MATH 316/257 HW # 5

PROBLEM 1 FOR THE "SAWTOOTH" FUNCTION

$$f(x) = \begin{cases} x & \text{IF } 0 \leq x \leq 1 \\ 2-x & \text{IF } 1 \leq x \leq 2 \end{cases}$$

DEFINED ON $0 \leq x \leq 2$ COMPUTE ITS

(i) FOURIER SINE SERIES

(ii) FOURIER COSINE SERIES

(iii) BY EVALUATING $f(1)$ USE YOUR RESULTS FROM (i) AND (ii) IN TURN TO CALCULATE A FORMULA FOR

$$1 + 1/3^2 + 1/5^2 + 1/7^2 + 1/9^2 + \cdots$$

PROBLEM 2 SOLVE THE FOLLOWING PROBLEM DESCRIBING HEAT CONDUCTION IN A CLOSED THIN CIRCULAR WIRE:

$$\begin{cases} u_t = u_{xx} & -1 < x < 1, \quad t > 0 \\ u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t) & \text{PERIODIC BC} \\ u(x,0) = |x| & \end{cases}$$

IN ADDITION CALCULATE $\lim_{t \to \infty} u(x,t)$.

PROBLEM 3 LET $f(x) = x^2$ ON $-\pi \leq x \leq \pi$. VERIFY THAT THE FOURIER SERIES FOR $f(x)$ WHEN f IS EXTENDED PERIODICALLY OUTSIDE $[-\pi, \pi]$ IS

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

THEN, USE PARSEVAL'S RELATION TO EVALUATE THE INFINITE SUM $\sum_{n=1}^{\infty} 1/n^4$.

PROBLEM 4 LET $f(x) = 1 - x$ ON $0 < x < 1$.

(i) FIND THE FOURIER SINE SERIES OF $f(x)$

(ii) WHAT DOES THE FOURIER SINE SERIES CONVERGE TO WHEN $x = 0$? WHEN $x = 1/2$? WHEN $x = 1$?

HINT: GRAPH THE EXTENDED FUNCTION ON $-\infty < x < \infty$.

PROBLEM 5 LET $f(x) = 3 - x$ ON $0 < x < 3$.

(i) SKETCH THE GRAPH OF THE EVEN EXTENSION OF $f(x)$ ON $-3 < x < 0$, WHICH IS THEN EXTENDED PERIODICALLY WITH PERIOD 6. WHAT ARE THE CORRESPONDING FOURIER COSINE COEFFICIENTS?

(ii) SKETCH THE GRAPH OF THE ODD EXTENSION OF $f(x)$ ON $-3 < x < 0$, WHICH IS THEN EXTENDED PERIODICALLY WITH PERIOD 6. WHAT ARE THE CORRESPONDING FOURIER SINE COEFFICIENTS?

(iii) WHICH OF THE TWO EXTENSIONS, I.e. (i) OR (ii), GIVES A SERIES THAT CONVERGES MORE RAPIDLY? BRIEFLY EXPLAIN WHY?

MATH 257/316 HW 5 SOLUTIONS

PROBLEM 1

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2-x & \text{if } 1 \le x \le 2 \end{cases}$$

(i) FOURIER SINE SERIES:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

so $$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = x \quad du = dx$

integrate by parts: $\quad dv = \sin\left(\frac{n\pi x}{2}\right)dx \quad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

$$b_n = -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

so $$b_n = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = (2-x) \quad du = -dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx \quad v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

so $$b_n = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \left[-\frac{2}{n\pi}(2-x)\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2\Big]$$

$$b_n = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n = \text{even} \\ \frac{8}{n^2\pi^2}(-1)^{(n-1)/2}, & n = \text{ODD} \end{cases}$$

THUS $$f(x) = \frac{8}{\pi^2}\sum_{\substack{n=1 \\ n \text{ ODD}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{n\pi x}{2}\right) \qquad \text{let } n = 2m-1$$

so $$f(x) = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\sin\left(\frac{(2m-1)\pi x}{2}\right). \Bigg\} (*)$$

(ii) FOURIER COSINE SERIES:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{2}\right)$$

WE CALCULATE $2a_0 = \int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$

$$= \frac{1}{2} + \left[2 - \frac{x^2}{2}\Big|_1^2\right] = 1.$$

SO $a_0 = 1/2$.

NOW $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx.$

WE INTEGRATE BY PARTS:

$u = x$, $du = dx$, $dv = \cos\left(\frac{n\pi x}{2}\right)dx$, $v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$u = 2-x$, $du = -dx$, $dv = \cos\left(\frac{n\pi x}{2}\right)dx$, $v = +\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

SO $a_n = \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx + (2-x)\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 + \frac{2}{n\pi}\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx\right.$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2\Bigg]$$

SO $a_n = \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - (-1)^n\frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}.$

SO $a_n = \frac{4}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right] = \begin{cases} 0 & \text{IF } n = \text{ODD} \\ \frac{4}{n^2\pi^2}\left[-2 + 2\cos\left(\frac{n\pi}{2}\right)\right] & \text{IF } n = \text{EVEN} \end{cases}$

THUS $f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n \text{ EVEN}}}^{\infty} \frac{\left(2\cos(n\pi/2) - 2\right)}{n^2}\cos\left(n\pi x/2\right)$

NOTICE IF $n = 2, 6, 10, \ldots$ THEN $2\cos(n\pi/2) - 2 = -4$

$n = 4, 8, \ldots$ THEN $2\cos(n\pi/2) - 2 = 0.$

SO LET $n = 4m+2$ WITH $m = 0, 1, 2, \ldots$

THEN $$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{-4}{(4m+2)^2}\cos\left((2m+1)\pi x\right).$$

THUS $$\left.f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos\left((2m+1)\pi x\right).\right\} \quad (**)$$

(iii) NOW EVALUATE THE F-SINE SERIES (*) AT $x=1$. SINCE f IS CONTINUOUS AT $x=1$, THEN

$$1 = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\sin\left(\frac{(2m-1)\pi}{2}\right) = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}(-1)^{m-1}$$

BUT $\left((-1)^{m-1}\right)^2 = 1 \ \forall m$ SO, $$1 = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}.$$

THUS $$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \pi^2/8. \quad (+)$$

NOW EVALUATE THE FOURIER COSINE SERIES AT $x=1$ IN (**) USING $\cos\left((2m+1)\pi\right) = -1 \ \forall m = 0, 1, 2, \ldots$ WE OBTAIN SINCE f IS CONTINUOUS AT $x=1$ THAT $f(1) = 1$ AND

$$1 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}(-1) \longrightarrow 1 = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$$

THUS $$\pi^2/8 = \sum_{m=0}^{\infty} 1/(2m+1)^2$$ WHICH IS SAME AS IN (+)

PROBLEM 2

$$u_t = u_{xx} \quad -1 < x < 1, \quad t > 0$$

$$u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t) \quad \text{BC}$$

$$u(x,0) = |x|$$

WE SEPARATE VARIABLES $u = T(t)\,\Phi(x)$ TO GET $\dfrac{T'}{T} = \dfrac{\Phi''}{\Phi} = -\lambda.$

THUS,
$$\Phi'' + \lambda \Phi = 0, \quad -1 < x < 1$$
$$\Phi(-1) = \Phi(1); \quad \Phi_x(-1) = \Phi_x(1)$$

THE EIGENFUNCTIONS ARE $\lambda_0 = 0$, $\Phi_0 = 1$, AND FOR $\lambda > 0$,

$$\Phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x).$$

SATISFYING THE BC GIVES $\begin{array}{l} 2B\sin(\sqrt{\lambda}) = 0 \\ 2A\sin(\sqrt{\lambda}) = 0 \end{array} \rightarrow \sin(\sqrt{\lambda}) = 0$ gives $\lambda = n^2\pi^2$

THUS $\lambda_n = n^2\pi^2$ AND $\Phi_n(x) = A_n\cos(n\pi x) + B_n\sin(n\pi x)$, WITH $n = 1, 2, \ldots$

NOW $T_n' = -\lambda_n T_n$ GIVES $T_0 = 1$ AND $T_n = e^{-\lambda_n t}$ FOR $n \geq 1$.

SUPERPOSITION YIELDS
$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left(A_n\cos(n\pi x) + B_n\sin(n\pi x) \right) e^{-n^2\pi^2 t}.$$

NOW IMPOSE $u(x,0) = |x| = A_0 + \sum_{n=1}^{\infty} \left(A_n\cos(n\pi x) + B_n\sin(n\pi x) \right).$

SINCE $|x|$ IS EVEN WE MUST HAVE $B_n = 0 \;\; \forall n = 1, 2, \ldots$

NOW TO FIND A_0 WE INTEGRATE FROM $-1 < x < 1$ TO GET
$$A_0 \int_{-1}^{1} 1\,dx = \int_{-1}^{1} |x|\,dx = 2\int_0^1 x\,dx = 1 \longrightarrow A_0 = 1/2.$$

SIMILARLY, MULTIPLYING BY $\cos(m\pi x)$ WE GET UPON INTEGRATING:
$$A_m \int_{-1}^{1} \cos^2(m\pi x)\,dx = \int_{-1}^{1} |x|\cos(m\pi x)\,dx = 2\int_0^1 x\cos(m\pi x)\,dx$$

NOW $\int_{-1}^{1} \cos^2(m\pi x)\,dx = \int_{-1}^{1}\left(\frac{1+\cos(2m\pi x)}{2}\right) = 1.$

THUS $A_m = 2\int_0^1 x\cos(m\pi x)\,dx = 2\left[\frac{x}{m\pi}\sin(m\pi x)\Big|_0^1 - \int_0^1 \frac{1}{m\pi}\sin(m\pi x)\,dx\right]$

$u = x \quad du = dx$

$dv = \cos(m\pi x)\,dx \qquad v = \frac{1}{m\pi}\sin(m\pi x)$

THUS $A_m = -\frac{2}{m\pi}\left[\frac{-1}{m\pi}\cos(m\pi x)\Big|_0^1\right] = \frac{2}{m^2\pi^2}\left[(-1)^m - 1\right] = \begin{cases} -\frac{4}{m^2\pi^2} & m \text{ ODD} \\ 0 & m \text{ EVEN} \end{cases}$

THUS $u(x,t) = \frac{1}{2} - \sum_{\substack{m=1 \\ m \text{ ODD}}}^{\infty} \frac{4}{m^2\pi^2}\cos(m\pi x)\,e^{-m^2\pi^2 t}.$

WE CAN WRITE THIS AS

$$u(x,t) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{4}{[(2n-1)\pi]^2}\cos((2n-1)\pi x)\,e^{-(2n-1)^2\pi^2 t}.$$

NOW AS $t \to \infty$ WE HAVE $u(x,t) \to 1/2$

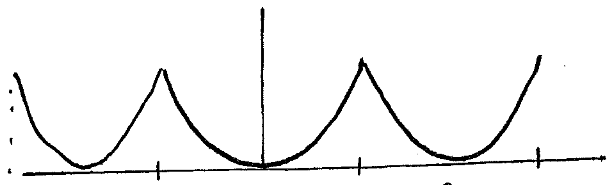
THUS $\lim_{t\to\infty} u(x,t) = 1/2$ IS THE STEADY-STATE.

PROBLEM 3

LET $f(x) = x^2$ ON $-\pi \le x \le \pi$. SINCE $f(x)$ IS EVEN WE HAVE WITH $T = 2L = 2\pi$

THAT $$x^2 = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx).$$

WE PLOT



NOW WE CALCULATE $$\int_{-\pi}^{\pi} x^2\,dx = A_0(2\pi) \quad \text{SO} \quad A_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx.$$

THUS $$A_0 = \frac{1}{3\pi} x^3\Big|_0^{\pi} = \pi^2/3.$$

NOW $$\int_{-\pi}^{\pi} A_n \cos^2(nx)\,dx = \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = 2\int_0^{\pi} x^2 \cos(nx)\,dx.$$

THUS $$A_n \pi = 2\int_0^{\pi} x^2 \cos(nx)\,dx = 2\Big[\underbrace{\frac{x^2}{n}\sin(nx)\Big|_0^{\pi}}_{=0} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx$$

$u = x^2,\ du = 2x\,dx$

$dv = \cos(nx)\,dx \qquad v = \frac{1}{n}\sin(nx)$

SO $$A_n \pi = -\frac{4}{n}\int_0^{\pi} x\sin(nx)\,dx = -\frac{4}{n}\Big[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \underbrace{\frac{1}{n}\int_0^{\pi}\cos(nx)\,dx}_{=0}$$

$u = x,\ du = dx$

$dv = \sin(nx)\,dx$

$v = -\frac{1}{n}\cos(nx)$

SO $$A_n \pi = \frac{4\pi}{n^2}(-1)^n \quad \text{OR} \quad A_n = \frac{4}{n^2}(-1)^n.$$

WE CONCLUDE THAT $$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx).$$

NOW RECALL PARSEVAL'S RELATION FOR

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx)$$

THAT

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 \, dx = \frac{c_0^2}{2} + \sum_{n=1}^{\infty} c_n^2 .$$

SINCE $c_0 = 2\pi^2/3$ AND $c_n = \dfrac{4(-1)^n}{n^2}$ WE GET

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (x^2)^2 \, dx = \frac{1}{2}\left(\frac{4\pi^4}{9}\right) + 16 \sum_{n=1}^{\infty} 1/n^4 .$$

$$\frac{1}{5\pi} x^5 \Big|_{-\pi}^{\pi} = \frac{2}{5\pi}(\pi)^5 = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} 1/n^4 .$$

OR

$$2\pi^4\left(\frac{1}{5} - \frac{1}{9}\right) = 16 \sum_{n=1}^{\infty} 1/n^4$$

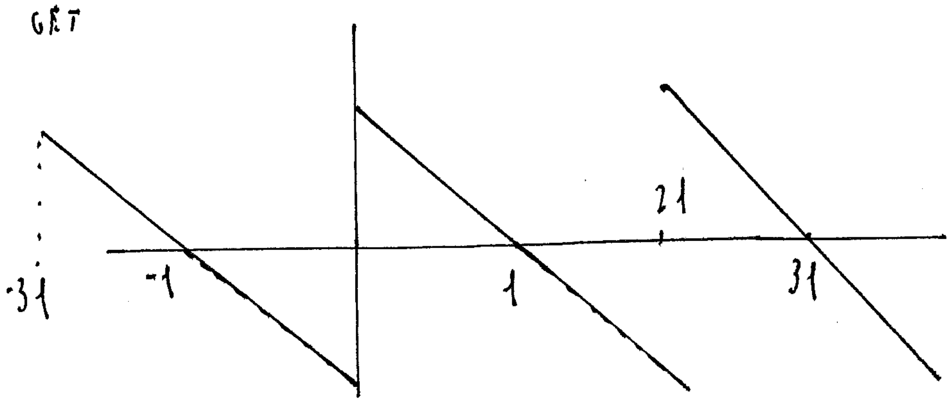
$$2\pi^4\left(\frac{4}{45}\right) = 16 \sum_{n=1}^{\infty} 1/n^4 .$$

WE GET

$$\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90 .$$

PROBLEM 4 $\quad f(x) = \ell - x$ ON $0 < x < \ell$.

(i) WE EXTEND AS ODD ON $-\ell < x < 0$ AND THEN EXTEND PERIODICALLY TO GET



NOW $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$ WITH $\quad b_n = \frac{2}{\ell}\int_0^{\ell} (\ell - x)\sin\left(\frac{n\pi x}{\ell}\right) dx.$

WE INTEGRATE

$$b_n = \frac{2}{\ell}\left[-\frac{\ell^2}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell} - \int_0^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right) dx \right]$$

$u = x \quad du = dx \quad dv = \sin\left(\frac{n\pi x}{\ell}\right) dx \quad v = -\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)$

$$b_n = \frac{2}{\ell}\left[-\frac{\ell^2}{n\pi}\left((-1)^n - 1 \right) - \left(-\frac{x\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell} + \underbrace{\frac{\ell}{n\pi}\int_0^{\ell}\cos\left(\frac{n\pi x}{\ell}\right) dx}_{=0} \right) \right]$$

$$b_n = \frac{2}{\ell}\left[-\frac{\ell^2}{n\pi}\left((-1)^n - 1 \right) + \frac{\ell^2}{n\pi}(-1)^n \right]$$

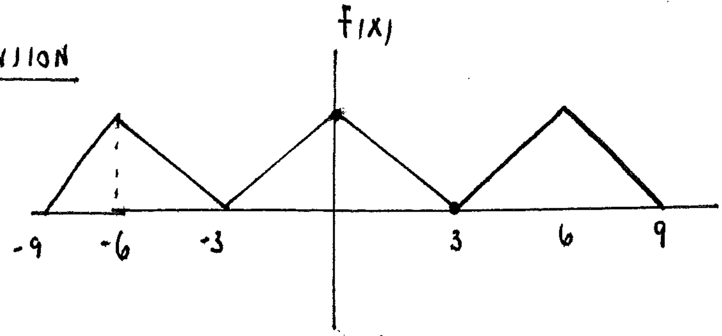
$$b_n = \frac{2}{\ell}\,\frac{\ell^2}{n\pi} = \frac{2\ell}{n\pi}.$$

WE CONCLUDE THAT $\quad f(x) = \ell - x = \sum_{n=1}^{\infty} \frac{2\ell}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right) \quad (*)$

(ii) BY FOURIER CONVERGENCE THEOREM THE FOURIER SINE SERIES CONVERGES TO $f(\ell/2)$ WHEN $x = \ell/2$ AND $f(\ell)$ WHEN $x = \ell$ SINCE $x = \ell/2$ AND $x = \ell$ ARE POINTS WHERE EXTENDED FUNCTION IS CONTINUOUS. IT CONVERGES TO $\frac{f(0^+) + f(0^-)}{2} = \frac{1}{2}(\ell - \ell) = 0$ AT THE JUMP DISCONTINUITY POINT $x = 0$. AT $x = \ell/2$ WE GET $\frac{\ell}{2} = \sum_{n=1}^{\infty} \frac{2\ell}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

PROBLEM 5 CONSIDER $f(x) = 3 - x$ ON $0 < x < 3$.

(i) EVEN EXTENSION



NOTICE THAT EXTENDED FUNCTION IS CONTINUOUS FOR ALL X.

WE HAVE $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$$

NOW $$a_0 \int_{-3}^{3} dx = \int_{-3}^{3} |3 - x| \, dx \quad \text{SO} \quad 6 a_0 = 2\int_0^3 (3-x)\,dx.$$

WE CALCULATE $$3 a_0 = \int_0^3 (3-x)\,dx = 9 - x^2/2 \Big|_0^3 = 9 - 9/2 = 9/2.$$

THUS $$a_0 = 3/2.$$

SIMILARLY $$a_n = \frac{2}{3}\int_0^3 (3-x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left[\frac{9}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^3 - \int_0^3 x\cos\left(\frac{n\pi x}{3}\right)dx\right]$$

THUS $$a_n = -\frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi x}{3}\right)dx = \frac{-2}{3}\left[\frac{3x}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^3 - \frac{3}{n\pi}\int_0^3 \sin\left(\frac{n\pi x}{3}\right)dx\right]$$

$u = x \quad du = dx$

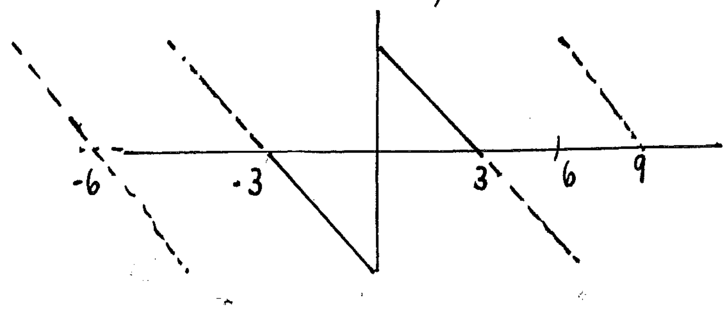
$dv = \cos\left(\frac{n\pi x}{3}\right)dx \quad v = \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)$

THUS $$a_n = \frac{2}{n\pi}\int_0^3 \sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{n\pi}\left(\frac{-3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big|_0^3\right)$$

$$a_n = \frac{6}{n^2\pi^2}\left[1 - (-1)^n\right] = \begin{cases} 12/n^2\pi^2 & \text{IF } n = \text{ODD} \\ 0 & n = \text{EVEN} \end{cases}$$

THUS $$f(x) = \frac{3}{2} + \frac{12}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2m-1)\pi x}{3}\right). \qquad (*)$$

(ii) ODD EXTENSION THE PLOT IS



NOTICE THAT THE EXTENDED FUNCTION IS DISCONTINUOUS AT $X = 6m$ WITH $m = 0, \pm 1, \pm 2, \dots$

NOW WE CALCULATE FOURIER SINE COEFFICIENTS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\pi x/3 \right)$$

WE HAVE

$$b_n = \frac{2}{3} \int_0^3 (3 - x) \sin\left(\frac{n\pi x}{3} \right) dx$$

$$= \frac{2}{3} \left[-\frac{9}{n\pi} \cos\left(\frac{n\pi x}{3} \right) \Big|_0^3 - \int_0^3 x \sin\left(\frac{n\pi x}{3} \right) dx \right]$$

$u = x,\ du = dx,\ dv = \sin\left(\frac{n\pi x}{3} \right) dx$

$v = \frac{-3}{n\pi} \cos\left(\frac{n\pi x}{3} \right)$

SO $$b_n = \frac{2}{3} \left[-\frac{9}{n\pi} \left((-1)^n - 1 \right) - \left(-\frac{3x}{n\pi} \cos\left(\frac{n\pi x}{3} \right) \Big|_0^3 + \underbrace{\frac{3}{n\pi} \int_0^3 \cos\left(\frac{n\pi x}{3} \right) dx}_{=0} \right) \right]$$

SO $$b_n = \frac{2}{3} \left[-\frac{9}{n\pi} \left((-1)^n - 1 \right) + \frac{9}{n\pi} (-1)^n \right] = \frac{6}{n\pi}$$

THUS $$f(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{3} \right) \qquad (**)$$

(iii) NOTICE THAT THE FOURIER COSINE SERIES IN (*) HAS COEFFICIENTS THAT DECAY AT $\approx 1/(2n-1)^2$ AS COMPARED TO FOURIER SINE COEFFICIENTS IN (**) THAT DECAY LIKE $1/n$. THUS FOURIER COSINE SERIES HAS <u>BETTER</u> CONVERGENCE PROPERTIES OWING TO THE FACT THAT ITS EXTENDED FUNCTION IS CONTINUOUS $\forall x$.