MATH 316 HW # 4

PROBLEM 1 FIND THE EXPLICIT SOLUTION TO THE HEAT EQUATION

$$u_t = 100\, u_{xx}, \quad 0 < x < 1, \quad t > 0$$

WITH $u(0,t) = u(1,t) = 0$

AND INITIAL CONDITION $u(x,0) = \sin(2\pi x) - 2\sin(5\pi x)$.

PROBLEM 2 FIND THE EXPLICIT SOLUTION TO THE HEAT EQUATION

$$u_t = \frac{1}{4} u_{xx}, \quad 0 < x < 2, \quad t > 0$$

WITH $u(0,t) = u(2,t) = 0$

AND $u(x,0) = 2\sin\left(\pi x/2\right) - \sin(\pi x) + 4\sin(2\pi x)$.

PROBLEM 3 (i) SUPPOSE THAT $u(x,t)$ SATISFIES

$$u_t = k\, u_{xx} - bu \qquad \text{FOR SOME } k > 0 \text{ AND } b > 0, \text{ CONSTANTS,}$$

SHOW THAT IF WE WRITE $u(x,t) = e^{\alpha t} v(x,t)$ FOR SOME α TO BE FOUND THAT WE CAN GET THAT v SATISFIES

$$v_t = k\, v_{xx}.$$

PROBLEM 4 USE EITHER THE RESULT IN PROBLEM 3 OR STANDARD SEPARATION OF VARIABLES FIND AN INFINITE SERIES REPRESENTATION FOR THE SOLUTION TO

$$u_t = k\, u_{xx} - u \quad \text{IN} \quad 0 < x < L, \quad t > 0$$

WITH $u(0,t) = u(L,t) = 0$

AND $u(x,0) = x(L-x)$.

PROBLEM 5 CONSIDER THE FOLLOWING HEAT EQUATION WITH INSULATED BOUNDARY CONDITIONS:

$$(*)\quad \begin{cases} u_t = \kappa u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad (\kappa > 0 \text{ CONSTANT}) \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = 1 + 2\cos^2(\pi x) + 4\cos^2(2\pi x) \end{cases}$$

(i) USING THE SEPARATION OF VARIABLES TECHNIQUE FIND THE FORM OF AN INFINITE SERIES REPRESENTATION FOR THE SOLUTION TO $(*)$

(ii) FIND THE COEFFICIENTS IN THE FOURIER COSINE SERIES EXPLICITLY BY USING $u(x,0)$.

(HINT: YOU NEED A STANDARD IDENTITY FOR $\cos^2(A)$).

(iii) CALCULATE $\lim_{t\to\infty} u(x,t) \equiv u_\infty$. THIS IS THE STEADY-STATE TEMPERATURE.

(iv) DEFINE $M(t) \equiv \int_0^1 u(x,t)\,dx$.

BY INTEGRATING THE PDE OVER $0 < x < 1$ SHOW THAT

$$M(t) = \int_0^1 u(x,0)\,dx \quad \text{FOR ALL } t.$$

THUS, $M(t)$ IS THE SAME FOR ALL TIME t.