MATH 300 HW 7 SOLUTIONS

SECTION 4.6 #4

IF $p(z) = a_0 + a_1 z + \dots + a_n z^n$ AND $\max |p(z)| = M$ FOR $|z|=1$ SHOW THAT $|a_k| \le M$.

PROOF $\quad p^{(k)}(z) = k!\, a_k$ AT $z=0$ AS SEEN BY DIFFERENTIATING.

SO $\quad p^{(k)}(0) = k!\, a_k$.

NOW BY CAUCHY'S INEQUALITY OVER A DISK $|z|=1$ WE HAVE

$$|p^{(k)}(0)| \le \frac{k!}{(1)^{k+1}} \max_{|z|=1} |p| \le \frac{M k!}{1}$$

SO $\quad |p^{(k)}(0)| = k!\, |a_k| \le \dfrac{M k!}{1!}$.

THIS IMPLIES THAT $|a_k| \le M$ FOR $k = 1, \dots, n$.

SECTION 4.6 #8

IF $f(z)$ IS ANALYTIC IN $1 \le |z| \le 2$ AND $|f(z)| \le 3$ ON $|z|=1$ AND $|f(z)| \le 12$ ON $|z|=2$ SHOW THAT $|f(z)| \le 3|z^2|^2$ FOR $1 \le |z| \le 2$.

PROOF LET $g(z) = \dfrac{f(z)}{z^2}$. THEN $g(z)$ IS ANALYTIC IN $1 \le |z| \le 2$.

MOREOVER, $\quad |g(z)| \le \left|\dfrac{f(z)}{3z^2}\right| = \dfrac{|f(z)|}{3} \le 1$ ON $|z|=1$

AND $\quad |g(z)| \le \dfrac{|f(z)|}{3|z|^2} \le \dfrac{12}{12} = 1$ ON $|z|=2$.

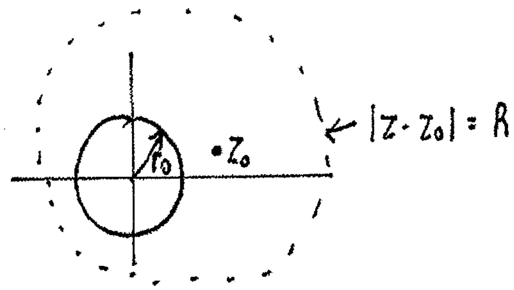
THEREFORE $|g(z)| \le 1$ ON THE BOUNDARY OF $D = \{z \mid 1 \le |z| \le 2\}$.

BY MAX-MODULUS PRINCIPLE IT FOLLOWS THAT $|g(z)| \le 1$ IN $1 \le |z| \le 2$.

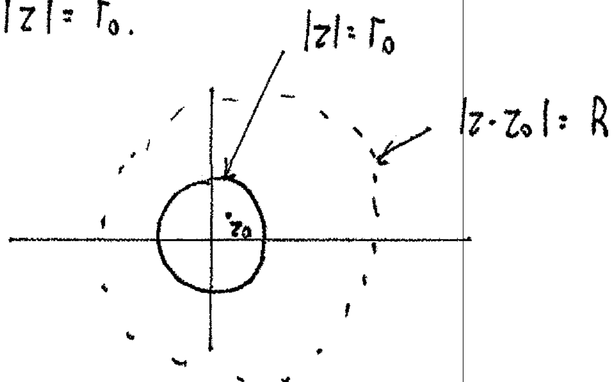
$\rightarrow \quad |f(z)| \le 3|z^2|$ IN ANNULUS $1 \le |z| \le 2$.

SECTION 4.6 #7 SUPPOSE f IS ENTIRE AND $|f(z)| \leq |z|^2$ FOR $|z|$ SUFFICIENTLY LARGE, i.e. FOR $|z| > r_0$. PROVE THAT f MUST BE A POLYNOMIAL OF DEGREE AT MOST 2.

PROOF CHOOSE ANY POINT z_0 AND CHOOSE R BY $R > |z_0| + r_0$ SO THAT $|z - z_0| = R$ IS OUTSIDE $|z| = r_0$.



OR



NOW RECALL THAT $\displaystyle |f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max_{C_R} |f(z)|$ (*) $\quad C_R : |z - z_0| = R$.

SINCE f IS ENTIRE.

NOW SINCE $C_R : |z - z_0| = R$ IS OUTSIDE $|z| = r_0$, THEN $|f(z)| \leq |z|^2$ ON C_R. WE CALCULATE $|z - z_0| \geq ||z| - |z_0|| = ||z| - |z_0||$ BY Δ INEQUALITY.

THUS $\quad |f(z)| \leq |z|^2 \leq (R + |z_0|)^2$. $\quad$ (SINCE $R \geq ||z| - |z_0||$

NOW SUBSTITUTE INTO (*)

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} (R + |z_0|)^2 . \quad (+)$$

THIS MUST HOLD FOR EACH z_0 AND R. IN PARTICULAR, FOR $R \to \infty$, IT MUST ALSO HOLD.

LET $R \to \infty$ AND LET $n \geq 3$, THEN $|f^{(n)}(z_0)| = 0$ FOR ANY $n \geq 3$.

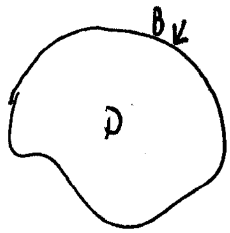
THIS IMPLIES $f^{(n)}(z_0) = 0$ FOR $n \geq 3$. SINCE z_0 IS ANY POINT,

$\Rightarrow f(z)$ MUST BE A POLYNOMIAL OF DEGREE ≤ 2.

SECTION 4.6 # 13

LET $f(z), g(z)$ BE ANALYTIC IN A DOMAIN D AND CONTINUOUS ON BOUNDARY OF D. SUPPOSE $g(z) \neq 0$. PROVE THAT IF $|f(z)| \leq |g(z)|$ HOLDS FOR ALL Z ON B, THEN IT MUST HOLD FOR ALL Z IN D.

PROOF



LET $h(z) = \dfrac{f(z)}{g(z)}$.

SINCE $g(z) \neq 0$ THEN $h(z)$ IS ANALYTIC AND BY ASSUMPTION

$$|h(z)| = \left| \frac{f(z)}{g(z)} \right| = \frac{|f(z)|}{|g(z)|} \leq 1 \text{ ON } B.$$

BY MAXIMUM MODULUS PRINCIPLE $\longrightarrow$ $|h(z)| \leq 1$ IN D.

$\longrightarrow$ $|f(z)| \leq |g(z)|$ IN D.

SECTION 4.6 # 14

LET $f(z)$ BE ANALYTIC IN A BOUNDED DOMAIN D AND CONTINUOUS ON THE BOUNDARY OF D. ASSUME $f \neq 0$ AT ANY Z IN D. PROVE THAT THE MODULUS OF $|f(z)|$ ATTAINS ITS MINIMUM VALUE ON THE BOUNDARY OF D.

PROOF SINCE $f(z)$ IS ANALYTIC IN D AND $f \neq 0$ IN D, IT FOLLOWS THAT $g(z) = \dfrac{1}{f(z)}$ IS ANALYTIC IN D. BY THE MAX-MODULUS PRINCIPLE

$$|g(z)| \leq \max_{Z \in \text{BOUNDARY}} |g(z)| \quad \text{FOR } Z \text{ IN } D.$$

THIS IMPLIES THAT $\quad |f(z)| \geq \min\limits_{Z \in \text{BOUNDARY}} |f(z)|$.

SECTION 4.6 # 16 SHOW THAT $\max\limits_{|z|\le 1} |az^n + b| = |a| + |b|$. n = integer ≥ 0.

PROOF $f(z) = az^n + b$ IS ANALYTIC AND SO BY MAX-MODULUS PRINCIPLE

$$\max_{|z|\le 1} |az^n + b| = \max_{|z|=1} |az^n + b|$$

NOW ON $|z| = 1$ WE HAVE $z = e^{i\varphi}$ SO $|az^n + b| = |ae^{in\varphi} + b|$.

SINCE a, b COMPLEX, WE WRITE $|az^n + b|^2 = \left| |a|e^{i\alpha}e^{in\varphi} + |b|e^{i\beta}\right|^2$

WHERE $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$. SO

$$|az^n + b|^2 = \left|e^{i\beta}\left(|a|e^{i(\alpha+n\varphi-\beta)} + |b|\right)\right|^2 = \left||a|e^{i(\alpha+n\varphi-\beta)} + |b|\right|^2$$

$$= \left||a|e^{iw} + |b|\right|^2 \quad \text{WHERE } w \equiv \alpha + n\varphi - \beta$$

$$= \left[(|a|\cos w + |b|)^2 + |a|^2 \sin^2 w\right]$$

$$|az^n + b|^2 = \left[|a|^2 + |b|^2 + 2|a||b|\cos w\right] \quad \text{ON } |z| = 1.$$

NOW THE MAX OF RHS OCCURS WHEN $w = 0$.

THUS $\max\limits_{|z|=1} |az^n + b|^2 = \left[|a|^2 + |b|^2 + 2|a||b|\right] = (|a| + |b|)^2$.

$$\longrightarrow \quad \max_{|z|\le 1} |az^n + b| = \max_{|z|=1} |az^n + b| = |a| + |b|.$$

SECTION 4.6 # 17 FIND $\max\limits_{|z|\le 1} |(z-1)(z+1/2)|$

NOW $f(z) = (z-1)(z+1/2)$ IS ANALYTIC. THUS BY MAX-MODULUS PRINCIPLE

$$\max_{|z|\le 1} |(z-1)(z-1/2)| = \max_{|z|=1} |(z-1)(z-1/2)| = \max_{|z|=1} |z-1||z-1/2|. \text{ LET } z = e^{i\varphi}$$

NOW $|z-1|^2|z-1/2|^2 = |e^{i\varphi}-1|^2|e^{i\varphi}-1/2|^2 = |(\cos\varphi - 1) + i\sin\varphi|^2 |(\cos\varphi - 1/2) + i\sin\varphi|^2$

$$= [(\cos\varphi - 1)^2 + \sin^2\varphi][(\cos\varphi - 1/2)^2 + \sin^2\varphi] = [2 + 2\cos\varphi][5/4 + \cos\varphi].$$

SO $\max\limits_{|z|=1} |z-1||z-1/2| = \max\limits_{0\le\varphi<2\pi} \sqrt{5/2 - 1/2\cos\varphi - 2\cos^2\varphi}$.

NOW DO CALCULUS

LET $h(\varphi) = -2\cos^2\varphi - 1/2 \cos\varphi + 5/2$ ON $0 \le \varphi < 2\pi$.

FIND MAX value FOR $h(\varphi)$. NOTE: $4\cos\varphi\sin\varphi = 2\sin 2\varphi$.

set $h'(\varphi) = +4\cos\varphi\sin\varphi + \sin\varphi/2 = 0 \rightarrow \sin\varphi = 0$ OR $\cos\varphi = -1/8$.

$h''(\varphi) = 4\cos 2\varphi + \frac{1}{2}\cos\varphi$

IF $\sin\varphi = 0 \rightarrow \varphi = 0, \pi \rightarrow h''(0) > 0,\ h''(\pi) = 4 - 1/2 > 0$.

THUS $\varphi = 0, \pi \rightarrow$ MIN POINTS.

IF $\cos\varphi = -1/8 \rightarrow h''(\varphi) < 0 \rightarrow$ MAX POINT.

THUS AT $\cos\varphi = -1/8$, THEN

$$h = -2\cos^2\varphi - 1/2\cos\varphi + 5/2 = -2\left(1/64\right) + 1/16 + 5/2$$

$$h_{MAX} = -1/32 + 2/32 + 80/32 = 81/32.$$

$\rightarrow$ FROM (*) ON PREVIOUS PAGE,

$$\max_{|z|\le 1} \left|(z-1)(z-1/2)\right| = \max_{|z|=1} |z-1|\,|z-1/2| = \max_{0\le\varphi<2\pi} \sqrt{h(\varphi)}$$

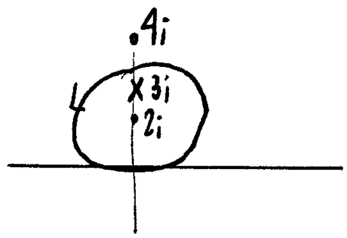
$$= 9/\sqrt{32} = 9/2\sqrt{8}.$$

$$= 9\sqrt{2}/8.$$

EXTRA 1 CALCULATE $I = \int_C \frac{e^z}{z^2+9}\, dz$ WITH $|z-2i| = 2$.

SOLUTION $(z^2+9) = (z+3i)(z-3i)$ HAS SINGULARITIES AT $z = 3i, -3i$.

WE PLOT



ONLY $z = 3i$ IS INSIDE CONTOUR.

SO DEFINE $f(z) = \frac{e^z}{z+3i}$ AND LET $I = \int_C \frac{f(z)}{z-3i}\, dz$.

BY C.I.F. WE HAVE $I = f(3i)\, 2\pi i = 2\pi i \left(\frac{e^{3i}}{6i} \right) = \frac{\pi}{3} \left[\cos 3 + i \sin 3 \right]$.

EXTRA 2 LET $n \geq 1$ AND $a_0, \ldots, a_n$ BE COMPLEX NUMBERS WITH $a_n \neq 0$. LET φ BE REAL WITH $0 \leq \varphi \leq 2\pi$ AND DEFINE $F(\varphi) = a_0 + a_1 e^{i\varphi} + \ldots + a_n e^{in\varphi}$.

PROVE THAT $\exists$ A φ IN $0 \leq \varphi \leq 2\pi$ SUCH THAT $|F(\varphi)| \geq |a_0|$.

PROOF DEFINE $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. LET D BE THE UNIT CIRCLE $|z| \leq 1$. THEN BY MAXIMUM MODULUS PRINCIPAL

$$|p(0)| = |a_0| \leq \max_{|z|=1} |p(z)|.$$

BUT ON $|z| = 1$, $z = e^{i\varphi}$ AND $p(e^{i\varphi}) = F(\varphi)$ SO

$$|p(0)| = |a_0| \leq \max_{0 \leq \varphi \leq 2\pi} |F(\varphi)|.$$

THUS PROVES THAT $\exists\ \varphi$ IN $0 \leq \varphi \leq 2\pi$ SUCH THAT $|a_0| \leq |F(\varphi)|$

EXTRA 3 LET D BE UNIT DISK $|z| \le 1$. SUPPOSE $f(z)$ IS ANALYTIC IN D AND SATISFIES $f(0) = 1$ AND $|f(z)| > 1$ WHEN $|z| = 1$. PROVE THERE IS A z_0 WITH $|z_0| < 1$ FOR WHICH $f(z_0) = 0$.

PROOF SUPPOSE INSTEAD THAT THERE IS NO z_0 WITH $|z_0| < 1$ FOR WHICH $f(z_0) = 0$. THEN LET $g(z) = \dfrac{1}{f(z)}$. WE HAVE THAT $g(z)$ IS ANALYTIC IN $|z| \le 1$ AND

$$|g(0)| = 1 \quad \text{WITH} \quad |g(z)| = \frac{1}{|f(z)|} < 1 \quad \text{FOR} \quad |z| = 1.$$

THIS VIOLATES THE MAX-MODULUS PRINCIPLE. $\rightarrow$ CONTRADICTION.

SO WE MUST HAVE A z_0 IN $|z_0| < 1$ FOR WHICH $f(z_0) = 0$.

EXTRA 4 LET $f(z), g(z)$ BE ANALYTIC IN $|z| \le 1$ AND ASSUME THAT

(i) $|f| \ge |g|$ ON $|z| = 1$

(ii) $f(z) \ne 0$ FOR ANY $|z| < 1$

PROVE THAT $|f| \ge |g|$ IN $|z| < 1$.

PROOF DEFINE $h(z) = \dfrac{g(z)}{f(z)}$. SINCE $f \ne 0$ IN $|z| < 1$ AND f, g ARE ANALYTIC IN $|z| \le 1 \longrightarrow h(z)$ IS Analytic IN $|z| \le 1$.

NOW $$|h(z)| = \frac{|g(z)|}{|f(z)|} \le 1 \quad \text{ON} \quad |z| = 1. \rightarrow \max_{|z|=1} |h(z)| \le 1.$$

BY MAX MODULUS PRINCIPLE

$$|h(z)| \le \max_{|z|=1} |h(z)| \le 1 \quad \text{FOR ANY } z \text{ IN } |z| < 1.$$

$$\rightarrow \quad |f| \ge |g| \quad \text{IN} \quad |z| < 1.$$