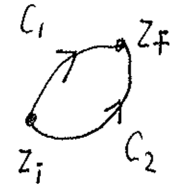
AT THIS STAGE WE HAVE THE FOLLOWING THEOREM:

<u>THEOREM</u> LET $f(z)$ BE CONTINUOUS ON A DOMAIN D. THE FOLLOWING STATEMENTS ARE <u>EQUIVALENT</u>:

(i) $f(z)$ HAS AN ANTI-DERIVATIVE $\mathcal{F}(z)$ IN D

(ii) IF C IS ANY LOOP IN D THEN $\int_C f(z)\,dz = 0$.

(iii) PATH INDEPENDENCE holds, i.e. $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$



<u>REMARKS</u> (1) THIS THEOREM IS RATHER USELESS IN THAT HOW CAN WE POSSIBLY CHECK WHETHER $\int_C f(z)\,dz = 0$ FOR EVERY CURVE C IF WE CANNOT CALCULATE EASILY THE ANTI-DERIVATIVE.

(2) IT IS ONLY USEFUL IF WE CAN SOMEHOW "SPOT" OR "GUESS" THE $\mathcal{F}(z)$. WE WOULD PREFER A THEOREM REFERRING TO A SPECIFIC PROPERTY OF $f(z)$ THAT IS EASILY CHECKED. (i.e. IMPRACTICAL TO USE FOR $\int_C z^5 \cos(z^2)\,dz$).

<u>PROOF OF THEOREM</u>

(ii) $\longleftrightarrow$ (iii) IS TRIVIAL

(i) $\longrightarrow$ (iii) IS TRIVIAL; (i) $\longrightarrow$ (ii) IS ALSO TRIVIAL.

THE ONLY TRICKY ISSUE IS TO PROVE THAT IF (iii) HOLDS THEN (i) MUST HOLD.

<u>PROOF THAT (iii) $\longrightarrow$ (i)</u> SUPPOSE THAT (iii) HOLDS. THEN DEFINE

$$\mathcal{F}(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

WHERE WE HAVE TAKEN <u>ANY</u> PATH FROM z_0 TO z

THUS, BY PATH-INDEPENDENCE $\mathcal{F}(z)$ IS WELL-DEFINED. WE CALCULATE

$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_{z}^{z+\Delta z} f(\zeta)\,d\zeta$$

WHERE WE TAKE STRAIGHT LINE FROM z TO Δz.

THUS IF $\zeta(t) = z + t\Delta z$ WITH $0 \le t \le 1 \rightarrow d\zeta/dt = \Delta z$ AND

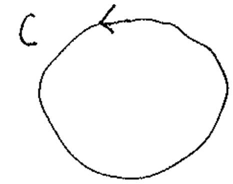
$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_0^1 f(z+t\Delta z)\,\Delta z\,dt \Rightarrow \frac{\mathcal{F}(z+\Delta z) - \mathcal{F}(z)}{\Delta z} = \int_0^1 f(z+t\Delta z)\,dt$$

NOW TAKE $\Delta z \rightarrow 0$ AND $\mathcal{F}'(z) = \int_0^1 f(z)\,dt = f(z)$.

THEOREM (CAUCHY'S INTEGRAL THEOREM)

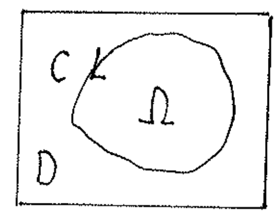
SUPPOSE THAT $f(z)$ IS ANALYTIC, WITH A CONTINUOUS DERIVATIVE f', INSIDE AND ON A SIMPLE CLOSED CURVE C. THEN,

$$(*) \qquad \int_C f(z)\,dz = 0$$



PROOF WE FIRST MUST RECALL GREEN'S THEOREM IN THE PLANE. LET P, Q, P_x, P_y, Q_x, Q_y BE CONTINUOUS IN A DOMAIN D THAT CONTAINS A SIMPLE CLOSED CURVE C ORIENTED COUNTERCLOCKWISE. THEN GREEN'S THEOREM IS

$$(+) \qquad \int_C (P\,dx + Q\,dy) = \iint_\Omega (Q_x - P_y)\,dx\,dy$$



WHERE Ω IS THE REGION INSIDE C. I WILL OUTLINE ON NEXT PAGE THE DERIVATION OF THIS FROM DIVERGENCE THEOREM.

TO PROVE $(*)$ WE LET $f = u + iv$ AND $dz = dx + i\,dy$.

THEN
$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad (1)$$

NOW SINCE f IS ANALYTIC $\longrightarrow$ $u_x = v_y$, $u_y = -v_x$ (CR) EQUATION

f' IS CONTINUOUS $\longrightarrow$ u_x, u_y, v_x, v_y CONTINUOUS

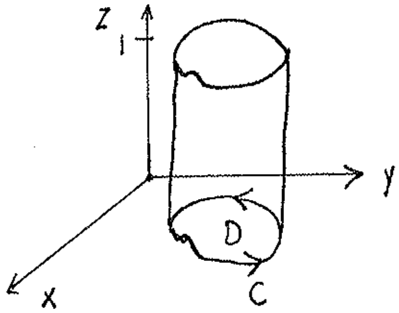
USING GREEN'S THEOREM ON EACH TERM IN (1) WE GET

$$\int_C f(z)\,dz = -\int_\Omega [v_x + u_y]\,dx\,dy + i\int_\Omega [u_x - v_y]\,dx\,dy = 0 \text{ BY CR.}$$

THUS
$$\int_C f(z)\,dz = 0.$$

GREEN'S THEOREM DERIVATION

WE CONSIDER A 3-D REGION V WITH BASE D IN X,Y PLANE THAT IS EXTENDED FROM $Z=0$ TO $Z=1$ TRIVIALLY.



LET C BOUND D WITH COUNTERCLOCKWISE ORIENTATION.

LET $\underline{F}$ BE THE VECTOR FIELD

$\underline{F} = F_1 \hat{i} + F_2 \hat{j} + 0\,\hat{k}$ WITH NO Z COMPONENT.

ASSUME THAT $F_1, F_2, F_{1x}, F_{1y}, F_{2x}, F_{2y}$ ARE CONTINUOUS IN D.

THEN THE DIVERGENCE THEOREM GIVES

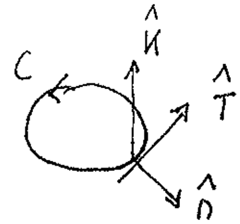
$$\iiint_V \nabla\cdot \underline{F}\, dv = \iint_S \hat{F}\cdot\hat{n}\, ds$$

WHERE S IS SURFACE AREA OF V AND $\hat{n}$ IS UNIT OUTWARD NORMAL TO S.

NOTICE THAT S CONSISTS OF THE SIDES OF THE "OIL CAN" AND THE TOP AND BOTTOM OF THE CAN. NOW THE FLUX ON THE TOP SURFACE $(Z=1)$ AND BOTTOM SURFACE $Z=0$ IS ZERO SINCE $\underline{F}\cdot\hat{k} = 0$ (i.e. $\hat{n} = \hat{k}$ ON $Z=1$)

THUS
$$\iiint_V \nabla\cdot\underline{F}\, dx\, dy\, dz = \int_0^1 \left(\iint_D (F_{1x}+F_{2y})\, dx\, dy\right) dz = \iint_D (F_{1x}+F_{2y})\, dx\, dy$$
$$= \int_0^1 \left(\oint_C \underline{F}\cdot\hat{n}\, ds\right) dz = \oint_C \underline{F}\cdot\hat{n}\, ds.$$

THIS GIVES
$$\iint_D (F_{1x}+F_{2y})\, dx\, dy = \oint_C \underline{F}\cdot\hat{n}\, ds$$



LET $\hat{T}$ TANGENT TO C, $\hat{k}$ IS $\perp$ TO D, SO

$\hat{n} = \hat{T}\times\hat{k}$ AND $\hat{T}\, ds = \underline{dr}$ WITH $\underline{dr} = (dx, dy) = \hat{i}\, dx + \hat{j}\, dy$.

THUS, $(*)$
$$\iint_D (F_{1x}+F_{2y})\, dx\, dy = \oint_C \underline{F}\cdot[\hat{T}\times\hat{k}]\, ds = \oint_C (\hat{k}\times\underline{F})\cdot\hat{T}\, ds = \int_C (\hat{k}\times\underline{F})\cdot\underline{dr}$$

HERE WE USED THE VECTOR IDENTITY $\underline{a}\cdot[\underline{b}\times\underline{c}] = (\underline{a}\times\underline{b})\cdot\underline{c}$. WE THEN CALCULATE

$$\hat{k}\times\bar{F} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ F_1 & F_2 & 0 \end{vmatrix} = -F_2\hat{i} + F_1\hat{j}.$$

THEN IN $(*)$
$$\left[\begin{aligned} &\iint_D (F_{1x}+F_{2y})\, dx\, dy = \oint_C (-F_2\hat{i} + F_1\hat{j})\cdot(\hat{i}\,dx + \hat{j}\,dy) \\ &\iint_D (F_{1x}+F_{2y})\, dx\, dy = \oint_C (F_1\, dy - F_2\, dx) \end{aligned}\right.$$

GREEN'S THEOREM.

<u>REMARKS</u> (i) WE ASSUMED

a) $f(z)$ IS ANALYTIC INSIDE Ω AND ON C (Analyticity $\rightarrow f'$ EXISTS)

b) $f'(z)$ IS CONTINUOUS IN Ω $\longleftarrow$ would like to remove this condition

c) C IS A simple closed curve $\leftarrow$ can we have figure 8? more general curves?

A MORE PRECISE VERSION OF CAUCHY'S THEOREM BY E. GOURSAT REMOVED CONDITION (ii). FOR MATHEMATICALLY INCLINED READERS SEE

- SECTION 2.3 OF THE BOOK "BASIC COMPLEX ANALYSIS" BY J. MARSDEN.
- THE REMOVAL OF CONDITION (ii) IS MENTIONED BUT NOT PROVED IN OUR BOOK BY SAFF AND SNIDER.

A REGION D IS CALLED SIMPLY-CONNECTED IF D IS CONNECTED AND EVERY CLOSED CURVE IN D CONTAINS ONLY POINTS IN D.

simply connected

not simply connected

ROUGHLY speaking, simply connected domains are connected domains with no holes.

<u>CAUCHY-GOURSAT THEOREM</u> LET $f(z)$ BE ANALYTIC ON A SIMPLY CONNECTED DOMAIN D AND LET C BE A SIMPLE CLOSED CURVE IN D. THEN $\int_C f(z)\,dz = 0$

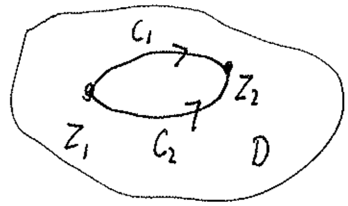
<u>REMARKS</u> (i) WE CAN APPLY THIS THEOREM TO FIGURE 8 TYPE LOOPS, BY CONSIDERING EACH LOOP SEPARATELY.

(ii) CAN ALSO GENERALIZE AS SHOWN IN THE NEXT DEFORMATION OF CONTOUR APPROACH.

WE NOW STATE A FEW CONSEQUENCES OF CAUCHY-GOURSAT THEOREM.

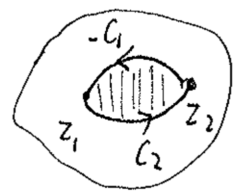
THEOREM (PATH INDEPENDENCE) SUPPOSE f IS ANALYTIC ON A SIMPLY CONNECTED REGION D. THEN FOR ANY TWO CURVES C_1 AND C_2 WITH SAME ORIENTATION JOINING THE SAME POINTS z_0 AND z_1 WE HAVE

$$\int_{C_1} f\, dz = \int_{C_2} f\, dz$$



i.e. path independence.

PROOF we make a closed curve C oriented counterclockwise and use cauchy-goursat theorem



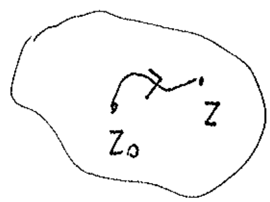
$$\int_C f(z)\, dz = \int_{C_2} f(z)\, dz + \int_{-C_1} f(z)\, dz = 0$$

since f analytic.

THUS, $\int_{C_2} f(z)\, dz = -\int_{-C_1} f(z)\, dz = \int_{C_1} f(z)\, dz.$ ∎

THEOREM (ANTI-DERIVATIVE) SUPPOSE $f(z)$ IS ANALYTIC IN A SIMPLY CONNECTED DOMAIN D. THEN $f(z)$ HAS AN ANTI-DERIVATIVE $\mathcal{F}(z)$ WHERE

$$\mathcal{F}(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$



ζ-plane.

PROOF SINCE $f(\zeta)$ IS ANALYTIC, WE HAVE THAT THE INTEGRAL FROM $\zeta = z_0$ TO $\zeta = z$ IS INDEPENDENT OF PATH AND HENCE IS WELL DEFINED. WE WANT TO SHOW THAT $\mathcal{F}'(z) = f(z)$. WE CALCULATE

$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_{z_0}^{z+\Delta z} f(\zeta)\, d\zeta - \int_{z_0}^{z} f(\zeta)\, d\zeta = \int_{z}^{z+\Delta z} f(\zeta)\, d\zeta.$$

NOW WRITING $f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, d\zeta$ WE GET BY ADDING AND SUBTRACTING $f(z)$

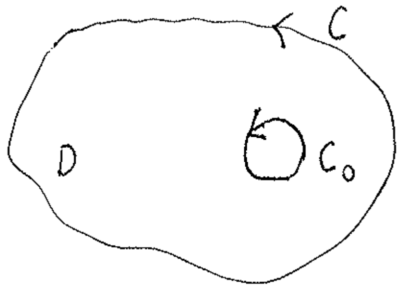
$$\frac{\mathcal{F}(z+\Delta z) - \mathcal{F}(z)}{\Delta z} = \int_{z}^{z+\Delta z} \frac{(f(\zeta) - f(z))}{\Delta z}\, d\zeta + f(z).$$

LET $\Delta z \to 0$ THEN

$$\mathcal{F}'(z) = \lim_{\Delta z \to 0} \frac{\mathcal{F}(z+\Delta z) - \mathcal{F}(z)}{\Delta z} \Rightarrow f(z).$$

DEFORMATION RESULT I

CONSIDER THE DOMAIN D AS SHOWN, THAT HAS A "HOLE".

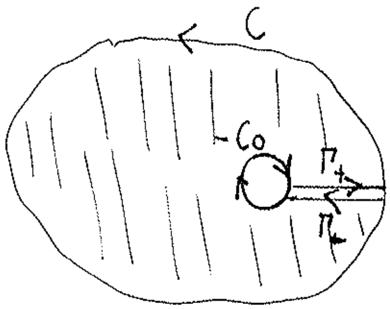


SUPPOSE THAT $F(z)$ IS ANALYTIC BETWEEN C_0 AND C, AND IS ANALYTIC ON C AND C_0.

THEN $$\int_C F(z)\,dz = \int_{C_0} F(z)\,dz$$

WITH BOTH CONTOURS ORIENTED COUNTERCLOCKWISE.

DERIVATION WE CONSIDER THE CONTOUR AS SHOWN. WE INTRODUCE A PATH



Γ_- FROM C TO $-C_0$, AND A PATH Γ_+ FROM $-C_0$ BACK TO C.

WE LET $\Gamma = C + \Gamma_- + (-C_0) + \Gamma_+$

INSIDE Γ (shaded region), $F(z)$ IS ANALYTIC

AND BY CAUCHY'S INTEGRAL THEOREM $\int_\Gamma F(z)\,dz = 0$

(NOTE: STRICTLY speaking Γ IS NOT a simple curve, but one can take Γ_- arbitrarily close to Γ_+).

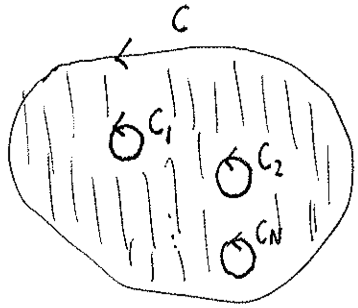
THEN $$\int_\Gamma f(z)\,dz = \int_C f(z)dz + \int_{\Gamma_-} f(z)\,dz + \int_{\Gamma_+} f(z)\,dz + \int_{-C_0} f(z)\,dz = 0.$$

BUT $\int_{\Gamma_-} F(z)\,dz = -\int_{\Gamma_+} F(z)\,dz$ (same curve traversed in different directions).

THUS $$\underbrace{\int_C F(z)\,dz}_{\text{counterclockwise}} = -\int_{-C_0} F(z)\,dz = \underbrace{\int_{C_0} F(z)\,dz}_{\text{counterclockwise}}$$ $\boxtimes$

DEFORMATION RESULT II

SUPPOSE THAT WE HAVE A DOMAIN WITH N HOLES AS SHOWN.

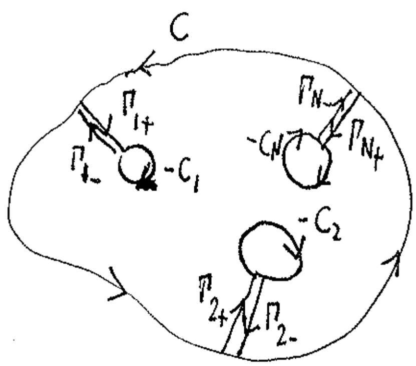


SUPPOSE THAT $f(z)$ IS ANALYTIC IN THE SHADED REGION INSIDE AND ON C, BUT OUTSIDE $C_1, .., C_N$. SUPPOSE f IS ANALYTIC ON $C_1, .., C_N$.

THEN $$\int_C f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz$$

counterclockwise orientation.

THE DERIVATION OF THIS IS TO USE CAUCHY'S THEOREM ON THE CONTOUR AS SHOWN



THEN

$$\left(\int_C + \sum_{j=1}^{N} \left(\int_{\Gamma_{j+}} + \int_{\Gamma_{j-}} \right) + \sum_{j=1}^{N} \int_{-C_j} \right) f(z)\,dz = 0$$

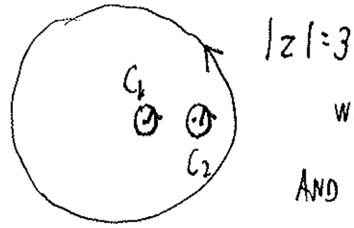
BUT $\int_{\Gamma_{j+}} = -\int_{\Gamma_{j-}}$ SO

$$\int_C f(z)\,dz = -\sum_{j=1}^{N} \int_{-C_j} f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz.$$

<u>REMARK</u> (i) TYPICALLY HOW THIS WILL BE USED IS THAT WE WILL PUT AN ISOLATED "SINGULARITY" OF $f(z)$ INSIDE A SMALL DISK.

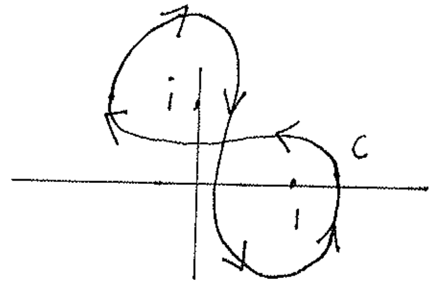
i.e. IF $f(z) = \dfrac{1}{(z-1)(z-2)}$ AND $C: \ |z| = 3$

THEN WE DRAW



WHERE C_1 CONTAINS $z=1$ AND C_2 CONTAINS $z=2$.

TO ILLUSTRATE THE USE OF THE DEFORMATION RESULT WE CONSIDER HERE A SIMPLE EXAMPLE. (MANY MORE EXAMPLES ARE CONSIDERED LATER)



<u>EXAMPLE 1</u> CALCULATE $I = \int_C \frac{z}{(z-i)(z-1)}\,dz$

WHERE C IS THE FIGURE 8 PATH AS SHOWN.

<u>SOLUTION</u> NOTICE THAT $f = \frac{z}{(z-i)(z-1)}$ IS <u>NOT</u> ANALYTIC AT $z = i$ AND AT $z = 1$.

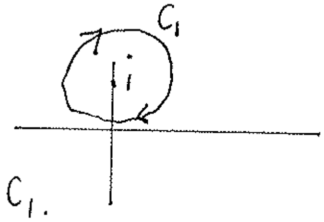
WE FIRST USE PARTIAL FRACTION DECOMPOSITION

$$\frac{z}{(z-i)(z-1)} = \frac{A}{z-i} + \frac{B}{z-1} \longrightarrow \quad z = A(z-1) + B(z-i)$$

$$z = i \longrightarrow \quad A = \frac{i}{i-1} = \frac{i(-i-1)}{2} = \frac{1-i}{2}$$

$$z = 1 \longrightarrow \quad B = \frac{1}{1-i} = \frac{1+i}{2}.$$

THUS $I = \int_C \left(\frac{A}{z-i} + \frac{B}{z-1}\right) dz = \int_C \frac{A}{z-i}\,dz + \int_C \frac{B}{z-1}\,dz.$

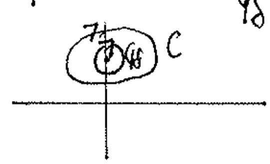


- DEFINE C_1 TO BE <u>CLOCKWISE LOOP</u> AROUND $z = i$

THEN $\int_{C_1} \frac{B}{z-1}\,dz = 0$ SINCE $1/_{z-1}$ IS <u>ANALYTIC</u> inside C_1.

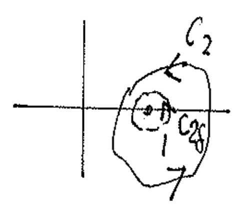
BY PATH DEFORMATION RESULT I, WE GET $\int_{C_1} \frac{A}{z-i}\,dz = \int_{C_\delta} \frac{A}{z-i}\,dz$

WHERE C_δ IS <u>CLOCKWISE</u> CONTOUR $|z-i| = \delta$.



BUT $\int_{C_\delta} \frac{A}{z-i}\,dz = -2\pi i\,(A)$.

HENCE $\int_{C_1} \frac{B}{z-1}\,dz = 0 \quad , \quad \int_{C_1} \frac{A}{z-i}\,dz = -2\pi i A.$



- NOW DEFINE C_2 TO BE <u>COUNTERCLOCKWISE</u> LOOP AROUND $z = 1$.

THEN $\int_{C_2} \frac{A}{z-i}\,dz = 0$ SINCE $1/_{z-i}$ IS ANALytic inside C_2

AND BY PATH DEFORMATION $\int_{C_2} \frac{B}{z-1}\,dz = \int_{C_{2\delta}} \frac{B}{z-1}\,dz = 2\pi i B \qquad C_{2\delta}$ counterclockwise

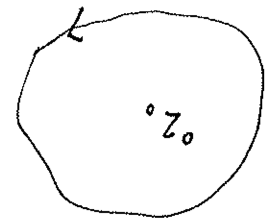
THUS, $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = -2\pi i A + 2\pi i B$

$$= 2\pi i\,(B - A) = 2\pi i\,(i) = -2\pi.$$

THEOREM (CAUCHY-INTEGRAL FORMULA)

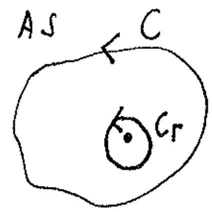
LET C BE A SIMPLE CLOSED CONTOUR ORIENTED COUNTERCLOCKWISE. IF $f(z)$ IS ANALYTIC INSIDE AND ON C. THEN IF z_0 IS ANY POINT INSIDE C, WE HAVE

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$



<u>PROOF</u> NOTICE THAT $\dfrac{f(z)}{z-z_0}$ IS ANALYTIC INSIDE AND ON C EXCEPT AT z_0.

THUS BY CAUCHY-GOURSAT WE CAN DEFORM AS



$$I \equiv \int_C \frac{f(z)}{z-z_0}\, dz = \int_{C_r} \frac{f(z)}{z-z_0}\, dz$$

WHERE $C_r = \{ z \mid |z - z_0| = r \}$.

NOW WE WRITE,
$$\int_{C_r} \frac{f(z)}{z-z_0}\, dz = \int_{C_r} \frac{f(z_0)}{z-z_0}\, dz + \int_{C_r} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz.$$

THIS YIELDS THAT
$$I = 2\pi i\, f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

SINCE I AND $2\pi i\, f(z_0)$ ARE INDEPENDENT OF r, WE CAN TAKE THE LIMIT $r \to 0$ TO OBTAIN

$$I - 2\pi i\, f(z_0) = \lim_{r \to 0} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (+)$$

BUT
$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \max_{C_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| 2\pi r = \max_{C_r} |f(z) - f(z_0)|\, 2\pi. \quad (*)$$

NOW DEFINE $M_r = \max_{C_r} |f(z) - f(z_0)|$. SINCE f IS CONTINUOUS AT z_0 IT FOLLOWS THAT $f(z) \to f(z_0)$ AS $r \to 0$, SO THAT $M_r \to 0$ AS $r \to 0$.

HENCE, $(*)$ IMPLIES $\left| \int_{C_r} \dfrac{f(z) - f(z_0)}{z - z_0}\, dz \right| \longrightarrow 0$ AS $r \to 0$. THEN $(+)$ YIELDS THAT

$$I = \int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) \longrightarrow f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

EXAMPLE 1 EVALUATE $I = \int_C \frac{1}{z}\, dz$ WHERE C IS ELLIPSE $x^2 + 4y^2 = 1$ ORIENTED COUNTERCLOCKWISE.

SOLUTION WE DEFORM TO CONTOUR AS SHOWN SINCE $1/z$ ANALYTIC EXCEPT AT $z = 0$.

THEN $\left(\int_C + \int_{C_\delta} \right) (1/z)\, dz = 0$ BY CAUCHY-GOURSAT

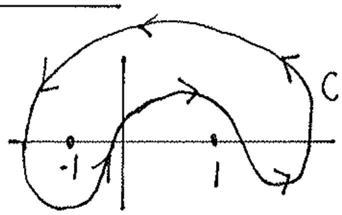
WHERE C_δ IS CIRCLE $|z| = \delta$ ORIENTED CLOCKWISE

THUS $\int_C \frac{1}{z}\, dz = -\int_{C_\delta} \frac{1}{z}\, dz = \int_{-C_\delta} \frac{1}{z}\, dz = 2\pi i$ BY PREVIOUS NOTES.

NOW $-C_\delta$ ORIENTED COUNTERCLOCKWISE

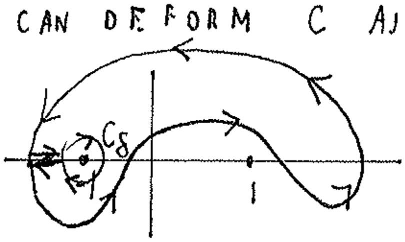
$$\int_{-C_\delta} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\delta e^{it}}\, i\delta e^{it}\, dt = i\, 2\pi . \qquad \text{THUS } I = 2\pi i.$$

EXAMPLE 2 CALCULATE $I = \int_C \frac{dz}{z^2 - 1}$ WHERE C IS PATH SHOWN BELOW.



NOTICE: $z = -1, 1$ ARE POINTS WHERE $f(z) = \frac{1}{z^2 - 1}$ IS NOT ANALYTIC, BUT ONLY $z = -1$ IS INSIDE C.

WE CAN DEFORM C AS SHOWN:



THUS $\left(\int_C + \int_{C_\delta} \right) \frac{1}{z^2 - 1}\, dz = 0$ BY CAUCHY-GOURSAT.

BUT $\frac{1}{z^2 - 1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$

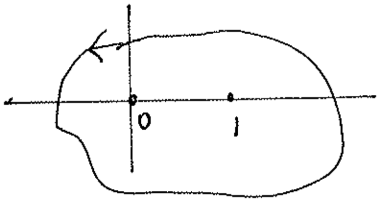
WE CONCLUDE THAT $\int_C \frac{1}{2(z-1)}\, dz = 0$ SINCE $f_1(z) = \frac{1}{2(z-1)}$ IS ANALYTIC INSIDE AND ON C. THUS $\int_C \frac{1}{z^2 - 1}\, dz = \int_C \frac{-1}{2(z+1)}\, dz$.

NOW BY PATH DEFORMATION $\int_C \frac{1}{2(z+1)}\, dz + \int_{C_\delta} \frac{1}{2(z+1)}\, dz = 0$. BY CAUCHY-GOURSAT.

HENCE, $\int_C \frac{-1}{2(z+1)}\, dz = +\int_{C_\delta} \frac{1}{2(z+1)}\, dz = \frac{-1}{2} \int_{-C_\delta} \frac{1}{z+1}\, dz = \frac{-1}{2}(2\pi i)$ SINCE $-C_\delta$ COUNTERCLOCKWISE

THUS, $\int_C \frac{dz}{z^2 - 1} = -\frac{1}{2}(2\pi i) = -\pi i$.

EXAMPLE 3 CALCULATE $I = \int_C \frac{3z-2}{z(z-1)}\,dz$ WITH C AS SHOWN.



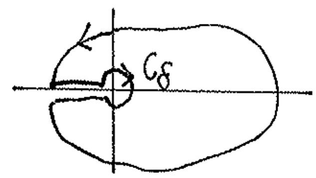
THE INTEGRAND $\frac{3z-2}{z(z-1)}$ IS ANALYTIC EXCEPT AT $z=0$ AND $z=1$.

WE USE PARTIAL FRACTIONS: $\frac{3z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

HENCE $3z-2 = A(z-1) + Bz$. IF $z=0 \to A=2$
IF $z=1 \to B=1$.

THUS $I = \int_C \frac{2}{z}\,dz + \int_C \frac{1}{z-1}\,dz$.

- FOR $I_1 = \int_C \frac{2}{z}\,dz$ WE DEFORM AS FOLLOWS



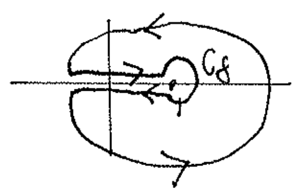
THEN BY CAUCHY-GOURSAT

$$\int_C \frac{2}{z}\,dz + \int_{C_\delta} \frac{2}{z}\,dz = 0.$$

$$I_1 = \int_C \frac{2}{z}\,dz = -\int_{C_\delta} \frac{2}{z}\,dz = \int_{-C_\delta} \frac{2}{z}\,dz = 2(2\pi i).$$

counterclockwise

- FOR $I_2 = \int_C \frac{1}{z-1}$ WE DEFORM AS



SO $\int_C \frac{1}{z-1}\,dz + \int_{C_\delta} \frac{1}{z-1}\,dz = 0$ BY CAUCHY-GOURSAT.

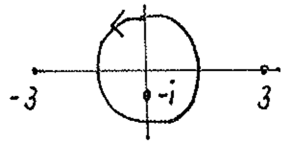
THUS $\int_C \frac{1}{z-1}\,dz = -\int_{C_\delta} \frac{1}{z-1}\,dz = \int_{-C_\delta} \frac{1}{z-1}\,dz = 2\pi i$

COUNTERCLOCKWISE

THEREFORE, $I = \int_C \frac{3z-2}{z(z-1)}\,dz = I_1 + I_2 = 4\pi i + 2\pi i = 6\pi i$.

EXAMPLE 4 CALCULATE $I = \int_C \frac{z\,dz}{(9-z^2)(z+i)}$ WHERE C IS THE DISK $|z|=2$

ORIENTED COUNTERCLOCKWISE.



NOTICE THAT $f(z) = \frac{z}{(9-z^2)(z+i)}$ IS ANALYTIC EXCEPT AT $z=-i, 3, -3$.

ONLY $z=-i$ IS INSIDE C. WE CAN USE PARTIAL FRACTIONS BUT INSTEAD

USE CAUCHY INTEGRAL FORMULA

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz$$

WHEN C counterclockwise CONTAINING z_0, AND f Analytic inside and on C.

LET $z_0 = -i$ SO $\quad f(-i) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z+i}\,dz$

LET $f(z) = z/9-z^2$ THEN $\quad 2\pi i\, f(-i) = \int_C \frac{f(z)}{z+i}\,dz = \int_C \frac{z}{(9-z^2)(z+i)}\,dz$

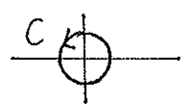
THUS $\quad \int_C \frac{z/(9-z^2)}{z+i}\,dz = 2\pi i \left(\frac{z}{9-z^2}\right)\Big|_{z=i} = 2\pi i\left(\frac{-i}{9+1}\right) = \pi/5$

SO $I = \pi/5$.

EXAMPLE 5 LET C BE THE UNIT CIRCLE $|z|=1$ ORIENTED COUNTERCLOCKWISE.

CALCULATE $\int_C \frac{e^{az}}{z}\,dz$ FOR ANY CONSTANT a, AND FROM THIS PROVE

THAT $\int_0^{\pi} e^{a\cos\varphi} \cos(a\sin\varphi)\,d\varphi = \pi$.



PROOF BY CAUCHY-INTEGRAL FORMULA SINCE $f(z) = e^{az}$ IS ANALYTIC

INSIDE AND ON C WE OBTAIN,

$f(0) = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z-0}\,dz$. THUS $\int_C \frac{e^{az}}{z}\,dz = 2\pi i$ SINCE $f(0)=1$.

NOW LET $z = e^{i\varphi} \rightarrow dz/d\varphi = i e^{i\varphi}$ $\quad \displaystyle\int_C \frac{e^{az}}{z}\, dz = \int_0^{2\pi} \frac{e^{a z(\varphi)}}{z(\varphi)} \frac{dz}{d\varphi}\, d\varphi$

THUS,
$$\int_C \frac{e^{az}}{z}\, dz = \int_0^{2\pi} \frac{e^{a(\cos\varphi + i\sin\varphi)}}{e^{i\varphi}}\, i e^{i\varphi}\, d\varphi = i\int_0^{2\pi} e^{a\cos\varphi}\left[\cos(a\sin\varphi) + i\sin(a\sin\varphi)\right] d\varphi$$

THUS
$$2\pi i = i\left[\int_0^{2\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\, d\varphi\right] - \int_0^{2\pi} e^{a\cos\varphi}\sin(a\sin\varphi)\, d\varphi$$

TAKE IMAGINARY PARTS OF BOTH SIDES,
$$(*) \quad \int_0^{2\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\, d\varphi = 2\pi.$$

SINCE INTEGRAND IN (*) IS 2π PERIODIC, THEN
$$\int_{-\pi}^{\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\, d\varphi = 2\pi.$$

HOWEVER, SINCE INTEGRAND IN (+) IS EVEN IN φ, THEN
$$\int_0^{\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\, d\varphi = \pi.$$

<u>EXAMPLE 6</u> CALCULATE $I = \displaystyle\int_0^{2\pi} \cos^{2n}\varphi\, d\varphi$ WITH n A POSITIVE INTEGER.

WE WRITE $\cos\varphi = \frac{1}{2}(z + 1/z)$ WITH $z = e^{i\varphi}$ AND $dz = i e^{i\varphi} d\varphi \rightarrow \dfrac{dz}{iz} = d\varphi$.

WE CONVERT I TO A LINE INTEGRAL OVER UNIT CIRCLE ORIENTED COUNTERCLOCKWISE.

WE WRITE
$$I = \int_0^{2\pi} \cos^{2n}\varphi\, d\varphi = \int_C \left[\tfrac{1}{2}(z + 1/z)\right]^{2n} \frac{dz}{iz} = \frac{1}{i 2^{2n}} \int_C (z + 1/z)^{2n} \frac{dz}{z}.$$

NOW THE BINOMIAL THEOREM IS $(a+b)^{2n} = \displaystyle\sum_{k=0}^{2n} a^{2n-k} b^k \binom{2n}{k}$ AND

WE APPLY IT WITH $a = z$, $b = 1/z$. WE CALCULATE,
$$I = \frac{1}{i 2^{2n}} \int_C \frac{1}{z} \sum_{k=0}^{2n} z^{2n-k}\left(\frac{1}{z}\right)^k \binom{2n}{k} dz = \frac{1}{i 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \int_C z^{2n-2k-1}\, dz.$$

WE NOW DEFORM TO A CIRCULAR DISK OF RADIUS δ CENTERED AT THE ORIGIN OR MORE SIMPLY NOTICE AND RECALL THAT
$$\int_C z^m\, dz = \begin{cases} 0 & \text{IF } m \neq -1 \\ 2\pi i & \text{IF } m = -1 \end{cases}$$

THUS, THE ONLY TERM IN THE SUM TO CONTRIBUTE IS WHEN $K=n$.

THUS,

$$I=\int_0^{2\pi}\cos^{2n}\varphi\,d\varphi=\frac{1}{i\,2^{2n}}\binom{2n}{n}2\pi i=\frac{2\pi}{2^{2n}}\,\frac{(2n)!}{(n!)^2}.$$

(WE RECALL $\binom{K}{j}=\dfrac{K!}{j!(K-j)!}$.).

THUS,

$$I=\frac{2\pi}{2^{2n}}\,\frac{2n(2n-1)(2n-2)\cdots 1}{(1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n)}=\frac{2\pi}{2^{2n}}\,\frac{[2n(2n-2)(2n-4)\cdots 2][(2n-1)(2n-3)\cdots 1]}{[1\cdot 2\cdots n][1\cdot 2\cdots n]}$$

$$I=2\pi\,\frac{[2n(2n-2)(2n-4)\cdots 2][(2n-1)(2n-3)\cdots 1]}{[2\cdot 4\cdots(2n)][2\cdot 4\cdots 2n]}$$

THUS

$$I=2\pi\,\frac{(1\cdot 3\cdots(2n-1))}{(2\cdot 4\cdots 2n)}.$$

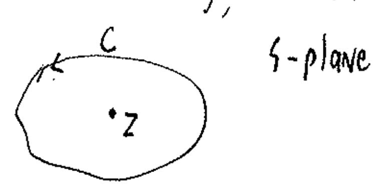
FOR INSTANCE

$$\int_0^{2\pi}\cos^{10}(\varphi)\,d\varphi=2\pi\left(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\right).$$

CONSEQUENCES OF THE CAUCHY INTEGRAL FORMULA

KEY POINT 1 IF f IS ANALYTIC WITHIN AND ON A SIMPLE CLOSED CONTOUR C, THEN FOR ANY POINT z INSIDE C (ORIENTED COUNTERCLOCKWISE), WE HAVE

$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(\varsigma)}{\varsigma-z}\,d\varsigma$$



AND THAT $f', f'', \ldots, f^{(n)}$ ARE ALL ANALYTIC AT z. IN OTHER WORDS, IF f IS ANALYTIC AT POINT z THEN ITS DERIVATIVES OF ALL ORDERS ARE ALSO ANALYTIC AT POINT z. MOREOVER,

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_C\frac{f(\varsigma)}{(\varsigma-z)^{n+1}}\,d\varsigma.$$

PROOF WE FIRST SHOW THAT $f'(z)$ EXISTS FOR ANY z IN C, AND THAT $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$ (i.e. we can differentiate under integral sign).

TO PROVE THIS, WE NOTE THAT SINCE z IS INSIDE C, THEN FOR $z+\Delta z$ IT MUST ALSO BE IN C FOR $|\Delta z|$ SMALL ENOUGH.

WE CALCULATE
$$J \equiv \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left[\frac{1}{\zeta-z-\Delta z} - \frac{1}{\zeta-z}\right]\frac{f(\zeta)}{\Delta z}\,d\zeta \quad (*)$$

THUS
$$J = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z-\Delta z)(\zeta-z)}\,d\zeta$$
OBTAINED BY COMBINING $(*)$.

NOW ADD AND SUBTRACT
$$J - \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,dz = \frac{1}{2\pi i}\int_C \left[\frac{1}{(\zeta-z-\Delta z)(\zeta-z)} - \frac{1}{(\zeta-z)^2}\right] f(\zeta)\,d\zeta$$
$$= \frac{1}{2\pi i}\int_C \frac{1}{(\zeta-z)}\left[\frac{1}{\zeta-z-\Delta z} - \frac{1}{\zeta-z}\right] f(\zeta)\,d\zeta$$
$$J - \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta-z)^2}\,dz = \frac{1}{2\pi i}\Delta z\int_C \frac{f(\zeta)}{(\zeta-z-\Delta z)(\zeta-z)^2}\,d\zeta. \quad (+)$$

NOW LET $|\Delta z|\to 0$ AND DEFINE $M = \max_C |f(\zeta)|$. WE WANT TO SHOW THAT $|\text{RHS}|$ OF $(+)$ TENDS TO 0 AS $|\Delta z|\to 0$.

NOW SINCE z IS <u>INSIDE</u> C, THEN THERE IS A POSITIVE CONSTANT d SUCH THAT $|\zeta - z| \geq d > 0$ FOR ALL ζ ON CONTOUR C. LET $|\Delta z| < d/2$.

THEN BY Δ INEQUALITY

$$|\zeta - z - \Delta z| \geq \big|\,|\zeta-z| - |\Delta z|\,\big| \geq d - d/2 = d/2$$

FOR ζ ON CONTOUR C. THEREFORE,
$$\frac{1}{|\zeta-z-\Delta z|\,|\zeta-z|^2} \leq \frac{1}{(d/2)\,d^2} \qquad \text{FOR } \zeta \text{ ON } C.$$

WE THEN ESTIMATE

$$\left| \frac{1}{2\pi i} \Delta z \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \left| \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} dz \right| \qquad (**)$$

BUT

$$\left| \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} \right| \leq \max_{\zeta \text{ on } C} |f| \frac{1}{(d/2) d^2}$$

THUS $(**)$ BECOMES,

$$\left| \frac{1}{2\pi i} \Delta z \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \max_{\zeta \text{ on } C} |f(\zeta)| \frac{2}{d^3} L$$

WHERE $L = \text{length}(C)$.

FINALLY IN $(+)$

$$\left| J - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \max_{\zeta \text{ on } C} |f(\zeta)| \frac{2}{d^3} L \longrightarrow 0 \text{ AS } \Delta z \to 0.$$

SINCE $J \to f'$ AS $\Delta z \to 0$, WE HAVE

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

WHICH JUSTIFIES DIFFERENTIATING UNDER THE INTEGRAL SIGN.

FURTHER DIFFERENTIATIONS YIELD THAT $f^{(n)}(z)$ EXISTS FOR ANY n

AND SO

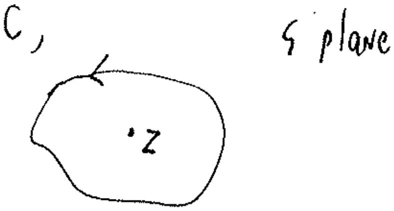
$$f^{(n)}(z) = \frac{1}{2\pi i} n! \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

SINCE z IS ARBITRARY INSIDE C, THEN $f^{(n)}$ IS ANALYTIC AT EACH POINT INSIDE C FOR ANY $n > 0$.

<u>NOTE</u>: THIS IS VERY DIFFERENT FROM CALCULUS. i.e. LET $f(x) = x^3 \sin(1/x)$

IT IS EASY TO SEE THAT $f'(x)$ EXISTS BUT $f''(x)$ DOES NOT AT $x = 0$.

<u>KEY POINT 2</u> LET C BE A SIMPLE CLOSED CONTOUR AND LET $f(\zeta)$ BE ANALYTIC INSIDE AND ON C. THEN IF z IS INSIDE C,



$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \quad (*)$$

TAKING C_R TO BE THE DISK $|\zeta - z| = R$, AND ASSUMING THAT $|f(\zeta)| \leq M$ ON C_R, WE ESTIMATE FROM $(*)$ THAT

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} (2\pi R) = \frac{n!\, M}{R^n}. \quad (+) \quad \text{CAUCHY'S INEQUALITY.}$$

THIS ESTIMATE LEADS TO AN IMPORTANT RESULT KNOWN AS LIOUVILLE'S THEOREM:

<u>LIOUVILLE'S THEOREM</u> LET $f(\zeta)$ BE AN ENTIRE FUNCTION THAT IS BOUNDED (i.e. $|f(\zeta)| \leq M$) FOR ALL ζ IN THE COMPLEX PLANE. THEN $f(\zeta)$ IS CONSTANT FOR ALL ζ.

<u>PROOF</u> PICK ANY POINT z IN COMPLEX PLANE, AND DEFINE C_R: $|\zeta - z| = R$. THEN FROM (+) WITH $n = 1$ WE HAVE

$$|f'(z)| \leq \frac{M}{R}.$$

SINCE R CAN BE MADE ARBITRARILY LARGE, WHILE $f'(z)$ IS A FIXED NUMBER, IT FOLLOWS THAT $f'(z) = 0$. SINCE z IS ARBITRARY, IT FOLLOWS THAT $f'(z) = 0$ FOR ALL z, AND CONSEQUENTLY

$$f(z) = c \qquad c = \text{CONSTANT} \ \forall z.$$

REMARKS (i) IF $f(\zeta)$ IS ENTIRE AND $|\mathrm{Re}\, f| \le M \ \forall \zeta \rightarrow f$ IS CONSTANT.

PROOF LET $g(\zeta) = e^{f(\zeta)}$. THEN $g(\zeta)$ IS ENTIRE SINCE f IS.

NOW LET $f = u + iv$. $\longrightarrow g = e^{u} e^{iv}$ AND $|g(\zeta)| = e^{u}$.

BUT $u \le |u| \le M$ IN ζ-plane $\Longrightarrow |g(\zeta)| \le e^{M}$.

THUS $g(\zeta)$ IS ENTIRE BOUNDED FUNCTION. LIOUVILLE'S THEOREM YIELDS THAT $g(\zeta)$ IS A CONSTANT $\rightarrow f(\zeta) =$ CONSTANT.

(ii) IF $f(\zeta)$ IS ENTIRE AND $|\mathrm{Im}\, f| \le M \ \forall \zeta \longrightarrow f$ IS CONSTANT.

TO PROVE THIS START WITH $g(\zeta) = e^{-if(\zeta)}$ AND FOLLOW (i).

KEY POINT 3 LET $f(z)$ BE ANALYTIC INSIDE AND ON A SIMPLE CLOSED CONTOUR C THAT IS A CIRCLE (COUNTERCLOCKWISE) OF RADIUS R CENTERED AT z_0. THEN

$$(*) \qquad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\varphi})\, d\varphi = \text{average of the values of } f \text{ on the circle.}$$

$(*)$ IS CALLED THE MEAN-VALUE PROPERTY.

PROOF

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z(\varphi))}{z(\varphi) - z_0} \frac{dz}{d\varphi}\, d\varphi .$$

NOW LET $z - z_0 = Re^{i\varphi} \longrightarrow dz/d\varphi = iRe^{i\varphi} \qquad z(\varphi) = z_0 + Re^{i\varphi}$.

THUS
$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\varphi})}{Re^{i\varphi}}\, iRe^{i\varphi}\, d\varphi$$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\varphi})\, d\varphi \qquad \blacksquare$$

THEOREM SUPPOSE $\exists$ REAL CONSTANTS $A>0$ AND $B>0$ SUCH IF f ENTIRE WE HAVE THAT $|f(z)| \leq A + B|z|^n$ FOR ALL z WHERE n IS A POSITIVE INTEGER. THEN $f(z)$ IS A POLYNOMIAL OF degree n.

PROOF SUPPOSE $f(z)$ IS ENTIRE AND $\forall z$, $|f(z)| \leq A + B|z|^n$.

PICK AND $z = z_0$ AND MAKE A DISK $|z - z_0| \leq r$.

NOW ON $|z-z_0| = r$ WE HAVE $|f(z)| \leq A + B|(z-z_0) + z_0|^n$

SO $|f(z)| \leq A + B\left[\,|z-z_0| + |z_0|\,\right]^n = A + B\left[r + |z_0|\right]^n \equiv M.$

BY CAUCHY'S INEQUALITY

$$|f^{(n+1)}(z_0)| \leq \frac{M\,(n+1)!}{r^{n+1}} = \frac{\left[A + B\left(r + |z_0|\right)^n\right](n+1)!}{r^{n+1}} \to 0$$

AS $r \to \infty$. THUS $|f^{(n+1)}(z_0)| = 0 \to f^{(n+1)}(z_0) = 0.$

SINCE THIS APPLIES FOR ALL z_0 WE HAVE

$$f^{(n+1)}(z) = 0 \quad \forall z.$$

$\to$ INTEGRATING n TIMES GIVES $f(z) =$ POLYNOMIAL OF degree n.

EXAMPLE LET $p(z) = a_0 + a_1 z + \ldots + a_{N-1} z^{N-1} + a_N z^N$ WITH $N \geq 1$ integer.

LET $C: |z| = 1$ AND SUPPOSE THAT $|p(z)| \leq M$ ON $|z| = 1$. $a_N \neq 0$.

PROVE THAT $|a_k| \leq M$. $\forall k = 0, \ldots, N$.

PROOF BY DIFFERENTIATING k TIMES $p^{(k)}(0) = k!\, a_k$.

THUS BY C.I. INEQUALITY WITH $R = 1$,

$$|p^{(k)}(0)| \leq \frac{M\,k!}{R^k} = M\,k!.$$

BUT $p^{(k)}(0) = k!\, a_k \longrightarrow |a_k|\, k! \leq M\,k! \to |a_k| \leq M$

$\forall k = 0, \ldots, N.$

THE NEXT APPLICATION OF LIOUVILLE'S THEOREM IS TO THE "FUNDAMENTAL THEOREM OF ALGEBRA".

<u>THEOREM</u> LET $a_0, .., a_n$ BE COMPLEX NUMBERS, $n \geq 1$ AND $a_n \neq 0$. DEFINE THE POLYNOMIAL $p(z) = a_0 + a_1 z + .. + a_n z^n$.

THEN THERE IS A POINT z_0 SUCH THAT $p(z_0) = 0$. □

<u>REMARK</u> BY REPEATEDLY FACTORING OUT ROOTS IT FOLLOWS THAT A POLYNOMIAL OF DEGREE n HAS n ROOTS (POSSIBLY NOT DISTINCT).

<u>PROOF</u> BY CONTRADICTION. SUPPOSE $p(z)$ IS A NONCONSTANT POLYNOMIAL WITH NO ROOTS. THEN $\frac{1}{p(z)}$ IS AN ENTIRE FUNCTION AND NECESSARILY CONTINUOUS. WRITING

$$p(z) = a_n z^n \left[1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + .. \right]$$

WE SEE THAT $\lim_{|z| \to \infty} \frac{|p(z)|}{|z|^n} = a_n$ AND SO $\lim_{|z| \to \infty} |p(z)| = \infty$.

THIS YIELDS $\lim_{|z| \to \infty} \frac{1}{|p(z)|} = 0$.

BY CONTINUITY $\exists R$ SUFFICIENTLY LARGE SO THAT

$$\frac{1}{|p(z)|} \leq 1 \quad \text{FOR } |z| \geq R.$$

BUT FOR $|z| \leq R$, $\frac{1}{|p(z)|}$ IS CONTINUOUS ON A CLOSED REGION AND HENCE IS BOUNDED ABOVE SO THAT $\frac{1}{|p(z)|} \leq M$ ON $|z| \leq R$.

AS SUCH $\frac{1}{|p(z)|} \leq \text{MAX}(1, M)$ ON ALL OF $\mathbb{C}$ AND SO IS A BOUNDED ENTIRE FUNCTION. BY LIOUVILLE'S THM THIS MEANS THAT $p(z) =$ CONSTANT FUNCTION. (CONTRADICTION!). HENCE $p(z)$ MUST HAVE A ROOT.

ANOTHER VERSION OF MAX-MODULUS

LEMMA (MAX-MODULUS PRINCIPLE: LOCAL)

SUPPOSE THAT $|f(z)| \le |f(z_0)|$ AT EACH POINT z IN SOME DISK $|z-z_0| < r$. ASSUME $f(z)$ IS ANALYTIC IN THE DISK. THEN $f(z) = f(z_0)$ EVERYWHERE IN THE DISK $|z-z_0| \le r$.

I.E. THERE IS NO DISK WHERE $|f(z_0)|$ IS A LOCAL MAXIMUM IN THE DISK UNLESS f = CONSTANT IN THE DISK.

PROOF LET z_1 BE ANY OTHER POINT IN DISK $|z-z_0| < r$ WITH $\rho = |z_1 - z_0| < r$. LET C_ρ BE COUNTERCLOCKWISE CIRCLE $|z-z_0| = \rho$ PASSING THROUGH z_1. BY C.I.F.

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-z_0}\, dz.$$

NOW PARAMETRIZE $C_\rho : z = z_0 + \rho e^{i\varphi} \longrightarrow dz/_{z-z_0} = i\, d\varphi$ SO THAT

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\varphi})\, d\varphi \quad \text{GAUSS'S MEAN-VALUE THM.}$$

BY OUR BASIC INTEGRAL ESTIMATE

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})|\, d\varphi. \qquad (1)$$

HOWEVER, $|f(z_0 + \rho e^{i\varphi})| \le |f(z_0)|$ IN $0 \le \varphi \le 2\pi$ BY ASSUMPTION.

INTEGRATING BOTH SIDES OVER $0 \le \varphi \le 2\pi$ WE GET

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\varphi = |f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})|\, d\varphi. \qquad (2)$$

COMBINING (1) AND (2) WE HAVE $|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})|\, d\varphi$.

THIS CAN BE WRITTEN AS $\int_0^{2\pi} h(\varphi)\, d\varphi = 0$ WITH $h(\varphi) \equiv -|f(z_0 + \rho e^{i\varphi})| + |f(z_0)|$.

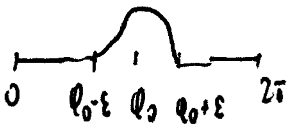
BUT $h(\varphi) \ge 0$ IS A CONTINUOUS FUNCTION. IT FOLLOWS THAT

$$h(\varphi) \equiv 0 \quad \text{FOR ANY } \rho \text{ IN } 0 \le \rho \le r.$$

REMARK IF $h(\varphi_0) > 0$ AT $\varphi = \varphi_0$. THEN BY CONTINUITY WE HAVE

$$h(\varphi) > 0 \quad \text{ON} \quad -\varepsilon + \varphi_0 < \varphi < \varphi_0 + \varepsilon$$

FOR SOME $\varepsilon > 0$.



$$\rightarrow \int_0^{2\pi} h(\varphi)\,d\varphi > 0$$

CONTRADICTION.

THEREFORE, WE HAVE

$$|f(z)| = |f(z_0)| \quad \text{FOR ALL POINTS ON } |z - z_0| = \rho.$$

NOW SINCE $0 \leq \rho < r$ IS ARBITRARY IT FOLLOWS THAT

$$|f(z)| = |f(z_0)| \quad \text{IN} \quad |z - z_0| < r.$$

SINCE THE BALL IS A DOMAIN, $|f(z)| =$ CONSTANT IMPLIES THAT $f(z)$ IS CONSTANT IN BALL.

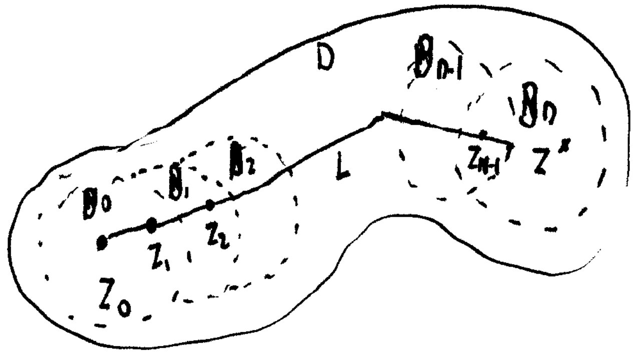
$$\longrightarrow \quad f(z) = f(z_0) \quad \text{IN} \quad |z - z_0| < r. \quad \square$$

NOW WE WANT TO EXTEND THIS RESULT TO A GLOBAL VERSION OF THE MAX-MODULUS PRINCIPLE.

<u>THEOREM</u> IF A FUNCTION $f(z)$ IS ANALYTIC AND NOT CONSTANT IN A DOMAIN D, THEN $|f(z)|$ HAS NO MAXIMUM VALUE IN D. THAT IS, THERE IS NO POINT z_0 IN D SUCH THAT $|f(z)| \leq |f(z_0)|$ FOR ALL POINTS IN D.

<u>PROOF</u> GIVEN THAT $f(z)$ IS ANALYTIC IN D WE PROVE THE THEOREM BY ASSUMING THAT $|f(z)|$ DOES HAVE A MAXIMUM VALUE AT SOME z_0 IN D AND THEN SHOWING THAT $f(z)$ MUST BE CONSTANT THROUGHOUT D.

SINCE D IS CONNECTED WE DRAW A POLYGONAL LINE LYING IN D THAT EXTENDS FROM z_0 TO ANY OTHER POINT P LABELED BY z^* IN D.



LET $d>0$ BE THE SHORTEST DISTANCE FROM POINTS ON L TO THE BOUNDARY OF D.

NEXT, OBSERVE THAT THERE IS A FINITE SEQUENCE OF POINTS

$$z_0, z_1, \ldots z_{N-1}, z_N = z^*$$

ALONG L SUCH THAT $z_N = z^*$ AND

$$|z_{k+1} - z_k| < d \qquad k = 0, \ldots, N-1.$$

THIS GIVES THE SEQUENCES OF NEIGHBOURHOODS $B_k(z_k)$ WITH

$$B_k(z_k) = \{ z \mid |z - z_k| < d \}, \quad k = 0, \ldots, N$$

IT FOLLOWS THAT $z_{k+1} \in B_k(z_k)$ SINCE $|z_{k+1} - z_k| < d$, $k = 0, N-1$ WE HAVE $f(z)$ IS ANALYTIC IN EACH $B_k(z_k)$ AND THAT $z_{k+1} \in B_k(z_k)$.

- NOW SINCE $|f(z)|$ HAS A MAXIMUM VALUE IN D AT z_0 IT HAS A MAXIMUM VALUE IN B_0 AT THAT POINT. HENCE BY LOCAL MAX-MODULUS $f(z) = f(z_0)$ EVERYWHERE IN B_0. IN PARTICULAR $f(z_1) = f(z_0)$.
- THIS MEANS THAT $|f(z)| \le |f(z_1)|$ AT EACH $z \in B_1$. THE LOCAL M-M LEMMA THEN GIVES $f(z) = f(z_1) = f(z_0)$ FOR $z \in B_1$. IN PARTICULAR, THIS GIVES $f(z_2) = f(z_0)$.
- HENCE $|f(z)| \le |f(z_2)|$ IN B_2, SO THAT $f(z) = f(z_2) = f(z_0)$ WHEN z IN B_2. CONTINUING IN THIS MANNER WE OBTAIN THAT

$$f(z_N) = f(z_0).$$

RECALLING THAT z_N COINCIDES WITH THE POINT P, WHICH IS ANY POINT OTHER THAN z_0 IN D. WE CONCLUDE THAT

$$f(z) = f(z_0) \text{ FOR EVERY POINT } z \text{ IN } D.$$

THEREFORE, $f(z)$ IS A CONSTANT EVERYWHERE IN D.

NOW LET $R = D \cup \bar{D}$ WHERE $\bar{D}$ IS BOUNDARY OF D.
NOW R IS A CLOSED BOUNDED REGION. SINCE ANALYTIC FUNCTIONS ARE CONTINUOUS IT FOLLOW THAT $M = \max_{R} |f(z)|$ EXISTS.

THAT IS $\exists\, M > 0$ S.T. $|f(z)| \leq M \ \forall z \in D \cup \bar{D}$, WITH EQUALITY HOLDING FOR AT LEAST ONE SUCH POINT.

SO IF $f(z)$ IS A CONSTANT FUNCTION $\longrightarrow |f(z)| = M \ \forall z \in D \cup \bar{D}$.
IF HOWEVER, $f(z)$ IS NOT CONSTANT, THEN BY THM JUST PROVED,

$$|f(z)| \neq M \text{ FOR ANY } z \in D.$$

<u>THEOREM</u> SUPPOSE THAT $f(z)$ IS ANALYTIC EVERYWHERE IN A DOMAIN D AND ON ITS BOUNDARY ∂D. ~~ASSUME~~

$$|f(z)| \leq \max_{z \in \partial D} |f(z)| \quad \text{FOR } z \in D.$$

I.E. FOR NON-CONSTANT FUNCTIONS A STRICT MAXIMUM OCCURS SOMEWHERE ON BOUNDARY ∂D AND NEVER IN INTERIOR D.

THEOREM (MAX-MODULUS HARMONIC FUNCTIONS)

LET $f(z)$ BE ANALYTIC EVERYWHERE IN A DOMAIN D AND ON ITS BOUNDARY ∂D. DEFINE $u = \mathrm{Re}[f(z)]$ SO THAT $u(x,y)$ IS HARMONIC. THEN

$$(*) \quad u(x,y) \le \max_{\partial D} u(x,y) \quad \text{FOR ANY } (x,y) \text{ IN } D.$$

IN OTHER WORDS, THE MAXIMUM OF u MUST OCCUR ON THE BOUNDARY, UNLESS u = CONSTANT EVERYWHERE.

PROOF LET V = HARMONIC CONJUGATE OF u AND WRITE $f = u + iV$.

DEFINE $g(z) = e^{f(z)}$. THEN $g(z)$ IS ANALYTIC IN D AND ON $\partial\Omega$ AND SO SATISFIES MAX-MODULUS PRINCIPLE $|g(z)| \le \max_{\partial D} |g(z)|$ IN D.

BUT $|g| = |e^{u+iV}| = |e^u||e^{iV}| = e^u$.

$$\text{SO} \quad e^u \le \max_{\partial D} e^u \quad \text{IN } D.$$

SINCE e^s IS MONOTONE INCREASING WE GET $(*)$ $\square$.

NOW IS THERE A MIN-MODULUS PRINCIPLE? YES BUT UNDER AN EXTRA CONDITION THAT $f(z)$ HAS NO ZEROES IN D.

THEOREM (MIN-MODULUS) LET $f(z)$ BE ANALYTIC EVERYWHERE IN D AND ON ∂D. MOREOVER, ASSUME THAT $f \neq 0$ AT ANY POINT IN D OR ON ∂D. THEN $\min |f(z)|$ OCCURS ON ∂D.

PROOF: LET $g(z) = \dfrac{1}{f(z)}$. THEN SINCE $f(z) \neq 0$ IN D OR ∂D, $g(z)$ IS ANALYTIC IN D AND ON ∂D. BY APPLYING MAX-MODULUS PRINCIPLE TO $|g(z)|$ WE OBTAIN THE MIN-MODULUS RESULT FOR $|f(z)|$ $\square$

REMARKS

(i) FOR MIN-MODULUS PRINCIPLE FOR $|f|$ WE MUST HAVE $f \neq 0$ IN D. TO SEE THIS CONSIDER

$$f(z) = z \text{ in } |z| \leq 1. \text{ HERE } D: |z| < 1 \text{ AND } \partial D: |z| = 1$$

THEN $|f(0)| = 0$ AND $|f(z)| = 1$ ON ∂D.

$\rightarrow$ MIN-MODULUS DOES NOT HOLD.

(ii) HOWEVER, MIN-MODULUS PRINCIPLE DOES HOLD FOR HARMONIC FUNCTIONS SINCE THE PROOF WAS BASED ON $g = e^f = e^{u+iv}$ AND SO $g \neq 0$ IN DOMAIN D. AS A RESULT WE HAVE THAT IF $f = u + iv$ IS ANALYTIC IN D AND ON ∂D, THEN

$$u \geq \min_{\partial D} u \quad \text{MUST HOLD AT EACH } (x,y) \text{ IN } D.$$

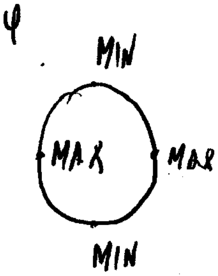
EXAMPLE 1 FIND MAX AND MIN OF $|e^{z^2}|$ IN $|z| \leq 1$.

SINCE $f(z) = e^{z^2}$ IS ENTIRE AND $f \neq 0$ IN $|z| \leq 1$, THE MAX AND MIN MUST OCCUR ON $|z| = 1$. ON $|z| = 1$ LET $z = e^{i\varphi}$.

WE HAVE $$|e^{z^2}| = |e^{e^{2i\varphi}}| = |e^{\cos 2\varphi + i \sin 2\varphi}| = e^{\cos 2\varphi}.$$

SO $$\max |e^{z^2}| = e^1 \quad (\text{WHEN } \varphi = 0, \pi;\ \text{i.e. } z = \pm 1)$$

$$\min |e^{z^2}| = e^{-1} \quad (\text{WHEN } \varphi = \pi/2, 3\pi/2,\ \text{i.e. } z = \pm i)$$



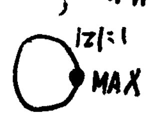
EXAMPLE 2 FIND MAX AND MIN OF $|z^3 e^z|$ ON $|z| \leq 1$.

SOLUTION LET $f(z) = z^3 e^z$. SINCE $f(0) = 0$ THE MIN-MODULUS PRINCIPAL DOES NOT APPLY AND WE HAVE TRIVIALLY THAT $\min_{|z| \leq 1} |f(z)| = |f(0)| = 0$.

BUT BY MAX-MODULUS PRINCIPLE, $|f(z)| \leq \max_{|z|=1} |f(z)|$ IN $|z| \leq 1$.

NOW ON $|z| = 1$, $|f(z)| = |z^3 e^z| = |z|^3 |e^z| = e^{\cos\varphi}$, WHICH HAS MAX AT $\varphi = 0$.

SO $|f(z)| \leq e^1$ IN $|z| \leq 1$



IMPORTANT!

EXAMPLE 3 SUPPOSE THAT $f(z)$ IS ANALYTIC INSIDE AND ON C AND THAT $|f(z)-1|<1$ FOR ALL z ON C. PROVE THAT $f(z)$ HAS NO ZEROES INSIDE C (i.e. no points z inside C where $f(z)=0$).

PROOF LET $g(z)=f(z)-1$. $g(z)$ is analytic inside and on C. THEN BY ASSUMPTION $|g(z)|<1$ ON C.

BY MAX-MODULUS PRINCIPLE, $|g(z)|<1$ FOR ALL z inside C. SINCE $f(z_0)=0$ FOR SOME z_0 inside C implies that $|g(z_0)|=1$, THE CONDITION $|g(z)|<1$ FOR ALL z inside C leads to a CONTRADICTION.

THUS, $f(z)=0$ HAS NO ROOTS inside C.

EXAMPLE 4 PROVE THAT THE POLYNOMIAL $p(z)=\frac{1}{3}z^6+\frac{1}{4}z^4+\frac{1}{6}z^3+1=0$ HAS NO ROOTS z inside $|z|\leq 1$.

SOL WE CALCULATE $|p(z)-1|\leq \frac{1}{3}+\frac{1}{4}+\frac{1}{6}=\frac{3}{4}<1$ FOR $|z|\leq 1$ BY Δ INEQUALITY.

THUS THE PROOF OF EXAMPLE 3 $\longrightarrow$ NO ROOTS TO $p(z)=0$ IN $|z|\leq 1$.

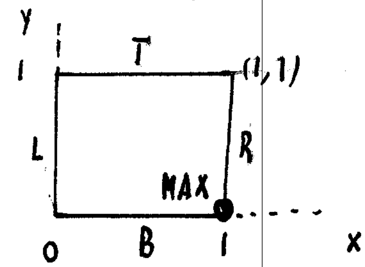
EXAMPLE 5 LET $f(z)$ BE ANALYTIC IN $1\leq |z|\leq 2$ AND $|f(z)|\leq 3$ ON $|z|=1$ AND $|f(z)|\leq 12$ ON $|z|=2$. PROVE THAT $|f(z)|\leq 3|z|^2$ FOR $1\leq |z|\leq 2$.

PROOF DEFINE $g(z)=f(z)/3z^2$. THIS IS ANALYTIC IN $1\leq |z|\leq 2$ AND THE DATA ON $|f(z)|$ ON $|z|=1$ AND $|z|=3$ PROVE THAT $|g(z)|<1$ ON $|z|=1$ AND ON $|z|=2$. THUS, BY MAX-MODULUS PRINCIPLE

$$|g(z)|\leq 1 \text{ ON } 1\leq |z|\leq 2 \longrightarrow |f(z)|\leq 3|z|^2 \text{ ON } 1\leq|z|\leq 2.$$

EXAMPLE 6 FIND MAXIMUM OF $u = \mathrm{Re}(z^3)$ ON THE SQUARE $[0,1] \times [0,1]$.

SOLUTION SINCE u IS HARMONIC AND THE REAL PART OF AN ANALYTIC FUNCTION THEN THE MAXIMUM OF u OCCURS ON THE BOUNDARY OF THE SQUARE BY MAX-MODULUS THM:



$$u = \mathrm{Re}\left[(x+iy)^3\right] = x^3 - 3xy^2$$

- ON L: $x=0 \rightarrow u=0$
- ON B: $y=0 \rightarrow u = x^3 \rightarrow$ MAX AT $x=1 \rightarrow u_{MAX}=1$
- ON R: $x=1 \rightarrow u = 1-3y^2 \rightarrow$ MAX AT $y=0 \rightarrow u_{MAX}=1$
- ON T: $y=1 \rightarrow u = x^3 - 3x = x^2(x-3) \rightarrow$ MAX AT $x=0$ $u_{MAX}=0$

$\Longrightarrow$ MAX OCCURS AT $(x,y) = (1,0)$ WITH $u_{MAX} = 1$.

EXAMPLE 7 FIND THE MAXIMUM OF $|z^2 + 4z - 2|$ FOR $|z| \leq 1$.

SOLUTION SINCE $g(z) \equiv z^2 + 4z - 2$ IS ANALYTIC FOR $|z| \leq 1$, THE MAXIMUM OF $|g(z)|$ OCCURS ON $|z|=1$. WE WORK WITH $|g(z)|^2$ AND PARAMETRIZE $z = e^{i\varphi}$. SO $|e^{2i\varphi} + 4e^{i\varphi} - 2|^2$ BECOMES USING $|w|^2 = w\bar{w}$ $\rightarrow$ $|g|^2 = (e^{2i\varphi} + 4e^{i\varphi} - 2)(e^{-2i\varphi} + 4e^{-i\varphi} - 2)$

THIS GIVES
$$h(\varphi) \equiv |g(e^{i\varphi})|^2 = 1 + 4e^{i\varphi} - 2e^{2i\varphi} + 4e^{-i\varphi} + 16 - 8e^{i\varphi} - 2e^{-2i\varphi} - 8e^{-i\varphi} + 4.$$

GROUPING TERMS:
$$h(\varphi) = 21 - 4e^{i\varphi} - 4e^{-i\varphi} - 2e^{2i\varphi} - 2e^{-2i\varphi}$$

BUT $e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$, $e^{2i\varphi} + e^{-2i\varphi} = 2\cos(2\varphi)$.

THUS $h(\varphi) = 21 - 8\cos\varphi - 4\cos(2\varphi)$. EXTREME VALUES WHERE $h'(\varphi) = 0 \rightarrow 8\sin\varphi + 8\sin(2\varphi) = 0 \rightarrow 8\sin\varphi + 16\sin\varphi\cos\varphi = 0$. THIS GIVES $\sin\varphi = 0 \rightarrow \varphi = 0, \pi$ OR $\cos\varphi = -1/2 \rightarrow \varphi = 2\pi/3, 4\pi/3$. SINCE $h''(\varphi) = 8\cos\varphi + 16\cos(2\varphi)$ WE HAVE $h''(0) > 0$ AND $h''(\pi) > 0$.

THEREFORE $\varphi = 0, \pi$ ARE LOCAL MIN.

HOWEVER $h''(2\pi/3) = 8\cos(2\pi/3) + 16\cos(4\pi/3) = -4 - 8 = -12 < 0.$

$h''(4\pi/3) = 8\cos(4\pi/3) + 16\cos(8\pi/3) = -12 < 0.$

WE CONCLUDE THAT $\varphi = 2\pi/3, 4\pi/3$ ARE LOCAL MAX.

WE CALCULATE WITH $h(\varphi) = 21 - 8\cos\varphi - 4\cos(2\varphi)$ THAT

$$h(2\pi/3) = 21 - 8(-1/2) - 4(-1/2) = 27$$

$$h(4\pi/3) = 21 - 8(-1/2) - 4(-1/2) = 27.$$

WE RECALL THAT $|g(e^{i\varphi})| = \sqrt{h(\varphi)}$

AND SO MAX-VALUE OF $|g(z)|$ IN $|z| \leq 1$

IS $|g| = \sqrt{27}$

AND IT OCCURS AT $z = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$

$z = e^{4\pi i/3} = -1/2 - i\sqrt{3}/2.$