MATH 300 HW 4

PROBLEM 5a) SECTION 3.2

$$e^{2+i\pi/4} = e^2\left[\cos(\pi/4) + i\sin(\pi/4)\right] = e^2\frac{\sqrt{2}}{2} + ie^2\frac{\sqrt{2}}{2}.$$

PROBLEM 5d) SECTION 3.2

FIND $\cos(1-i)$.

RECALL $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$. SO $\cos(1-i) = \dfrac{e^{i(1-i)}+e^{-i(1-i)}}{2}$

SO $\cos(1-i) = \dfrac{e^1e^i + e^{-i}e^{-1}}{2} = \frac{1}{2}e^1(\cos 1 + i\sin 1) + \frac{e^{-1}}{2}(\cos 1 - i\sin 1)$

SO $\cos(1-i) = \cos 1\left(\dfrac{e^1+e^{-1}}{2}\right) + i\sin 1\left(\dfrac{e^1-e^{-1}}{2}\right) = \cos 1\cosh 1 + i\sin 1\sinh 1.$

PROBLEM 5e) SECTION 3.2

FIND $\sinh[1+i\pi]$.

RECALL $\sinh z = \dfrac{e^z - e^{-z}}{2}$. LET $z = 1 + i\pi$

$$\sinh[1+i\pi] = \frac{1}{2}\left(e^{1+i\pi} - e^{-1-i\pi}\right) = \frac{1}{2}\left[e^1\cos\pi - e^{-1}\cos(\pi)\right]$$

$$= -\frac{1}{2}e^1 + \frac{1}{2}e^{-1} = -\frac{1}{2}(e^1 - e^{-1}) = -\sinh 1.$$

PROBLEM 11 SECTION 3.2

WHY IS $\mathrm{RE}\left[\dfrac{\cos z}{e^z}\right]$ HARMONIC IN WHOLE PLANE.

<u>SOLUTION</u> SINCE $e^z \neq 0$ FOR ANY z WE HAVE

$$f(z) = e^{-z}\cos z \text{ IS ANALYTIC IN WHOLE PLANE}$$

SINCE IT IS THE PRODUCT OF TWO ANALYTIC FUNCTIONS. AS A RESULT $\mathrm{RE}[f(z)]$ IS A HARMONIC FUNCTION.

PROBLEM 15 SECTION 3.2

PROVE THAT $\cos Z = 0$ IFF $Z = \pi/2 + K\pi$, K IS AN INTEGER.

PROOF: $\cos Z = \dfrac{e^{iZ} + e^{-iZ}}{2} = 0$. THUS $e^{iZ} = e^{-iZ}$ SO THAT

$$e^{2iZ} = -1 \rightarrow e^{2iZ} = e^{i\pi}.$$

IT FOLLOWS THAT $2iZ = i\pi + 2K\pi i$ FOR $K = 0, \pm 1, \pm 2, \ldots$

WE CONCLUDE THAT $Z = \dfrac{\pi}{2} + K\pi$, $K = 0, \pm 1, \pm 2, \ldots$

PROBLEM 17a) SECTION 3.2

SOLVE $e^{4Z} = 1$

SOLUTION: $e^{4Z} = e^{2K\pi i}$

SO $4Z = 2K\pi i$, $K = 0, \pm 1, \pm 2, \ldots$ SO $Z = \dfrac{K\pi i}{2}$, $K = 0, \pm 1, \pm 2, \ldots$

PROBLEM 17 b) SECTION 3.2

SOLVE $e^{iZ} = 3$.

SOLUTION WE WRITE $e^{iZ} = e^{\ln 3}$

WE CONCLUDE THAT $iZ = \ln 3 + 2K\pi i$, $K = 0, \pm 1, \pm 2, \ldots$

WE CONCLUDE THAT $Z = 2K\pi - i\ln 3$, $K = 0, \pm 1, \pm 2, \ldots$

PROBLEM 17 c) SECTION 3.2

SOLVE $\cos Z = i\sin Z$.

SOLUTION USE $\cos Z = (e^{iZ} + e^{-iZ})/2$, $\sin Z = (e^{iZ} - e^{-iZ})/2i$

SO THAT $\cos Z = i\sin Z$ GIVES $(e^{iZ} + e^{-iZ})/2 = (e^{iZ} - e^{-iZ})/2$

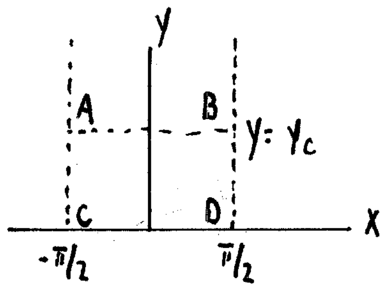
SO THAT $e^{-iZ} = 0$. THIS HAS NO SOLUTION. NO SOLUTION TO $\cos Z = i\sin Z$.

PROBLEM 21 b) SECTION 3.2

FOR $W = \sin Z$ FIND IMAGE OF $S = \{ z \mid -\pi/2 < \operatorname{Re} z < \pi/2,\ \operatorname{Im} z > 0 \}$.

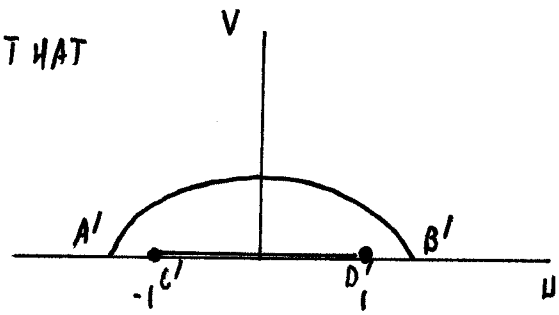
SOLUTION $\quad W = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

LET $W = U + iV$ SO $\quad U = \sin x \cosh y$
$\quad V = \cos x \sinh y$ $\quad$ WITH $y > 0,\ |x| < \pi/2$.



- IF $y > 0$ THEN $V > 0$.
  AND $|x| < \pi/2$
- $U^2/\cosh^2 y_c + V^2/\sinh^2 y_c = 1$ FOR y_c FIXED.
  THUS IS AN ELLIPSE IN UPPER 1/2 PLANE
- IF $y_c = 0$, $U = \sin x$, $V = 0$ SO $|U| \le 1$, $V = 0$.

WE HAVE THAT



IT FOLLOWS THAT THE IMAGE IS THE UPPER 1/2 PLANE $-\infty < U < \infty$ WITH $V > 0$.

PROBLEM 23 SECTION 3.2

PROVE THAT $\sin^2 z + \cos^2 z = 1 \quad \forall z$.

(a) LET $f(z) = \sin^2 z + \cos^2 z$. THE SUMS AND PRODUCTS OF ENTIRE FUNCTIONS ARE ENTIRE. SO $f(z)$ IS ENTIRE.

(b) $f'(z) = 2 \sin z \cos z - 2 \cos z \sin z = 0$ SO $f'(z) = 0$

(c) WE CONCLUDE THAT SINCE $f(z)$ IS ANALYTIC everywhere AND $f'(0) = 0$

WE MUST HAVE $f(z) = \text{CONSTANT} \quad \forall z$.

(d) TO FIND THE CONSTANT WE SET $z = 0$ AND GET

$$f(0) = (\cos 0)^2 + (\sin 0)^2 = 1.$$

(e) HENCE BY (c) AND (d), $f(z) = 1 \quad \forall z$.

PROBLEM 5a) SECTION 3.3

SOLVE $e^z = 2i$.

SOLUTION WE WRITE $i = e^{\pi i/2}$. THEN

$$e^z = 2i = e^{\ln 2 + \pi i/2}.$$

WE CONCLUDE THAT $z = \ln 2 + \frac{\pi i}{2} + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots$

PROBLEM 5 b) SECTION 3.3

SOLVE $\text{LOG}(z^2 - 1) = i\pi/2$ WHERE $\text{LOG}(w)$ IS PRINCIPAL BRANCH.

SOLUTION LET $w = z^2 - 1$. THEN $\text{LOG}(w) = i\pi/2$

WE LET $w = re^{i\varphi} \longrightarrow \text{LOG}(w) = \ln r + i\varphi = i\pi/2, \quad -\pi < \varphi \leq \pi$

THUS $r = 1$, $\varphi = \pi/2$. SO $w = e^{i\pi/2} = z^2 - 1$.

THUS $z^2 = 1 + e^{i\pi/2} = 1 + i = \sqrt{2}\, e^{\pi i/4}$.

NOW PUT $z = re^{i\varphi} \rightarrow r^2 e^{2i\varphi} = \sqrt{2}\, e^{\pi i/4}$.

TAKING MODULUS $\rightarrow r^2 = \sqrt{2}$ SO $r = 2^{1/4}$. ALSO $i2\varphi = i\pi/4 + 2k\pi i$,

$k = 0, 1$.

THUS $\varphi_1 = \pi/8$, $\varphi_2 = 9\pi/8$.

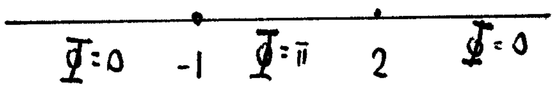
THUS, $z_1 = 2^{1/4} e^{i\pi/8}$ AND $z_2 = 2^{1/4} e^{9\pi i/8}$.

SECTION 3.4 # 4

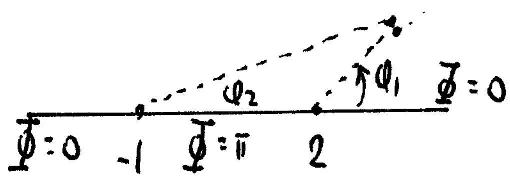
FIND THE SOLUTION TO THE FOLLOWING PROBLEM:

$$\Phi_{xx} + \Phi_{yy} = 0$$

ALSO CALCULATE $\phi(2,3)$.



SOLUTION SINCE $\log z = \ln r + i\varphi$ IS ANALYTIC ~~WE~~ EXCEPT FOR SOME BRANCH CUT, THEN φ IS HARMONIC. HOWEVER, WE NEED THE ANGLES CENTERED AT $z = -1, 2$.



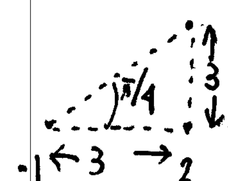
TRY $\Phi = A\varphi_1 + B\varphi_2 + C$.

- IF $\varphi_1 = \varphi_2 = 0 \rightarrow \Phi = 0 \rightarrow C = 0$.
- IF $\varphi_1 = \pi,\ \varphi_2 = 0 \rightarrow \Phi = \pi = A\pi \rightarrow A = 1$
- IF $\varphi_1 = \pi,\ \varphi_2 = \pi \rightarrow \Phi = 0 \rightarrow 0 = A\pi + B\pi$

SO $B = -1$.

THUS $\Phi = \varphi_1 - \varphi_2$.

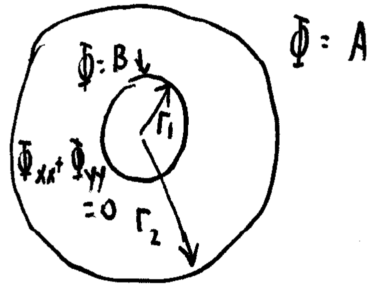
NOW AT $(x,y) = (2,3)$ THEN $\varphi_1 = \pi/2$ AND $\varphi_2 = \pi/4$



SO $\Phi(2,3) = \pi/2 - \pi/4 = \pi/4$.

SECTION 3.4 # 5

SOLVE THE FOLLOWING



SOLUTION WE RECALL THAT $\ln r$ IS THE REAL PART OF AN ANALYTIC FUNCTION IN $r_1 \leq r \leq r_2$, WITH $r = (x^2 + y^2)^{1/2}$.

WE TRY $\Phi = C_1 \ln r + C_2$ WHERE C_1, C_2 TO BE FOUND.

NOW $\Phi = B$ AT $r = r_1 \rightarrow C_1 \ln r_1 + C_2 = B$

$\Phi = A$ AT $r = r_2 \rightarrow C_1 \ln r_2 + C_2 = A$.

SUBTRACTING WE GET $C_1 = \dfrac{B - A}{\ln r_1 - \ln r_2}$

SIMILARLY WITH
$$C_1 + C_2/\ln r_1 = B/\ln r_1$$
$$C_1 + C_2/\ln r_2 = A/\ln r_2$$

SUBTRACTING GIVES: $C_2 \left[\dfrac{1}{\ln r_1} - \dfrac{1}{\ln r_2} \right] = \dfrac{B}{\ln r_1} - \dfrac{A}{\ln r_2}$

$$\rightarrow \quad C_2 = \frac{B \ln r_2 - A \ln r_1}{\ln r_2 - \ln r_1} = \frac{B \ln r_2 - A \ln r_1}{\ln (r_2/r_1)}$$

SO $\Phi = C_1 \ln r + C_2$ WITH

$$C_1 = \frac{B - A}{\ln (r_1/r_2)}, \quad C_2 = \frac{B \ln r_2 - A \ln r_1}{\ln (r_2/r_1)}.$$