MATH 316 HW # 2

PROBLEM 1 SOLVE THE INITIAL VALUE PROBLEMS:

(i) $x^2 y'' + 3xy' + 5y = 0$ WITH $y(1) = 1$, $y'(1) = -1$

(ii) $2(x-1)^2 y'' + (x-1)y' - 3y = 0$ WITH $y(2) = 1$, $y'(2) = 4$.

PROBLEM 2 FIND ALL VALUES OF α, WITH α A REAL CONSTANT, FOR WHICH ANY SOLUTION OF $x^2 y'' + \alpha x y' + \frac{5}{2} y = 0$ TENDS TO ZERO AS $x \to 0^+$.

PROBLEM 3 FOR EACH OF THE FOLLOWING FIND ALL SINGULAR POINTS OF THE GIVEN EQUATION AND DETERMINE WHETHER EACH ONE IS REGULAR OR IRREGULAR:

(i) $(x+3)y'' - 2xy' + (1-x^2)y = 0$

(ii) $x^2 y'' - 3\sin x \, y' + (1+x^2)y = 0$

(iii) $x^2(1-x)y'' - (1+x)y' + 2xy = 0$

PROBLEM 4 FOR EACH OF THE PROBLEMS IN PROBLEM 3 THAT HAVE A REGULAR SINGULAR POINT AT SOME $x = x_0$ FIND THE INDICIAL EQUATION AND DETERMINE THE EXPONENTS r IN THE APPROXIMATION $y \cong (x-x_0)^r$.

PROBLEM 5 CONSIDER $2x^2 y'' + 3xy' + (2x^2 - 1)y = 0$.

(i) SHOW THAT $x = 0$ IS A REGULAR SINGULAR POINT

(ii) FIND THE INDICIAL EQUATION AND THE ROOTS OF THE INDICIAL EQUATION.

(iii) SUBSTITUTING $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ FIND THE RECURRENCE RELATION FOR a_n AS WELL AS THE SERIES SOLUTION FOR EACH OF THE ROOTS r IN (ii).

PROBLEM 6 REPEAT PROBLEM 5 FOR

$$x^2 y'' + (6x + x^2) y' + x y = 0.$$

PROBLEM 7 THE LAGUERRE DIFFERENTIAL EQUATION IS

$$x y'' + (1 - x) y' + \lambda y = 0 \quad \text{WITH } \lambda \text{ A PARAMETER.}$$

(i) SHOW THAT $x = 0$ IS A REGULAR SINGULAR POINT.

(ii) FIND THE INDICIAL EQUATION, ITS ROOTS, AND THE RECURRENCE RELATION

(iii) FIND ONE SOLUTION FOR $x > 0$. SHOW THAT IF $\lambda = m$ WHERE m IS A POSITIVE INTEGER THEN THIS SOLUTION REDUCES TO A POLYNOMIAL.

MATH 316 HW # 2 SOLUTIONS

PROBLEM 1 SOLVE

(i) $x^2 y'' + 3xy' + 5y = 0$ WITH $y(1) = 1$, $y'(1) = -1$.

WE PUT $y = x^r$ TO GET $r(r-1) + 3r + 5 = 0$ SO $r^2 + 2r + 5 = 0$

THUS $r = -1 \pm 2i$. THEN GENERAL SOLUTION IS

$$y = c_1 x^{-1} \cos(2 \log x) + c_2 x^{-1} \sin(2 \log x).$$

NOW $y(1) = 1 \rightarrow c_1 = 1$

$$y'(x) = -x^{-2}\cos(2\log x) + 2x^{-2}\sin(2\log x) - c_2 x^{-2}\sin(2\log x) + c_2 2x^{-2}\cos(2\log x)$$

SO $y'(1) = -1 + 2c_2 = -1$ SO $c_2 = 0$. THUS $y = x^{-1}\cos(2\log x)$.

(ii) $2(x-1)^2 y'' + (x-1)y' - 3y = 0$ WITH $y(2) = 1$ AND $y'(2) = 4$.

WE PUT $y = (x-1)^r$ SO $2r(r-1) + r - 3 = 0$

THUS $2r^2 - r - 3 = 0 \rightarrow (r - \frac{3}{2})(r+1) = 0$

SO $r = 3/2$ AND $r = -1$. THUS

$$y = c_1(x-1)^{-1} + c_2(x-1)^{3/2}.$$

NOW $y(2) = 1 \rightarrow c_1 + c_2 = 1$

$y'(2) = 4 \rightarrow -c_1 + \frac{3c_2}{2} = 4$.

SO $c_2 = 2$, $c_1 = -1$. THUS $y = -(x-1)^{-1} + 2(x-1)^{3/2}$.

PROBLEM 2 $x^2 y'' + \alpha x y' + 5/2\, y = 0$. WE PUT $y = x^r$

TO GET $r(r-1) + \alpha r + \frac{5}{2} = 0$.

THUS $r^2 + (\alpha - 1)r + \frac{5}{2} = 0$.

NOW CONSIDER A QUADRATIC $(r - r_0)(r - r_1) = 0$.

WE IDENTIFY $\left.\begin{aligned} r_0 r_1 &= 5/2 \\ r_0 + r_1 &= 1 - \alpha. \end{aligned}\right\} (*)$

NOW FOR $y \to 0$ AS $x \to 0$ WE MUST FIND α SO THAT

$$\mathrm{RE}(r_0) > 0 \quad \text{AND} \quad \mathrm{RE}(r_1) > 0.$$

- IF r_0, r_1 COMPLEX THEN $r_0 = \lambda + i\mu$ $r_1 = \lambda - i\mu$ AND FROM $(*)$ WE HAVE $2\lambda = 1 - \alpha. \to$ NEED $\alpha > 1$ FOR $\lambda < 0$.

- IF r_0, r_1 REAL THEN $r_0 r_1 = 5/2 > 0$ IMPLY BOTH r_0, r_1 HAVE SAME SIGN.

  SO IF $r_0 + r_1 = 1 - \alpha < 0 \longrightarrow$ BOTH r_0, r_1 MUST BE negative

  IF $r_0 + r_1 = 1 - \alpha > 0 \longrightarrow$ " " " " " positive.

  SO IF $\alpha > 1$ WE CONCLUDE THAT $r_0 < 0$, $r_1 < 0$.

IN SUMMARY, IFF $\alpha > 1$ WE HAVE $y \to 0$ AS $x \to 0$

PROBLEM 3

(i) $(x+3)y'' - 2xy' + (1 - x^2)y = 0.$

SO $P = x + 3$, $Q = -2x$, $R = 1 - x^2$.

$P = 0$ AT $x = -3$ SO $x = -3$ IS A SINGULAR POINT.

WE CALCULATE $\lim_{x \to -3} (x+3)\dfrac{Q}{P} = 6$ $\quad \lim_{x \to -3} (x+3)^2 \dfrac{R}{P} = 0.$

THUS $x = -3$ IS A REG. SING. POINT.

(ii) $x^2 y'' - 3\sin x \; y' + (1 + x^2)y = 0$ $\quad P = x^2$, $Q = -3\sin x$, $R = 1 + x^2$.

SO $x = 0$ IS ONLY SINGULAR POINT. $\quad \lim_{x \to 0} x\dfrac{Q}{P} = \lim_{x \to 0} \dfrac{[-3\sin x]}{x} = -3$

AND $\lim_{x \to 0} x^2 \dfrac{R}{P} = \lim_{x \to 0} x^2 (1 + x^2)/x^2 = 1$. THUS $x = 0$ IS A REG. SING. POINT.

(iii) $x^2(1-x)y'' - (1+x)y' + 2xy = 0$

$P = x^2(1-x)$, $Q = -(1+x)$, $R = 2x$

SO $x=0$ AND $x=1$ ARE SINGULAR POINTS.

CHECK $x=0$ $\quad \lim_{x\to 0} \frac{xQ}{P} = \lim_{x\to 0} \frac{-x(1+x)}{x^2} = \lim_{x\to 0}\left(-\frac{(1+x)}{x}\right) = \infty.$

THUS $x=0$ IS AN IRREGULAR SINGULAR POINT.

CHECK $x=1$ $\quad \lim_{x\to 1} (x-1)\frac{Q}{P} = \lim_{x\to 1} \frac{-(x-1)(1+x)}{x^2(1-x)} = \lim_{x\to 1} \frac{(1+x)}{x^2} = 2.$

$$\lim_{x\to 1} (x-1)^2 \frac{R}{P} = \lim_{x\to 1} \frac{(x-1)^2\, 2x}{x^2(1-x)} = 0.$$

THUS $x=1$ IS A REGULAR SINGULAR POINT.

PROBLEM 4

(i) WE WRITE $(x+3)y'' - 2xy' + (1-x^2)y = 0$ A)

$$(x+3)^2 y'' + (x+3)\left[-2x\right]y' + \left[(1-x^2)(x+3)\right]y = 0$$

let $x \to -3$ IN TERMS IN $[\,..\,]$ $\rightarrow$ $(x+3)^2 y'' - 6(x+3)y' \mathrel{\hat{=}} 0.$

PUT $y = (x+3)^r \rightarrow$ $r(r-1) - 6r = 0$ SO $(r-7)r = 0 \rightarrow r = 0, 7.$

(ii) $x^2 y'' - 3\sin x\, y' + (1+x^2)y = 0$

$$x^2 y'' + x\left[\frac{-3\sin x}{x}\right]y' + \left[1+x^2\right]y = 0$$

Let $x \to 0$ IN $[\,..\,]$ $\rightarrow$ $x^2 y'' - 3xy' + y \mathrel{\hat{=}} 0.$

PUT $y = x^r \rightarrow$ $r(r-1) - 3r + 1 = r^2 - 4r + 1 = 0$ SO

$$r_{\pm} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

(iii) $X=1$ IS REG SING POINT.

$$x^2(1-x)y'' - (1+x)y' + 2xy = 0$$

SO $$(x-1)y'' + \frac{(1+x)}{x^2}y' - \frac{2x}{x^2}y = 0$$

$$(x-1)y'' + \frac{(1+x)}{x^2}y' - \frac{1}{x}y = 0$$

NOW $$(x-1)^2y'' + (x-1)\left[\frac{(1+x)}{x^2}\right]y' + \frac{(x-1)}{x}y = 0$$

NOW let $X \to 1$ IN $[\,..\,]$ TERMS.

$$(x-1)^2y'' + 2(x-1)y' \mathrel{\hat{=}} 0.$$

LET $y = (x-1)^r \longrightarrow r(r-1)+2r = r^2+r = 0$ SO $r = 0, -1.$

PROBLEM 5 $\quad 2x^2y''+3xy' + (2x^2-1)y = 0. \qquad P = 2x^2,\ Q = 3x,\ R = 2x^2-1.$

(i) WE CALCULATE $$\lim_{x\to 0}\frac{xQ}{P} = \lim_{x\to 0}\frac{3x^2}{2x^2} = \frac{3}{2} < \infty$$

$$\lim_{x\to 0}\frac{x^2R}{P} = \lim_{x\to 0}\frac{x^2(2x^2-1)}{2x^2} = \lim_{x\to 0}\frac{(2x^2-1)}{2} = -\frac{1}{2} < \infty$$

THUS $X=0$ IS A REGULAR singular POINT.

(ii) WE SHOULD EXPECT THAT $r(r-1)+\frac{3}{2}r-\frac{1}{2} = 0$ AS INDICIAL EQUATION BASED ON (i). LET'S CHECK:

$$x^2y'' + \left[\frac{3}{2}\right]xy' + \left[x^2-\tfrac{1}{2}\right]y = 0$$

LET $X \to 0$ IN $[\,..\,]$ TERMS. WE GET $x^2y'' + \frac{3x}{2}y' - \frac{1}{2}y = 0.$

NOW PUT $y = x^r \longrightarrow r(r-1) + \frac{3r}{2} - \frac{1}{2} = 0$ ✓. (same as above

$$\left(r^2 + r/_2 - 1/_2\right) = (r+1)(r-1/_2) = 0 \quad \text{SO} \quad r = -1,\ 1/_2$$

(iii) WE PUT $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ INTO $2x^2 y'' + 3xy' + (2x^2 - 1)y = 0$.

THIS GIVES $$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

SHIFT INDICES ON LAST TERM SO THAT $2\sum_{n=0}^{\infty} a_n x^{n+r+2} = 2\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$.

THIS YIELDS
$$\sum_{n=0}^{\infty} \left[2a_n(n+r)(n+r-1) + 3a_n(n+r) - a_n \right] x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0.$$

NOW THIS YIELDS
$$x^r a_0 \left[2r(r-1) + 3r - 1 \right] + x^{r+1} \left[2a_1 r(r+1) + 3a_1(r+1) - a_1 \right]$$
$$+ \sum_{n=2}^{\infty} \left[2a_n(n+r)(n+r-1) + 3a_n(n+r) - a_n + 2a_{n-2} \right] x^{n+r} = 0.$$

SETTING THE POWERS TO ZERO:

$$2r(r-1) + 3r - 1 = 0 \rightarrow r^2 + r/2 - 1/2 = 0 \quad \underline{\text{INDICIAL EQUATION}}$$

SO $r = -1, 1/2$.

THEN $a_1 \left[2r(r+1) + 3(r+1) - 1 \right] = 0 \rightarrow a_1 \left[2r^2 + 5r + 2 \right] = 0$

NOW WE HAVE $\left[2(n+r)(n+r-1) + 3(n+r) - 1 \right] a_n = -2a_{n-2}$.

SO $$a_n = -\frac{2a_{n-2}}{\left[(n+r)\left(2(n+r-1)+3\right) - 1\right]}, \quad n \geq 2 \quad \text{RECURSION RELATION.}$$

<u>TAKE $r_1 = 1/2$</u> THEN $a_1 = 0$ AND $a_3 = a_5 = a_7 = \dots = 0$.

NOW $$a_n = -\frac{2a_{n-2}}{n(2n+3)}, \quad n \geq 2$$

$$a_2 = -\frac{2}{2 \cdot 7}, \quad a_4 = -\frac{2a_2}{4 \cdot 11} = \frac{2 \cdot 2}{2 \cdot 4 \cdot 7 \cdot 11} = \frac{1}{1 \cdot 2 \cdot 7 \cdot 11}$$

NOW $\quad a_6 = -\dfrac{2a_4}{6\cdot 15} = -\dfrac{a_4}{3\cdot 15} = \dfrac{-1}{1\cdot 2\cdot 3\cdot 7\cdot 11\cdot 15}$

WE HAVE $\quad a_{2n} = \dfrac{(-1)^n}{n!\,[7\cdot 11\cdots(4n+3)]} \quad , \quad n = 0,1,2,\ldots$

$$Y_1(X) = X^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{n!\,[7\cdot 11\cdots(4n+3)]}$$

TAKE $r_2 = -1$ THEN $a_0 = 1,\ a_1 = 0$ AND $a_3 = a_5 = a_7 = \cdots = 0.$

NOW $\quad a_n = -\dfrac{2a_{n-2}}{n(2n-3)} \quad , \quad n \geq 2$

SO $\quad a_2 = -\dfrac{2a_0}{2\cdot 1} \quad , \quad a_4 = -\dfrac{2a_2}{4\cdot 5} = \dfrac{2\cdot 2}{1\cdot 2\cdot 4\cdot 5} \quad , \quad a_6 = \dfrac{-2a_4}{6\cdot 9} = \dfrac{-2\cdot 2\cdot 2}{1\cdot 2\cdot 4\cdot 5\cdot 6\cdot 9}$

SO $\quad a_6 = \dfrac{-2}{2\cdot 6\cdot 1\cdot 5\cdot 9} = \dfrac{-1}{1\cdot 2\cdot 3\cdot 5\cdot 9}$

CONTINUING $\quad a_{2n} = \dfrac{(-1)^n}{n!\,[1\cdot 5\cdots(4n-3)]} \quad n = 1,2,3\ldots$

SO $$Y_2(X) = X^{-1}\sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{n!\,[1\cdot 5\cdots 4n-3]} .$$

PROBLEM 6 $\quad x^2 y'' + (6x + x^2) y' + x y = 0$

(i) $P = x^2, \quad Q = 6x + x^2, \quad R = x.$

SO $\displaystyle\lim_{x\to 0} \frac{xQ}{P} = 6 \qquad \lim_{x\to 0} \frac{x^2 R}{P} = 0$

SO $x = 0$ IS A REGULAR SINGULAR POINT.

NOW $r(r-1) + 6r = r^2 + 5r = 0$ IS INDICIAL EQUATION SO $r = 0, -5$ ARE ROOTS. NOW WE DEVELOP A SERIES APPROXIMATION.

(ii) PUT $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. THEN

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 6 a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

THUS $$\sum_{n=0}^{\infty} a_n \left[(n+r)(n+r-1) + 6(n+r)\right] x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0.$$

THIS GIVES $$\sum_{n=0}^{\infty} a_n \left[(n+r)(n+r+5)\right] + \sum_{n=1}^{\infty} a_{n-1} (n+r) x^{n+r} = 0.$$

THIS GIVES $$a_0 \, r(r+5) + \sum_{n=1}^{\infty} \left(a_n (n+r)(n+r+5) + a_{n-1}(n+r) \right) x^{n+r} = 0.$$

THIS GIVES $r(r+5) = 0$ SO $r = 0, -5$.

NOW $$a_n = -\frac{a_{n-1}}{(n+r+5)}, \quad n = 1, 2, 3, \ldots$$

NOW TAKE $r = 0$ $\quad$ WE PUT $a_0 = 1$ THEN $\displaystyle a_n = \frac{-a_{n-1}}{(n+5)}, \quad n = 1, 2, \ldots$

$$a_1 = -\frac{a_0}{6}, \quad a_2 = -\frac{a_1}{7} = \frac{a_0}{6\cdot 7}, \quad a_3 = -\frac{a_2}{8} = -\frac{a_0}{6\cdot 7\cdot 8}$$

THUS $\displaystyle a_n = \frac{(-1)^n}{6\cdot 7\cdot 8 \cdots [n+5]} a_0 \qquad$ FOR $n \geq 1$

WE CAN WRITE THIS AS

$$a_n = \frac{5!\,(-1)^n\, a_0}{(n+5)!}, \quad \text{FOR } n = 1, 2, \ldots$$

THUS $y_1(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ WITH a_n GIVEN ABOVE.

<u>NOW TAKE $r = -5$</u> THEN $a_n = -\frac{a_{n-1}}{n}$ WITH $n = 1, 2, 3, \ldots$

WE HAVE $a_1 = -a_0$, $a_2 = -\frac{a_1}{2} = \frac{a_0}{2}$, $a_3 = -\frac{a_2}{3} = -\frac{a_0}{2 \cdot 3}$

THUS $a_n = \frac{(-1)^n}{n!} a_0$ FOR $n = 1, 2, \ldots$

WE CONCLUDE THAT $y_2(x) = a_0 x^{-5} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = x^{-5} e^{-x} a_0.$

PROBLEM 7 $\qquad xy'' + (1-x)y' + \lambda y = 0 \qquad \lambda$ PARAMETER

(i) $P = x$, $Q = 1-x$, $R = \lambda$ SO SINCE $P(0) = 0$ AND $R \neq 0$, $Q(0) \neq 0$, $x = 0$

IS A SINGULAR POINT. WE CALCULATE

$$\lim_{x\to 0} \frac{xQ}{P} = 1 < \infty \qquad \lim_{x\to 0} \frac{x^2 R}{P} = 0 < \infty$$

SO $x = 0$ IS A REGULAR SINGULAR POINT.

WE EXPECT $r(r-1) + r = 0 \longrightarrow r^2 = 0$ AS INDICIAL EQUATION.

(ii) NOW WE PUT $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. THEN $x^2 y'' + xy' - x^2 y' + \lambda xy = 0$.

WE HAVE THAT

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} + \lambda \sum_{n=0}^{\infty} \lambda a_n x^{n+r+1} = 0$$

THEN SHIFTING INDICES:

$$\sum_{n=0}^{\infty} \left[a_n (n+r)(n+r-1) + a_n (n+r) \right] x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} + \lambda \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0.$$

THIS GIVES

$$a_0\left([r][r-1] + r \right) + \sum_{n=1}^{\infty} \left[a_n (n+r)(n+r-1) + a_n (n+r) - a_{n-1}(n+r-1) + \lambda a_{n-1} \right] x^{n+r} = 0$$

NOW THIS SIMPLIFIES TO

$$a_0 r^2 x^r + \sum_{n=1}^{\infty} \left[a_n (n+r)^2 - a_{n-1}(n+r-1) + \lambda a_{n-1} \right] x^{n+r} = 0.$$

THUS $a_0 r^2 = 0 \rightarrow r = 0$ IS double ROOT OF INdicial eQUATION

AND $$a_n = \frac{\left[(n+r-1) - \lambda \right] a_{n-1}}{(n+r)^2}, \qquad n = 1, 2, 3, \ldots$$

WITH a_0 = ARBITRARY AND $r = 0$.

(iii) NOW SET $r = 0$ SO THAT

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1} \quad \text{FOR } n = 1, 2, 3, \ldots \quad \text{WITH } a_0 = 1.$$

THEN WE CALCULATE

$$a_1 = -\lambda,$$

$$a_2 = \frac{(1-\lambda)}{4}\, a_1 = -\frac{\lambda(1-\lambda)}{4}$$

$$a_3 = \frac{(2-\lambda)}{9}\, a_2 = -\frac{\lambda(1-\lambda)(2-\lambda)}{2^2 \cdot 3^2}$$

SO

$$a_n = \frac{-\lambda(1-\lambda)(2-\lambda)\cdots(n-1-\lambda)}{(n!)^2} \quad \text{FOR } n = 1, 2, 3, \ldots$$

THUS WE GET

$$y(x) = 1 - \lambda x + \frac{(-\lambda)(1-\lambda)\, x^2}{(2!)^2} + \cdots + \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)\, x^n}{(n!)^2} + \cdots$$

SO IF $\lambda = m$ A POSITIVE INTEGER THEN WE GET A POLYNOMIAL OF degree m SINCE $a_{m+1} = a_{m+2} = \cdots = 0$.

THUS,

$$y(x) = 1 - mx + \frac{(-m)(1-m)\, x^2}{2!} + \cdots + \frac{(-m)(1-m)\cdots(m-1-m)\, x^m}{(m!)^2}.$$