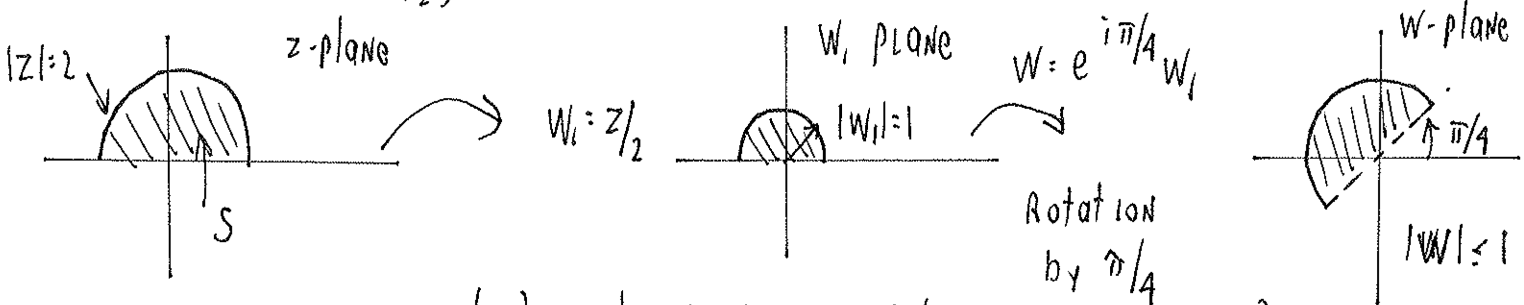
NOW $|e^{-i\pi/4} w - i| = |e^{-i\pi/4}(w - i e^{i\pi/4})| = |w - i e^{i\pi/4}| = |w - e^{3\pi i/4}|$.

THUS $$S' = \{ w \mid |w - e^{3\pi i/4}| \le 2, \ IM(e^{-i\pi/4} w) \ge 1 \}.$$

SECTION 2.1 10b)

WE LET $G = e^{i\pi/4} z$, $H = z/2$. THEN, $W = G[H(z)]$.

WE WRITE $W_1 = z/2$, $W = e^{i\pi/4} W_1$ AS EQUIVALENT TO $W = G[H(z)]$.



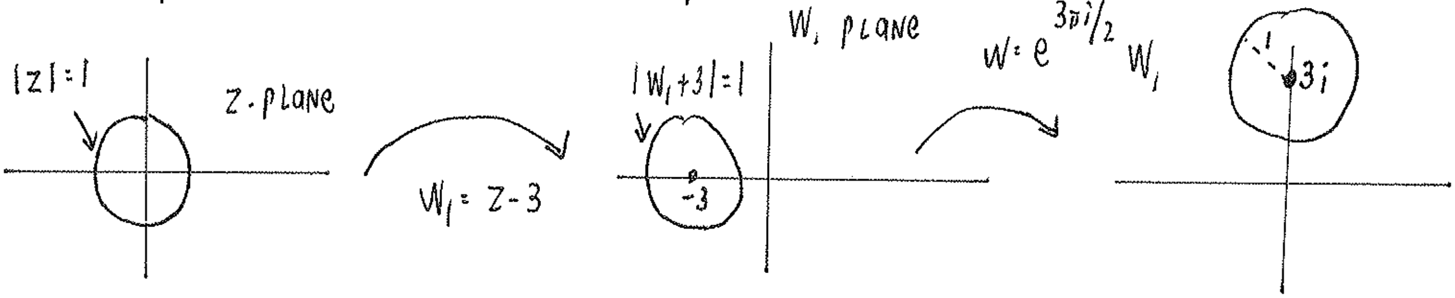
THUS $S' = \{ w \mid |w| \le 1, \ \pi/4 \le \arg(w) \le 5\pi/4 \}$

SECTION 2.1 # 11a

LET $F(z) = z-3$, $G(z) = -iz$. FIND IMAGE OF $|z| = 1$ UNDER $w = F[G(z)]$.

SO $w = -i(z-3) = e^{3\pi i/2}(z-3)$.

LET $w_1 = z-3$ AND $w = e^{3\pi i/2} w_1$



THUS $|e^{-3\pi i/2} w + 3| = 1 \longrightarrow |w + 3e^{+3\pi i/2}| = 1$

IMAGE IS $|w - 3i| = 1$.

PROBLEM 1

PROVE $|e^{-z^3}| \leq 1$ IN $|\arg z| < \pi/6$.

PROOF LET $z = re^{i\varphi}$. SO $z^3 = r^3\left(\cos(3\varphi) + i\sin(3\varphi)\right)$, WITH $r = |z| > 0$.

THEN $$e^{-z^3} = e^{-r^3\cos(3\varphi)}\, e^{-ir^3\sin 3\varphi}$$

THUS $$|e^{-z^3}| = e^{-r^3\cos(3\varphi)} \quad \text{SINCE} \quad |e^{-ir^3\sin 3\varphi}| = 1.$$

NOW $\cos(3\varphi) \geq 0$ IN $-\pi/2 \leq 3\varphi \leq \pi/2 \rightarrow |\varphi| = |\arg z| \leq \pi/6$.

THEN FOR ANY $r > 0$

$$|e^{-z^3}| \leq e^0 = 1 \quad \text{IN} \quad |\arg z| \leq \pi/6$$

PROBLEM 2 CALCULATE

$$I = \cos(\pi/9)\cos(2\pi/9)\cos(4\pi/9)$$

SOLUTION

LET $z = \cos(\pi/9) + i\sin(\pi/9) = e^{i\pi/9}$

THEN
$$\cos(\pi/9) = \frac{e^{i\pi/9} + e^{-i\pi/9}}{2} = \frac{z + 1/z}{2}$$
$$\cos(2\pi/9) = \frac{e^{2\pi i/9} + e^{-2\pi i/9}}{2} = \frac{z^2 + 1/z^2}{2}$$
$$\cos(4\pi/9) = \frac{e^{4\pi i/9} + e^{-4\pi i/9}}{2} = \frac{z^4 + 1/z^4}{2}$$

THUS $$I = \cos(\pi/9)\cos(2\pi/9)\cos(4\pi/9) = \frac{1}{8}\left(z + 1/z\right)\left(z^2 + 1/z^2\right)\left(z^4 + 1/z^4\right)$$

WE MULTIPLY IT OUT

$$(*) \quad I = \frac{1}{8}\left(z^3 + \frac{1}{z} + z + \frac{1}{z^3}\right)\left(z^4 + \frac{1}{z^4}\right) = \frac{1}{8}\left(z^7 + z^3 + z^5 + z + \frac{1}{z} + \frac{1}{z^5} + \frac{1}{z^3} + \frac{1}{z^7}\right)$$

NOW SINCE $z = e^{\pi i/9}$ THEN $z^9 = -1$.

IF $z^9 = -1 \longrightarrow z^7(z^2) = -1 \longrightarrow z^2 = -1/z^7$

SIMILARLY, FOR THE INVERSE POWERS

$$(+) \quad z^2 = -1/z^7, \quad z^6 = -1/z^3, \quad z^4 = -1/z^5, \quad z^8 = -1/z.$$

THUS SUBSTITUTING (+) INTO (*) WE GET

$$I = \frac{1}{8}\left(z^7 + z^3 + z^5 + z - z^8 - z^4 - z^6 - z^2\right)$$

$$(+) \quad I = \frac{1}{8}\left(z - z^2 + z^3 - z^4 + z^5 - z^6 + z^7 - z^8\right)$$

NOW RECALL $1 + w + \ldots + w^8 = \dfrac{1 - w^9}{1 - w}$.

LET $w = -z$ SO THAT $1 + \left(-z + z^2 \cdots + z^8\right) = \dfrac{1 + z^9}{1 + z}$

SO $z - z^2 \ldots - z^8 = 1 - \left(\dfrac{1 + z^9}{1 + z}\right)$

BUT SINCE $z = e^{\pi i/9}$, THEN $z^9 + 1 = 0$ SO THAT

$$z - z^2 \ldots - z^8 = 1$$

<u>FINALLY</u>, (+) GIVES $I = \dfrac{1}{8}\left(1 - \left(\dfrac{1 + z^9}{1 + z}\right)\right) = \dfrac{1}{8}$

THUS $\cos(\pi/9)\cos(2\pi/9)\cos(4\pi/9) = 1/8$.