MATH 300 SOLUTIONS HW #6

SECTION 4.4 # 10c)

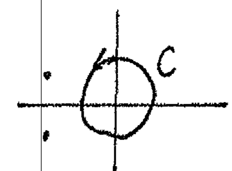
$f(z) = \dfrac{\cos z}{z^2 - 6z + 10}$ Analytic except when $z^2 - 6z + 10 = 0$

so $z^2 - 6z + 9 = -1$

$\rightarrow (z-3)^2 = -1 \rightarrow z = 3 \pm i$.

SINGULAR POINTS AT $z_{\pm} = 3 \pm i$ WITH $|z_{\pm}| = \sqrt{3^2 + 1^2} = \sqrt{10} > 2$.

- THUS ANALYTIC EXCEPT AT $z = z_{\pm}$

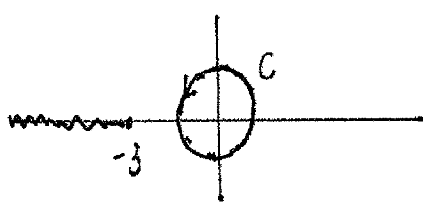


- NOTE $0 = \int_C f(z)\,dz$ WITH $C: |z| = 2$ SINCE singularities are outside C. (CG THEOREM)

SECTION 4.4 # 10d)

$f(z) = \mathrm{LOG}(z+3)$ HAS A BRANCH CUT ALONG $z = -3 + \gamma$ WITH $\gamma < 0$ AND REAL.

I.e. $\mathrm{RE}(z+3) < 0$ WITH $\mathrm{IM}(z+3) = 0$.



IT IS ANALYTIC ELSEWHERE.

IN particular, analytic inside $|z| = 2$

$\longrightarrow \int_C f(z)\,dz = 0$ BY CG THEOREM

SECTION 4.4 # 10e)

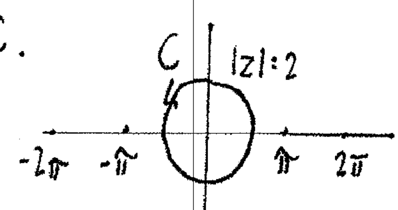
$f(z) = \dfrac{1}{\cos(z/2)}$ LET $\zeta = z/2$ THEN $\cos \zeta = 0$ when

$\zeta = \pi/2 + k\pi \quad k = 0, \pm 1, \pm 2, ..$

THUS $f(z)$ IS NOT ANALYTIC WHEN $z_k = \pi + 2k\pi \quad k = 0, \pm 1, \pm 2, ..$

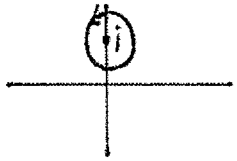
NOTICE THAT $|z_k| > 2$ AND HENCE BY CG $\int_C f(z)\,dz = 0$

WHERE $C: |z| = 2$ SINCE f ANALYTIC INSIDE AND ON C.

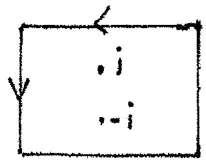


SECTION 4.4 # 13 EVALUATE $\int_C \frac{1}{z^2+1}\,dz$ ALONG 3 CONTOURS AS SHOWN.

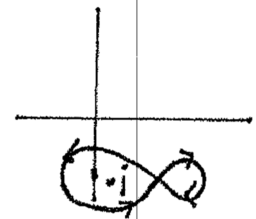
a)



b)



c)



WE WRITE $\frac{1}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i} \longrightarrow 1 = A(z+i) + B(z-i)$

$z = i \longrightarrow A = 1/2i$

$z = -i \longrightarrow B = -1/2i$

$$\int_C \frac{1}{z^2+1}\,dz = \frac{1}{2i}\int_C \frac{1}{z-i}\,dz - \frac{1}{2i}\int_C \frac{1}{z+i}\,dz$$

FOR a) ONLY $z = i$ IS INSIDE CONTOUR. $\longrightarrow \int_C \frac{1}{z+i}\,dz = 0$ AND

THUS $\int_C \frac{1}{z^2+1}\,dz = \frac{1}{2i}\int_C \frac{1}{z-i}\,dz. = \frac{1}{2i}(2\pi i) = \pi.$

FOR b) BOTH $z = i$ AND $z = -i$ ARE INSIDE CONTOUR.

THUS $\int_C \frac{1}{z^2+1}\,dz = \underbrace{\frac{1}{2i}\int_C \frac{1}{z-i}\,dz}_{\text{DEFORM TO A SMALL CURVE AROUND } i} - \underbrace{\frac{1}{2i}\int_C \frac{1}{z+i}\,dz}_{\text{DEFORM TO A SMALL CURVE AROUND } -i} = \frac{1}{2i}(2\pi i) - \frac{1}{2i}(2\pi i) = 0.$

FOR c) ONLY $z = -i$ IS INSIDE C. THUS $\int_C \frac{1}{z-i}\,dz = 0$ AND

WE CAN IGNORE THE SECOND LOOP SINCE IT IS CLOSED AND

$\frac{1}{z+i}$ IS ANALYTIC INSIDE.

THUS $\frac{-1}{2i}\int_C \frac{1}{z+i}\,dz = \int_C \frac{1}{z^2+1}\,dz$

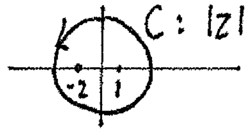
$= -\frac{1}{2i}(2\pi i) = -\pi.$

SO $\int_C \frac{1}{z^2+1}\,dz = -\pi.$

SECTION 4.4 # 15

CALCULATE $I = \int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz$ $\quad \Gamma: |z|=4$ traversed twice in clockwise direction.

CLEARLY $\frac{I}{2} = -\int_C \frac{z}{(z+2)(z-1)}\,dz \;(*)$



C: $|z|=4$

WHERE $C: |z|=4$ counterclockwise. THE singularities are inside C.

partial FRACTIONS: $\frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} \longrightarrow z = A(z-1) + B(z+2)$

$z=1 \longrightarrow B = 1/3$

$z=-2 \longrightarrow A = 2/3$

SO $\int_C \frac{z}{(z+2)(z-1)}\,dz = \int_C \frac{2}{3(z+2)}\,dz + \frac{1}{3}\int_C \frac{1}{z-1}\,dz$

$$= \frac{2}{3}(2\pi i) + \frac{1}{3}(2\pi i) = 2\pi i.$$

NOW FROM $(*)$ $\longrightarrow$ $I/2 = -2\pi i \longrightarrow I = -4\pi i.$

SECTION 4.5 # 16

SUPPOSE $f(z) = \frac{A_K}{z^K} + \frac{A_{K-1}}{z^{K-1}} + \dots + \frac{A_1}{z} + g(z) \quad K \geq 1$

WHERE $g(z)$ is ANALYTIC INSIDE AND ON $|z|=1$. SHOW $\oint_{|z|=1} f(z)\,dz = 2\pi i A_1$

PROOF WE CALCULATE $\oint_{|z|=1} \frac{1}{z^j}\,dz = \begin{cases} 2\pi i, & j=1 \\ 0 & j \neq 1 \end{cases} \quad (*)$

THUS $\oint_{|z|=1} \left(\frac{A_K}{z^K} + \frac{A_{K-1}}{z^{K-1}} + \dots + \frac{A_1}{z} + g(z)\right) dz = A_K \underbrace{\oint_{|z|=1} \frac{1}{z^K}\,dz}_{=0} + \dots + A_1 \overbrace{\oint_{|z|=1} \frac{1}{z}\,dz}^{=2\pi i} + \oint_{|z|=1} g(z)\,dz$

USING $(*)$ $\oint_{|z|=1} f(z)\,dz = 2\pi i A_1 + \underbrace{\oint_{|z|=1} g(z)\,dz}_{=0} = 2\pi i A_1.$

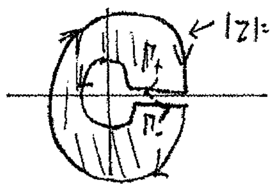
0 since g analytic on and inside $|z|=1$

SECTION 4.4 # 18

LET $I = \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}$ PROVE THAT $I = 0$.

(i) WE DEFORM THE CONTOUR USING CAUCHY-GOURSAT THEOREM

SINCE $f(z) = 1/z^2(z-1)^3$ IS ANALYTIC OUTSIDE $|z| = 2$. BY CG THEOREM,



NOW $\oint_{|z|=R \atop \text{clockwise}} \frac{dz}{z^2(z-1)^3} dz + \int_{\Gamma_+} + \int_{\Gamma_-} + \underbrace{\int_{|z|=2 \atop \text{counterclockwise}} \frac{dz}{z^2(z+1)^3}}_{I} = 0.$

BUT $\int_{\Gamma_+} + \int_{\Gamma_-} = 0$ SO $I = -\oint_{|z|=R \atop \text{clockwise}} \frac{dz}{(z-1)^3 z^2}$

THIS IMPLIES $I = I(R) = \oint_{|z|=R \atop \text{counterclockwise}} \frac{dz}{(z-1)^3 z^2}$ FOR EACH $R > 2$.

(ii) NOW $|I(R)| \leq \max_{|z|=R} \left(\frac{1}{|z-1|^3 |z|^3}\right) 2\pi R$. (*)

BUT $|z-1| \geq (|z|-1) = R-1$ FOR $R > 1$. $\rightarrow \frac{1}{|z-1|^3} \leq \frac{1}{(R-1)^3}$

THUS $\frac{1}{|z-1|^3 |z|^3} \leq \frac{1}{R^2(R-1)^3}$

(*) THEN IMPLIES THAT $|I(R)| \leq \frac{2\pi R}{R^2(R-1)^3}$ FOR ANY $R > 2$.

(iii) CLEARLY IF WE LET $R \to \infty$ THEN $2\pi R / R^2(R-1)^3 = 2\pi / R(R-1)^3 \to 0$

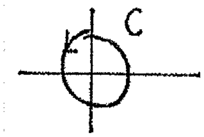
AS $R \to \infty$. THUS $\lim_{R \to \infty} I(R) = 0$.

(iv) SINCE $I = I(R)$ AND $I(R) \to 0$ AS $R \to \infty$ IT

FOLLOWS THAT $I = 0$.

SECTION 4.5 # 3b)

CALCULATE $I = \int_C \frac{ze^z}{2z-3}\,dz \quad C: |z| = 2$ counterclockwise



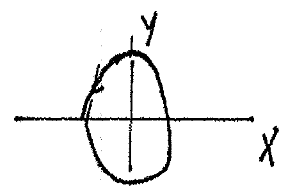
NOTICE $z = 3/2$ singular point is inside C.

RECALL $2\pi i\, f(z_0) = \int_C \frac{f(z)}{z-z_0}\,dz$ IF C simple closed curve containing z_0.

NOW $I = \frac{1}{2}\int_C \frac{(ze^z)}{z-3/2}\,dz = \frac{1}{2} f(3/2)\, 2\pi i, \quad f(z) = ze^z$

THUS $I = \frac{1}{2}\left(3/2\, e^{3/2}\right) 2\pi i = \frac{3\pi i}{2} e^{3/2}$.

SECTION 4.5 # 5) LET C BE ELLIPSE $x^2/4 + y^2/9 = 1$ ORIENTED COUNTERCLOCKWISE.



LET $G(z) = \int_C \frac{s^2 - s + 2}{s-z}\,dz \quad z$ inside C.

FIND $G(1)$, $G'(i)$ AND $G''(-i)$. (NOTICE $i, -i, 1$ ARE INSIDE C)

SOLUTION COMPARE $\int_C \frac{f(s)}{s-z}\,ds = 2\pi i\, f(z)$. HERE $f(s) = s^2 - s + 2$.

THUS $G(z) = 2\pi i\,(z^2 - z + 2)$ AND $G'(z) = 2\pi i\,(2z-1)$, $G''(z) = 4\pi i$

NOW $G(1) = 4\pi i$, $G'(i) = 2\pi i\,(2i-1) = -4\pi - 2\pi i$, $G''(-i) = 4\pi i$.

SECTION 4.5 # 9 SUPPOSE $f(z)$ IS ANALYTIC INSIDE AND ON $C: |z| = 1$. PROVE THAT IF $|f(z)| \le M$ ON C THEN $|f(0)| \le M$, $|f'(0)| \le M$, AND FIND A BOUND FOR $|f^{(n)}(0)|$.

SOLUTION RECALL BY DIFFERENTIATING CAUCHY-INTEGRAL FORMULA THAT

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds \qquad n = 1, 2, \ldots$$

LET $z = 0$ AND $C: |s| = 1 \longrightarrow |f^{(n)}(0)| \le \frac{n!}{|2\pi i|} \max_{\text{on } C} \left|f(s)/s^{n+1}\right| (\text{length of } C)$.

THUS $|f^{(n)}(0)| \le \frac{n!}{2\pi} M (2\pi) = Mn! \longrightarrow |f^{(n)}(0)| \le Mn!$.

IF $n = 1$ THEN $|f'(0)| \le M$.

PROBLEM SECTION 4.5 #4

$$\int_C \frac{z+i}{z^2(z+2)}\,dz$$

singularities at $z=0$, $z=-2$.

(a) THE CIRCLE $C: |z|=1$ (COUNTER CLOCKWISE) SO ONLY $z=0$ IS INSIDE.

LET $f(z) = \dfrac{z+i}{z+2}$. THEN $f(z)$ IS ANALYTIC IN $|z|\le 1$.

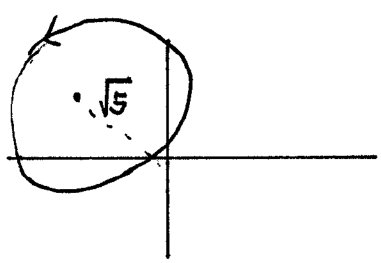
SO $$f'(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-0)^2}\,dz$$

THEN $$\int_C \frac{f(z)}{z^2}\,dz = 2\pi i\, f'(0) \quad \text{WITH } f(z) = (z+i)(z+2)^{-1}$$

THEN, $$f'(0) = \left[(z+2)^{-1} - (z+i)(z+2)^{-2}\right]\Big|_{z=0} = \frac{1}{2} - \frac{i}{4}.$$

THUS, $$\int_C \frac{f(z)}{z^2}\,dz = 2\pi i\left(\frac{1}{2} - \frac{i}{4}\right) = \frac{\pi}{2} + \pi i.$$

(b) NOW LET $C: |z-(i-2)| = 2$. CENTER IS AT $z_0 = i-2$ WITH RADIUS 2.



NOTICE $z=2$ IS inside DOMAIN BUT $z=0$ IS OUTSIDE.

SO LET $f(z) = (z+i)/z^2$. $$\int_C \frac{f(z)}{(z-(-2))} = 2\pi i\, f(-2).$$

BUT $f(-2) = \frac{1}{4}(i-2)$ SO $$\int_C \frac{f(z)}{z+2}\,dz = \frac{2\pi i}{4}(i-2) = -\frac{\pi}{2} - \pi i.$$

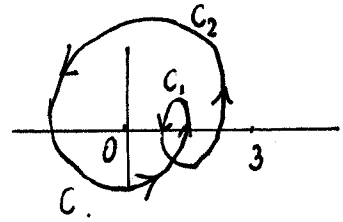
(c) NOW LET $C: |z-2i| = 1$ COUNTERCLOCKWISE. OBSERVE NOW THAT $z=0$ AND $z=-2$ ARE OUTSIDE C.

THUS $f(z) = \dfrac{z+i}{z^2(z+2)}$ IS ANALYTIC INSIDE C.

BY CAUCHY-GOURSAT $$\int_C f(z)\,dz = 0.$$

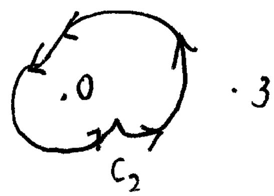
SECTION 4.5 # 7

CALCULATE $I = \int_C \frac{\cos z}{z^2(z-3)}\, dz$



SOLUTION LET $C = C_1 \cup C_2$. THEN SINCE $z = 0, 3$ OUTSIDE INNER LOOP C_1

WE HAVE $$I = \int_{C_2} \frac{\cos z}{z^2(z-3)}\, dz$$



SINCE ONLY $z = 0$ IS INSIDE WE HAVE

$f(z) = \frac{\cos z}{z-3}$ IS ANALYTIC INSIDE C_2.

BY C.I.F. $$I = \oint_{C_2} \frac{f(z)}{z^2} = 2\pi i\, f'(0).$$

NOW $f(z) = \frac{\cos z}{z-3}$ SO $f'(z) = -\frac{\sin z}{z-3} - \frac{\cos z}{(z-3)^2} \rightarrow f'(0) = \frac{-1}{9}.$

THUS $$I = \int_C \frac{\cos z}{z^2(z-3)}\, dz = -\frac{2\pi i}{9}$$ ▢