MATH 257/316 HW 1

PROBLEM 1 FIND THE RADIUS OF CONVERGENCE OF $\sum_{n=0}^{\infty} \frac{n!\, x^n}{n^n}$.

PROBLEM 2 FIND THE TAYLOR SERIES OF $f(x) = \frac{1}{1-x}$ ABOUT $x_0 = 2$.

PROBLEM 3 CONSIDER $L(y) \equiv x y'' + y' + x y$.

LET y_1 AND y_2 SOLVE $L(y_1) = 0$, $L(y_2) = 0$ WITH $y_1(1) = 1$, $y_1'(1) = 0$ AND $y_2(1) = 0$, $y_2'(1) = 1$. FIND THE FIRST 4 TERMS IN EACH OF THESE TWO FUNDAMENTAL SOLUTIONS BY EXPANDING IN POWERS OF $(x-1)$.

PROBLEM 4 DO THE SAME AS IN PROBLEM 3 FOR $L(y) = (1-x)y'' + xy' - y$ BUT NOW EXPANDING AROUND $x = 0$, i.e. $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = 1$.

PROBLEM 5 FIND THE FIRST FIVE TERMS IN THE POWER SERIES SOLUTION ABOUT $x = 0$ FOR

$$y'' + x y' + 2y = 0 \quad \text{WITH} \quad y(0) = 4 \quad \text{AND} \quad y'(0) = -1.$$

PROBLEM 6 DETERMINE A LOWER BOUND ON THE RADIUS OF CONVERGENCE OF SERIES SOLUTIONS ABOUT $x = x_0$ FOR EACH OF THE FOLLOWING:

(i) $(x^2 - 2x - 3)\, y'' + x y' + 4 y = 0$ WITH $x_0 = 4$

(ii) $(1 + x^3)\, y'' + 4 x y' + y = 0$ WITH $x_0 = 2$.

PROBLEM 7 FIND THE FIRST 3 TERMS IN EACH OF TWO LINEARLY INDEPENDENT SOLUTIONS FOR

$$y'' + \sin x \; y = 0$$

(HINT: RECALL $\sin x = x - x^3/3! + x^5/5! + \ldots$)

PROBLEM 8

THE EQUATION $y'' - 2xy' + \lambda y = 0$ ON $-\infty < x < \infty$

WITH λ A CONSTANT IS CALLED HERMITE'S EQUATION.

(i) BY SUBSTITUTING $y = \sum_{n=0}^{\infty} a_n x^n$ SHOW THAT

$$a_{n+2} = \frac{(2n - \lambda)\, a_n}{(n+2)(n+1)} \quad \text{FOR } n = 2, 3, 4,$$

(ii) WHEN $\lambda = 2m$ WITH m A NON-negative integer show that the corresponding solution is a polynomial.

(iii) FIND the corresponding polynomials when $\lambda = 0, 2, 4, 6, 8, 10.$