Lecture 11: Fourier Cosine Series

(*Compiled 3 March 2014*)

In this lecture we use separation of variables to solve the heat equation subject to Neumann boundary conditions. In this case we reduce the problem to expanding the initial condition function $f(x)$ in an infinite series of cosine functions - known as the Fourier Cosine Series.

Key Concepts: Heat Equation; Neumann Boundary Conditions; separation of variables; Fourier Cosine Series.

11 The heat equation subject to Homogenous Neumann Boundary Conditions

We consider the heat equation subject to the following initial and boundary conditions:

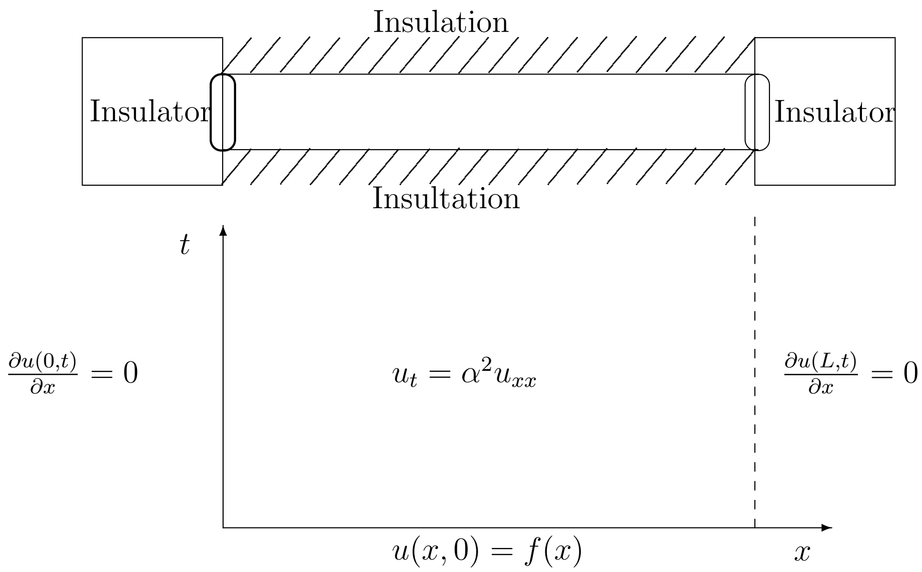


Figure 1. Consider a conducting bar with thermal conductivity α^2 that has an initial temperature distribution $u(x,0) = f(x)$ and whose endpoints are insulated

$$\text{Heat Equation}: \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L \tag{11.1}$$

$$\text{Boundary Conditions}: \quad \frac{\partial u(0,t)}{\partial x} = 0 = \frac{\partial u(L,t)}{\partial x} \tag{11.2}$$

$$\text{Initial Condition}: \quad u(x,0) = f(x) \tag{11.3}$$

11.1 Separation of Variables - Fourier sine Series:

Consider the heat conduction in an insulated rod whose endpoints are insulate for all time and within which the initial temperature is given by $f(x)$ as shown in figure 1.

Fourier's Guess:

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u_t = X(x)\dot{T}(t) &= \alpha^2 u_{xx} = \alpha^2 X''(x)T(t) \end{aligned} \tag{11.4}$$

$\div \alpha^2 XT$:

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{\alpha^2 T(t)} = \text{ Constant } = -\lambda^2. \tag{11.5}$$

Time equation

$$\begin{aligned} \dot{T}(t) = -\alpha^2\lambda^2 T(t) \quad \frac{dT}{T} &= -\alpha^2\lambda^2\, dt \\ \ln|T| &= -\alpha^2\lambda^2 t + c \\ T(t) &= D\mathrm{e}^{-\alpha^2\lambda^2 t}. \end{aligned} \tag{11.6}$$

Case I: Spatial equation assuming that $\lambda \neq 0$:

$$\begin{aligned} X''(x) + \lambda^2 X(x) = 0 \\ \text{Guess} \quad X(x) = \mathrm{e}^{rx} \Rightarrow (r^2+\lambda^2)\mathrm{e}^{rx} = 0 \quad r = \pm\lambda i \end{aligned} \tag{11.7}$$

$$\begin{aligned} X &= c_1\mathrm{e}^{i\lambda x} + c_2\mathrm{e}^{-i\lambda x} \\ &= A\cos\lambda x + B\sin\lambda x \\ X' &= -A\lambda\sin\lambda x + B\lambda\cos\lambda x \end{aligned}$$

Now impose the boundary conditions:

$$\begin{aligned} 0 &= \tfrac{\partial u(0,t)}{\partial x} = X'(0)T(t) \Rightarrow X'(0) = 0 \\ 0 &= \tfrac{\partial u(L,t)}{\partial x} = X'(L)T(t) \Rightarrow X'(L) = 0. \end{aligned} \tag{11.8}$$

Now substitute the solution from (11.8) and use the fact that we have assumed that $\lambda \neq 0$

$$\begin{aligned} 0 &= X'(0) = -A\lambda.0 + B\lambda \Rightarrow B = 0 \\ 0 &= X'(L) = -A\lambda sin\lambda L\lambda \Rightarrow \lambda_n = \left(\tfrac{n\pi}{L}\right) \quad n = 1,2,\ldots \end{aligned} \tag{11.9}$$

Therefore for the case $\lambda \neq 0$ we have the countably infinite set of eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right) \quad n = 1,2,\ldots \text{and} \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right). \tag{11.10}$$

Case II: Spatial equation assuming that $\lambda = 0$:

In this case the spatial ODE reduces to

$$X''(x) = 0 \tag{11.11}$$

which has a general solution

$$\begin{aligned} X(x) &= A.1 + Bx \\ X'(x) &= B \end{aligned} \tag{11.12}$$

Now imposing the boundary conditions

$$\begin{aligned} 0 &= X'(0) = B \Rightarrow B = 0 \\ 0 &= X'(L) = B \Rightarrow B = 0 \end{aligned} \tag{11.13}$$

The complete set of eigenvalues and eigenfunctions are thus:

$$\lambda_n = \left(\frac{n\pi}{L}\right) \quad n = 0, 1, 2, \ldots \text{and} \quad X_0(x) = 1,\ X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \ldots. \tag{11.14}$$

Thus $u_n(x,t) = \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2, \ldots$
are all solutions of $u_t = \alpha^2 u_{xx}$ that satisfy the boundary conditions (11.2). (11.15)

Since (11.1) is linear, a linear combination of solutions is again a solution. Thus the most general solution is for the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{11.16}$$

What about the initial condition $u(x,0) = f(x)$? If we let $t = 0$ in (11.16), then to complete the solution process we are reduced to determining the coefficients A_n in the series

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right). \tag{11.17}$$

As in the last lecture we use the inner product $< .,. >$ to project $f(x)$ onto the basis functions in the series:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \tag{11.18}$$

$$\langle f, \cos\left(\frac{k\pi x}{L}\right)\rangle = \int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx = A_0\int_0^L \cos\left(\frac{k\pi x}{L}\right)\,dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)\,dx. \tag{11.19}$$

Recall the identity $\cos(A)\cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$. Therefore

$$\begin{aligned}
J_{nk} &= \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)\,dx \\
&= \frac{1}{2}\int_0^L \cos(n-k)\frac{\pi x}{L} + \cos(n+k)\frac{\pi x}{L}\,dx \qquad n \neq k \\
&= \frac{1}{2}\left[\frac{\sin(n-k)\pi x/L}{(n-k)\pi/L} + \frac{\sin(n+k)\pi x/L}{(n+k)\pi/L}\right]_0^L \\
&= 0 \\
J_{nn} &= \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{2}\int_0^L 1 + \cos\left(\frac{2n\pi x}{L}\right)\,dx \\
&= L/2 \\
J_{00} &= \int_0^L 1 dx = L
\end{aligned} \tag{11.20}$$

Substituting these integrals into (11.19) we obtain the following expressions for the Fourier Coefficients A_k

$$A_0 = \frac{1}{L}\int_0^L f(x)dx. \tag{11.21}$$

$$A_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\, dx. \tag{11.22}$$

Finally the solution of the initial boundary value problem (11.1) is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{11.23}$$

where A_n are defined in (11.21)-(11.22). We observe that as $t \to \infty$ it follows that $u(x,t) \to A_0$, which is just the average value of the initial heat $f(x)$ distributed in the bar as can be seen from (11.21). This is consistent with physical intuition.

It is sometimes convenient to re-define the Fourier coefficients as follows:

$$a_0 = 2A_0$$

$$a_k = A_k,\ k = 1, 2, \ldots$$

so that the a_k assume the unified form $a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\, dx \quad k = 0, 1, 2, \ldots$ (11.24)

In terms of the new coefficients a_k defined in (11.24) the Fourier expansion for the initial condition function $f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{11.25}$$

while the solution of the heat equation (11.1) is of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{11.26}$$

Example 11.1 *Fourier Cosine Expansion: Determine the Fourier coefficients a_k for the function*

$$f(x) = x, //0 < x < 1 = L \tag{11.27}$$

and use the resulting Fourier Cosine expansion to prove the identity

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots + \frac{1}{(2k+1)^2} + \ldots$$

Solution:

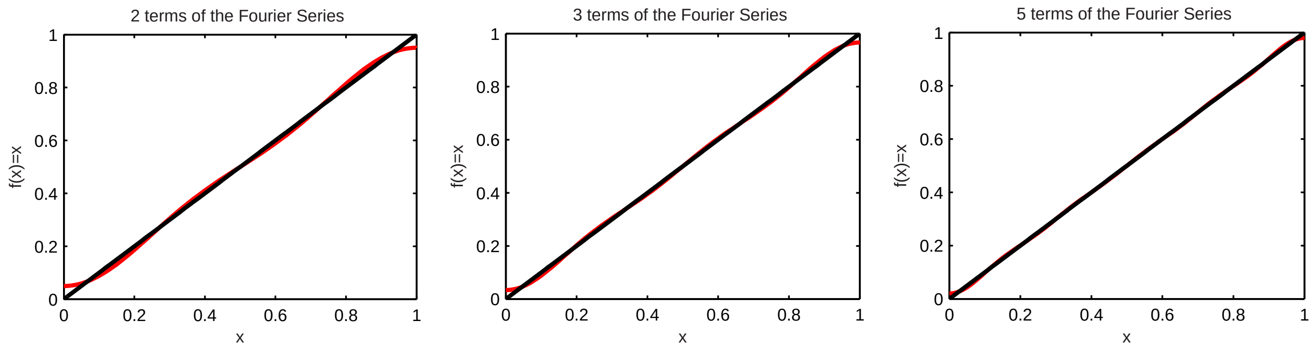
$$a_0 = 2\int_0^1 x dx = 2\left[\frac{x}{2}\right]_0^1 = 1$$

$$a_n = 2\int_0^1 x\cos(n\pi x)dx = 2\frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} -\frac{4}{n^2\pi^2},\ n \text{ odd} \\ 0,\ n \text{ even} \end{cases}$$

substituting these expressions for the a_n into (11.25), we obtain

$$f(x) = x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left((2k+1)\pi x\right) \tag{11.28}$$

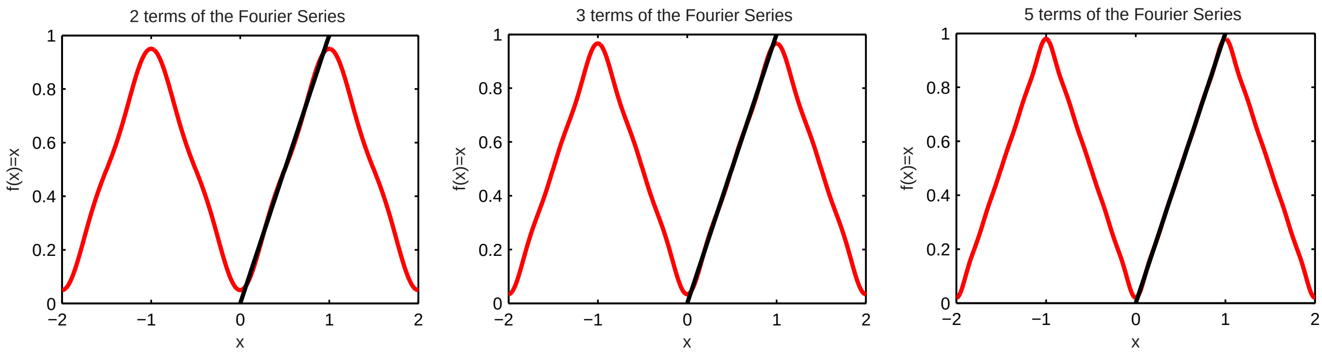
To obtain the required identity we set $x = 1$ in and rearrange terms. The partial sums are shown in figure 2



(a) Sum till $n = 2$ terms (b) Sum till $n = 3$ terms (c) Sum till $n = 5$ terms

FIGURE 2. These figures show the partial sums of the Fourier Cosine Series

In figure 3 we plot the same graphs but on a larger domain than $[0, L] = [0, 1]$.



(a) Sum till $n = 2$ terms (b) Sum till $n = 3$ terms (c) Sum till $n = 5$ terms

FIGURE 3. These figures show the partial sums of the Fourier Cosine Series

MATLAB Code:

```
%fourier cos example
clear;clf;dx=0.001;dt=0.001;
x=-2:dx:2;xr=0:dx:1;nterms=10;ntime=100;
for nt=1:ntime
    t = (nt-1)*dt;
    for n=1:nterms
        K=0:1:n;
        u(:,n+1)=0.5-4*(cos(pi*(2*K+1)'*x)'*(exp(-pi^2*(2*K+1).^2*t)./(2*K+1).^2)')/pi^2;
        plot(x',u(:,n+1),'r-',xr',xr','k-','linewidth',2);ax=axis;ax=[0 1 0 1.2];axis(ax);
        tit=[num2str(n+1),' terms of the Fourier Series '];title(tit);xlabel('x');ylabel('u(x,t), f(x)=x');pause(.01)
    end
    if mod(nt,5)==0,pause(.02);end
end
```