MATH 257/316 HW # 7

PROBLEM 1 FIND AN INFINITE SERIES REPRESENTATION FOR THE SOLUTION TO

$$u_t = u_{xx} - xe^{-t}, \quad 0 < x < 1, \ t > 0$$

$$u_x(0,t) = 0, \quad u(1,t) = 1$$

$$u(x,0) = 1$$

(HINT: YOU CAN USE THE INTEGRAL $\int_0^1 x \cos\left[(n+\frac{1}{2})\pi x\right] dx = \frac{(-1)^n}{(n+1/2)\pi} - \frac{1}{(n+1/2)^2\pi^2}$)

PROBLEM 2 THERE IS A GAS LEAK AT THE END $x = \pi$ OF A CORRIDOR $0 \le x \le \pi$. THE CONCENTRATION OF GAS SATISFIES

$$\begin{cases} u_t = u_{xx} & 0 < x < \pi, \ t > 0 \\ u_x(0,t) = 0, & u_x(\pi, t) = 1 \\ u(x,0) = 0 \end{cases}$$

(i) SHOW FROM THE PDE THAT THERE IS NO STEADY-STATE SOLUTION AND THAT $\int_0^\pi u(x,t)\,dx = t$ FOR ALL $t \geq 0$.

(ii) FIND THE SOLUTION $u(x,t)$ BY FIRST FINDING A PARTICULAR SOLUTION $w(x,t) = ax^2 + bx + ct$ FOR SOME a, b, c WHERE w SATISFIES THE PDE AND THE BC: $w_x(0,t) = 0$, $w_x(\pi, t) = 1$.

THEN WRITE $u(x,t) = w(x,t) + v(x,t)$ AND SOLVE THE HOMOGENEOUS PDE PROBLEM FOR $v(x,t)$ BY EIGENFUNCTION EXPANSIONS.

(HINT: THE INTEGRAL $\int_0^\pi x^2 \cos(nx)\,dx = \frac{2\pi(-1)^n}{n^2}$ FOR $n \geq 1$ MAY BE USEFUL).

PROBLEM 3 SOLVE THE WAVE EQUATION FOR THE DEFLECTION OF A STRING:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \ t > 0$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = \sin(2x) + 4\sin(4x), \quad u_t(x,0) = 0.$$

PROBLEM 4 CONSIDER THE DAMPED WAVE EQUATION FOR STRING VIBRATIONS WHERE THE DEFLECTION SATISFIES

$$(*) \begin{cases} u_{tt} + 2u_t = u_{xx}, & 0 < x < 1, \ t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \end{cases}$$

(i) FIND AN INFINITE SERIES REPRESENTATION FOR $u(x,t)$.

(ii) FIND THE EXPLICIT SOLUTION WHEN $f(x) \equiv 0$ AND $g(x) \equiv \sin(4\pi x)$

(iii) FOR THE GENERAL PDE $(*)$, DEFINE THE TOTAL ENERGY $E(t)$ BY

$$E(t) \equiv \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)\,dx.$$

SHOW THAT $dE/dt \leq 0 \quad \forall\, t > 0.$

PROBLEM 5 FIND THE SOLUTION TO THE WAVE EQUATION WITH AN IMPOSED EXTERNAL FORCING MODELED BY

$$u_{tt} = c^2 u_{xx} + \sin(2x)\sin(wt) \qquad 0 < x < \pi, \ t > 0$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = 0, \quad u_t(x,0) = 3\sin(2x).$$

ALSO, FOR WHAT VALUE OF w DOES A RESONANCE PHENOMENA OCCUR?