MATH 257/316 HW 10

PROBLEM 1 FIND THE SOLUTION TO LAPLACE'S EQUATION

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 0 \le r \le 1, \quad 0 \le \varphi \le \pi$$

WITH $u(1, \varphi) = 2 \sin(\varphi/2) + 4 \sin(3\varphi/2)$; u BOUNDED AS $r \to 0$

AND $u(r, 0) = 0$, $u_\varphi(r, \pi) = 0$.

PROBLEM 2 SOLVE

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 0 \le r \le 2, \quad 0 \le \varphi \le 2\pi$$

WITH $u(2, \varphi) = 1 + 4 \cos^2 \varphi$; u BOUNDED AS $r \to 0$

AND u, u_φ ARE 2π PERIODIC IN φ.

PROBLEM 3 SOLVE THE EXTERIOR DOMAIN PROBLEM

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 1 \le r < \infty, \quad 0 \le \varphi \le 2\pi$$

WITH $u(1, \varphi) = 2 \cos^2 \varphi$; u BOUNDED AS $r \to +\infty$

u, u_φ ARE 2π PERIODIC IN φ.

PROBLEM 4 CONSIDER THE EIGENVALUE PROBLEM

$$(*) \begin{cases} \Phi'' + \lambda \Phi = 0 \quad \text{IN} \quad 0 \le x \le L \\ \Phi'(0) - \Phi(0) = 0, \quad \Phi(L) = 0 \quad \text{WITH} \quad L > 0. \end{cases}$$

(i) PROVE THAT ANY EIGENVALUE MUST BE REAL AND POSITIVE.

(ii) SHOW GRAPHICALLY FROM SOLVING (*) THAT THERE ARE AN INFINITE NUMBER OF POSITIVE EIGENVALUES.

(iii) SHOW HOW TO SOLVE THE HEAT CONDUCTION PROBLEM

$$u_t = u_{xx} \quad \text{IN} \quad 0 \le x \le L, \quad t > 0 \text{ ; WITH IC } u(x, 0) = f(x)$$

AND WITH $u_x(0,t) = u(0,t) - A$ AND $u(L,t) = B$ WHERE A, B CONSTANTS

PROBLEM 5 SOLVE THE WAVE EQUATION WHERE THE TENSION IN THE STRING VARIES WITH POSITION:

$$u_{tt} = c^2 \left[x^2 u_x \right]_x \quad \text{IN} \quad 1 \leq x \leq 2, \; t > 0$$

WITH IC: $u(x,0) = f(x)$, $\quad u_t(x,0) = g(x)$

AND BC: $\quad u(1,t) = 0, \quad u(2,t) = 0.$

(HINT: YOU NEED TO EXPLICITLY FIND THE EIGENVALUES OF THE EULER EQUATION $\left(x^2 \Phi' \right)' + \lambda \Phi = 0$ IN $1 \leq x \leq 2$

WITH $\Phi(1) = 0, \quad \Phi(2) = 0$).

PROBLEM 6 (NOT TO BE TURNED IN).

SUPPOSE $\quad u_{xx} + u_{yy} = 0,$

DEFINE $\quad x = r\cos\varphi, \; y = r\sin\varphi \quad$ AND $\quad U(r,\varphi) \equiv u\left[r\cos\varphi, r\sin\varphi \right].$

SHOW THAT $\quad U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} = 0.$

MATH 257/316 SOLUTIONS

PROBLEM 1 $\quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \;; \quad u(r,0) = u_\varphi(r,\pi) = 0$

WE SEPARATE VARIABLES: $u = R(r)\,\Phi(\varphi)$ TO GET $\dfrac{r^2 R'' + rR'}{R} = \dfrac{-\Phi''}{\Phi} = \lambda.$

THUS
$$\Phi'' + \lambda \Phi = 0, \quad 0 < \varphi < \pi$$
$$\Phi(0) = 0, \quad \Phi'(\pi) = 0$$

SO $\quad \Phi = \sin[\sqrt{\lambda}\,\varphi]$ AND $\Phi'(\pi) = 0 \rightarrow \sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) = 0.$

THUS $\quad \sqrt{\lambda}\,\pi = \frac{(2n-1)}{2}\pi, \quad n = 1, 2, \ldots$ OR $\lambda_n = (n - 1/2)^2, \quad n = 1, 2, \ldots$

THUS $\quad \Phi_n(\varphi) = \sin[(n - 1/2)\varphi].$

NOW $\quad r^2 R_n'' + r R_n' - \lambda_n R_n = 0$ SO $R_n = c_1 r^{\sqrt{\lambda_n}} + c_2 r^{-\sqrt{\lambda_n}}.$

FOR BOUNDEDNESS AS $r \rightarrow 0$ SET $c_2 = 0$ AND $c_1 = 1$ WLOG.

THUS $\quad R_n(r) = r^{n - 1/2}.$

WE HAVE BY SUPERPOSITION:
$$u(r,\varphi) = \sum_{n=1}^{\infty} c_n r^{n-1/2} \sin[(n - 1/2)\varphi].$$

FINALLY $\quad u(1,\varphi) = 2\sin(\varphi/2) + 4\sin(3\varphi/2) = \sum_{n=1}^{\infty} c_n \sin[(n - 1/2)\varphi].$

WE GET $c_1 = 2, \; c_2 = 4$ AND OTHER $c_n = 0$ FOR $n \geq 3.$

THUS $\quad u(r,\varphi) = 2 r^{1/2} \sin(\varphi/2) + 4 r^{3/2} \sin(3\varphi/2)$

PROBLEM 2 SOLVE $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}=0$ IN $0\le r\le 2,\ 0\le\varphi\le 2\pi$

$$u(2,\varphi)=1+4\cos^2\varphi,\quad u \text{ BOUNDED AS } r\to 0$$

u, u_φ ARE 2π PERIODIC IN φ.

BY SEPARATION OF VARIABLES, WE FOUND FROM CLASS THAT

$$u(r,\varphi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\varphi)+b_n\sin(n\varphi)\right]r^n$$

NOW SINCE $\cos^2\varphi=\dfrac{1+\cos(2\varphi)}{2}$ WE GET

$$u(2,\varphi)=3+2\cos(2\varphi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}2^n\left[a_n\cos(n\varphi)+b_n\sin(n\varphi)\right].$$

WE CONCLUDE THAT $b_n=0$ FOR $n=1,2,3\ldots$; $a_0=6$, $2^2a_2=2$

SO $a_2=1/2$. THUS

$$u(r,\varphi)=3+\frac{1}{2}r^2\cos(2\varphi).$$

PROBLEM 3 $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}=0$ IN $1\le r<\infty,\ 0\le\varphi\le 2\pi$

$$u(1,\varphi)=2\cos^2\varphi;\quad u \text{ BOUNDED AS } r\to+\infty$$

u, u_φ ARE 2π PERIODIC IN φ.

UPON SEPARATING VARIABLES WE GET $\Phi''+\lambda\Phi=0$, Φ,Φ' 2π PERIODIC SO $\lambda=n^2$, $n=0,1,$

THEN $r^2R_n''+rR_n'-n^2R_n=0$ SO $R_n=C_1r^n+C_2r^{-n}$, $n\ge 1$

FOR BOUNDEDNESS AS $r\to+\infty$ WE SET $C_1=0$.

FOR $n=0$ WE GET $R_0=C_1+C_2\log r$ SO $C_2=0$ IS NEEDED.

BY SUPERPOSITION

$$u(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n \cos(n\varphi) + b_n \sin(n\varphi)\right]$$

NOW $\quad u(1,\varphi) = 2\cos^2\varphi = 1 + \cos(2\varphi) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos(n\varphi) + b_n \sin(n\varphi)\right)$

THUS SET $a_0 = 2$, $a_2 = 1$ AND all other a_n, b_n TO ZERO.

THUS $\quad u(r,\varphi) = 1 + \dfrac{1}{r^2}\cos(2\varphi).$

PROBLEM 4

CONSIDER THE EIGENVALUE PROBLEM

$$\Phi'' + \lambda\Phi = 0 \quad \text{IN} \quad 0 \le x \le L$$

$$\Phi'(0) - \Phi(0) = 0, \quad \Phi(L) = 0.$$

(i) DEFINE $\quad \mathcal{L}\Phi \equiv -\Phi''.$

THEN $\quad \mathcal{L}\Phi = \lambda\Phi$

WITH $\quad \Phi'(0) = \Phi(0), \quad \Phi(L) = 0.$

WE HAVE BY INTEGRATION BY PARTS THAT

$$(1) \quad \int_0^L (u\mathcal{L}v - v\mathcal{L}u)\,dx = -uv'\Big|_0^L + u'v\Big|_0^L.$$

TO PROVE EIGENVALUES ARE REAL SET $u = \Phi$ AND $v = \bar{\Phi}$.

THEN $\quad \displaystyle\int_0^L (\Phi\mathcal{L}\bar{\Phi} - \bar{\Phi}\mathcal{L}\Phi)\,dx = -\Phi\bar{\Phi}'\Big|_0^L + \Phi'\bar{\Phi}\Big|_0^L$

SO $\quad \displaystyle\int_0^L (\Phi\bar{\lambda}\bar{\Phi} - \bar{\Phi}\lambda\Phi)\,dx = \Phi(0)\bar{\Phi}'(0) - \Phi'(0)\bar{\Phi}(0)$

SINCE $\Phi(L) = \bar{\Phi}(L) = 0.$

BUT $\Phi'(0) = \Phi(0)$, $\bar{\Phi}'(0) = \bar{\Phi}(0)$ SO SINCE $\Phi\bar{\Phi} = |\Phi|^2$

$$(\bar{\lambda} - \lambda)\int_0^L |\Phi|^2\,dx = \Phi(0)\,\bar{\Phi}(0) - \Phi(0)\,\bar{\Phi}(0) = 0.$$

THUS SINCE $\int_0^L |\Phi|^2\,dx > 0 \rightarrow \lambda = \bar{\lambda}$ SO λ IS REAL.

NOW WE PROVE EIGENFUNCTIONS CORRESPONDING TO DIFFERENT EIGENVALUES ARE ORTHOGONAL:

SUPPOSE $\mathcal{L}\Phi_n = \lambda_n \Phi_n$ AND $\mathcal{L}\Phi_m = \lambda_m \Phi_m$ WITH $\lambda_n \neq \lambda_m$.

THEN, (1) GIVES $\int_0^L (\Phi_n \mathcal{L}\Phi_m - \Phi_m \mathcal{L}\Phi_n)\,dx = 0$

OR $(\lambda_m - \lambda_n)\int_0^L \Phi_m \Phi_n\,dx = 0.$

SINCE $\lambda_n \neq \lambda_m \rightarrow \int_0^L \Phi_n \Phi_m\,dx = 0.$ (2)

NOW FINALLY SHOW $\lambda > 0$. WE MULTIPLY

$$\Phi\,\mathcal{L}\Phi = \lambda\Phi^2.$$

INTEGRATE $\int_0^L -\Phi\,\Phi''\,dx = \lambda\int_0^L \Phi^2\,dx$

NOW USE IBP. $u = \Phi$, $du = \Phi'\,dx$

$dv = \Phi''\,dx$ $\quad v = \Phi'$

SO $$-\Phi\,\Phi'\Big|_0^L + \int_0^L (\Phi')^2\,dx = \lambda\int_0^L \Phi^2\,dx$$

BUT $\Phi(L) = 0$ SO

$$\Phi(0)\,\Phi'(0) + \int_0^L (\Phi')^2\,dx = \lambda\int_0^L \Phi^2\,dx$$

$\rightarrow$ SINCE $\Phi'(0) = \Phi(0)$ THEN

$$(\Phi(0))^2 + \int_0^L (\Phi')^2\,dx = \lambda\int_0^L \Phi^2\,dx$$ SO $\lambda > 0$.

(ii) THE SOLUTION TO $\Phi'' + \lambda \Phi = 0$ WITH $\Phi(L) = 0$ IS

$$\Phi = \sin\left[\sqrt{\lambda}\,(x - L)\right].$$

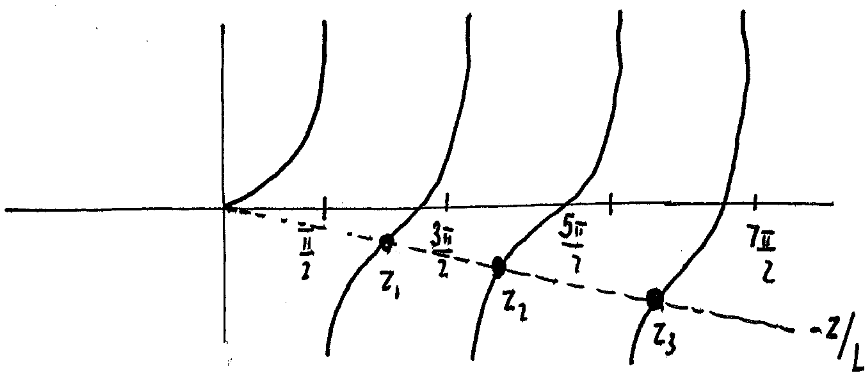
NOW $\Phi'(0) = \Phi(0)$ YIELDS $\sqrt{\lambda}\,\cos\left[-\sqrt{\lambda}\,L\right] = \sin\left[-\sqrt{\lambda}\,L\right]$.

BUT $\cos(z) = \cos(-z)$ AND $\sin(z) = -\sin(-z)$ SO THAT

$$\tan\left(\sqrt{\lambda}\,L\right) = -\sqrt{\lambda}.$$

WE WRITE THIS AS $\tan(z) = -\dfrac{z}{L}$, WHERE $z = \sqrt{\lambda}\,L > 0$.

WE PLOT



SO THAT $\exists$ AN INFINITE # ROOTS z_n WITH $\lambda_n = z_n^2/L^2$, $n = 1, 2, \ldots$

THUS $\dfrac{(2n-1)\pi}{2} < z_n < \dfrac{(2n+1)\pi}{2}$ FOR $n = 1, 2, \ldots$

(iii) NOW SOLVE $u_t = u_{xx}$ IN $0 \le x \le L$, $t > 0$

$$u_x(0,t) = u(0,t) - A, \qquad u(L,t) = B$$

$$u(x,0) = f(x).$$

THE STEADY-STATE SATISFIES $u_s'' = 0$

$$u_s'(0) = u_s(0) - A, \qquad u_s(L) = B.$$

WE GET $u_s = ax + b$ AND $a = b - A$

$aL + b = B$

WE OBTAIN THAT: $a = \dfrac{B - A}{1 + L}$,

$b = (B + AL)/\ldots$

NOW WRITE $u(x,t) = u_s(x) + v(x,t)$

SO THAT
$$\begin{cases} v_t = v_{xx} \\ v_x(0,t) = v(0,t), \quad v(L,t) = 0 \\ v(x,0) = f(x) - u_s(x) \end{cases}$$

SEPARATE VARIABLES: $v = \Phi(x)\, T(t)$

SO THAT $\dfrac{T'}{T} = \dfrac{\Phi''}{\Phi} = -\lambda$

$$\Phi'' + \lambda \Phi = 0$$
$$\Phi'(0) = \Phi(0), \quad \Phi(L) = 0$$
$\longrightarrow$ $\Phi_n(x) = \sin\left[\sqrt{\lambda_n}\,(x - L)\right]$

WITH $\lambda_n = z_n^2 / L^2$ AS SHOWN IN (ii)

NOW $T_n = e^{-\lambda_n t}$

BY SUPERPOSITION,
$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\, \Phi_n(x).$$

WE SET $v(x,0) = f(x) - u_s(x) = \sum_{n=1}^{\infty} c_n \Phi_n(x).$

BY ORTHOGONALITY $c_n = \dfrac{\int_0^L \left(f(x) - u_s(x)\right) \Phi_n(x)\, dx}{\int_0^L \left(\Phi_n(x)\right)^2 dx}$

SO THAT $u(x,t) = u_s(x) + \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\, \Phi_n(x).$

PROBLEM 5 $u_{tt} = c^2 [x^2 u_x]_x$ in $1 \le x \le 2$, $t > 0$

$u(1,t) = u(2,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

WE SEPARATE VARIABLES $u(x,t) = T(t)\,\Phi(x)$ TO OBTAIN

$$\frac{T''}{c^2 T} = \frac{(x^2 \Phi')'}{\Phi} = -\lambda .$$

THUS
$$\left.\begin{array}{l} (x^2 \Phi')' + \lambda \Phi = 0 \quad \text{IN } 1 \le x \le 2 \\ \Phi(1) = \Phi(2) = 0. \end{array}\right\} *$$

AFTER FINDING EIGENVALUES λ_n FROM $(*)$, THEN

$$T_n'' + w_n^2 T_n = 0 \qquad \text{WHERE} \qquad w_n \equiv c\sqrt{\lambda_n}.$$

THE SOLUTION IS $T_n = A_n \cos(w_n t) + B_n \sin(w_n t)$.

BY SUPERPOSITION WE HAVE

$$(1) \qquad u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(w_n t) + B_n \sin(w_n t) \right] \Phi_n(x).$$

NOW WE HAVE TO IMPOSE

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \Phi_n(x)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n w_n \Phi_n(x)$$

NOW BY ORTHOGONALITY WE HAVE $A_n = \dfrac{\int_1^2 f(x)\,\Phi_n(x)\,dx}{\int_1^2 (\Phi_n(x))^2\,dx}$, $B_n w_n = \dfrac{\int_1^2 g(x)\,\Phi_n(x)\,dx}{\int_1^2 (\Phi_n(x))^2\,dx}$

THUS WE HAVE

$$A_n = \frac{\int_1^2 f(x)\,\Phi_n(x)\,dx}{\int_1^2 (\Phi_n(x))^2\,dx}, \qquad B_n = \frac{\int_1^2 g(x)\,\Phi_n(x)\,dx}{w_n \int_1^2 (\Phi_n(x))^2\,dx}.$$

THIS GIVES THE SOLUTION IN (1).

OUR LAST ISSUE IS TO CALCULATE EIGENVALUES FROM (*).

WE WRITE $\mathcal{L}\Phi = -(x^2\Phi')'$

SO THAT
$$\begin{cases} \mathcal{L}\Phi = \lambda\Phi \\ \Phi(1) = \Phi(2) = 0, \end{cases}$$

THIS IS A STURM-LIOUVILLE PROBLEM FOR WHICH λ IS REAL, $\lambda \geq 0$, AND $\int_1^2 \Phi_n \Phi_m \, dx = 0$ FOR $n \neq m$. THE PROOF OF REALNESS OF λ AND ORTHOGONALITY IS BASED ON THE IDENTITY $\int_1^2 v\mathcal{L}w \, dx = \int_1^2 w\mathcal{L}v \, dx$.

NOW SOLVE IT DIRECTLY:

$$x^2\Phi'' + 2x\Phi' + \lambda\Phi = 0. \quad \text{PUT} \quad \Phi = x^B \rightarrow B(B-1) + 2B + \lambda = 0.$$

THUS $B^2 + B + \lambda = 0$ SO $B_\pm = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$.

- NOW SUPPOSE $\lambda > 1/4$ SO THAT $B = \dfrac{-1 \pm i\sqrt{4\lambda - 1}}{2}$.

THIS GIVES $\Phi = c_1 x^{-1/2} \sin\left[\dfrac{\sqrt{4\lambda-1}}{2}\log x\right] + c_2 x^{-1/2}\cos\left[\dfrac{\sqrt{4\lambda-1}}{2}\log x\right]$

NOW $\Phi(1) = 0$ YIELDS $c_2 = 0$. ALSO $\Phi(2) = 0$ GIVES,

$$\frac{\sqrt{4\lambda-1}}{2}\log 2 = n\pi, \quad n = 1, 2, .. \quad \text{SO} \quad 4\lambda - 1 = \frac{4n^2\pi^2}{(\log 2)^2}.$$

THUS $\lambda_n = \dfrac{n^2\pi^2}{(\log 2)^2} + \dfrac{1}{4}$. WE OBTAIN $\Phi_n(x) = x^{-1/2}\sin\left[\dfrac{n\pi \log x}{\log 2}\right]$.

- NOW SUPPOSE $0 < \lambda < 1/4$. THEN $\Phi = c_1 x^{B_+} + c_2 x^{B_-}$. NOW $\Phi(1) = \Phi(2) = 0$ GIVES

$$\left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 2^{B_+} + c_2 2^{B_-} = 0 \end{array}\right\}$$ ONLY SOLUTION IS $c_1 = c_2 = 0 \rightarrow \Phi = 0$ SO NO EIGENFUNCTION.

- SIMILARLY $\lambda = 1/4$ IS NOT AN EIGENVALUE.

IN CONCLUSION IN (1) WE HAVE $\Phi_n(x) = x^{-1/2}\sin\left(\dfrac{n\pi\log x}{\ldots}\right)$, $\lambda_n = \dfrac{n^2\pi^2}{\ldots} + \dfrac{1}{4}$

PROBLEM 6

$$u_{xx} + u_{yy} = 0.$$

LET $x = r\cos\varphi$, $y = r\sin\varphi$. SO $r^2 = x^2 + y^2$, $\tan\varphi = y/x$.

NOW $2r r_x = 2x$ SO $r_x = x/r$, $r_y = y/r$.

NOW $r_{xx} = 1/r - x r_x/r^2 = 1/r - x^2/r^3 = \frac{1}{r}\left[1 - \frac{x^2}{x^2+y^2}\right] = \frac{y^2}{r^3}$.

SIMILARLY $r_{yy} = x^2/r^3$. NOW $\sec^2\varphi\ \varphi_x = -y/x^2$ SO $(1+\tan^2\varphi)\varphi_x = -y/x^2$

OR $(1 + y^2/x^2)\varphi_x = -y/x^2 \longrightarrow \varphi_x = -y/r^2$ SIMILARLY $\varphi_y = x/r^2$.

THEN $r_{xy} = -\frac{x}{r^2} r_y = -\frac{xy}{r^3}$. NOW $\varphi_{xx} = +\frac{2y}{r^3} r_x = \frac{2yx}{r^4}$, $\varphi_{yy} = -\frac{2x}{r^3} r_y = \frac{-2xy}{r^4}$.

ALSO $\varphi_{xy} = -1/r^2 + 2y/r^3\ y/r = -1/r^2 + 2y^2/r^4 = 1/r^2\left[-1 + 2y^2/r^2\right]$

IN SUMMARY, WE HAVE

$$r_x = x/r,\quad r_y = y/r,\quad r_{xx} = y^2/r^3,\quad r_{yy} = x^2/r^3,\quad r_{xy} = -xy/r^3$$

$$\varphi_x = -y/r^2,\quad \varphi_y = x/r^2,\quad \varphi_{xx} = 2xy/r^4,\quad \varphi_{yy} = -2xy/r^4,\quad \varphi_{xy} = \frac{1}{r^4}\left[y^2 - x^2\right]$$

NOW CHANGE VARIABLES:

$$u_x = U_r r_x + U_\varphi \varphi_x$$

$$u_{xx} = U_{rr} r_x^2 + U_r r_{xx} + U_{\varphi\varphi}\varphi_x^2 + U_\varphi \varphi_{xx} + U_{r\varphi} r_x \varphi_x + U_{r\varphi} r_x \varphi_x$$

$$u_{yy} = U_{rr} r_y^2 + U_r r_{yy} + U_{\varphi\varphi}\varphi_y^2 + U_\varphi \varphi_{yy} + 2U_{r\varphi} r_y \varphi_y.$$

THUS $\Delta u = u_{xx} + u_{yy} = U_{rr}(r_x^2 + r_y^2) + U_r\left[r_{xx} + r_{yy}\right] + U_{\varphi\varphi}(\varphi_x^2 + \varphi_y^2)$
$+ U_\varphi(\varphi_{xx} + \varphi_{yy}) + 2U_{r\varphi}(r_x\varphi_x + r_y\varphi_y)$.

WE CALCULATE $r_x^2 + r_y^2 = \frac{1}{r^2}(x^2+y^2) = 1$. $r_{xx} + r_{yy} = (x^2+y^2)/r^3 = 1/r$.

$\varphi_x^2 + \varphi_y^2 = \frac{1}{r^4}(x^2+y^2) = 1/r^2$, $\varphi_{xx} + \varphi_{yy} = 0$, $r_x\varphi_x + r_y\varphi_y = -\frac{xy}{r^3} + \frac{xy}{r^3} = 0$.

THIS GIVES $u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\varphi\varphi}$.