Lecture 13: Full Range Fourier Series

(*Compiled 3 March 2014*)

In this lecture we consider the Full Range Fourier Series for a given functions defined on an interval $[-L, L]$. Otside this interval we see that the Fourier Series represents the *periodic extension* of the function $f(x)$.

Key Concepts: Full Range Fourier Series; Periodic Extension; Complex Fourier Series.

13.1 Fourier Series

We consider the expansion of the function $f(x)$ of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = S(x) \tag{13.1}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \qquad \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \text{ average value of } f.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{13.2}$$

Observations:

(1) Note that $\cos\left(\frac{n\pi}{L}(x+T)\right) = \cos\left(\frac{n\pi x}{L}\right)$ provided $\frac{n\pi T}{L} = 2\pi$, $T = \frac{2L}{n}$ and similarly $\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi x}{L}\right)$. Thus each of the terms of the Fourier Series $S(x)$ on the RHS of (13.1) is a periodic function having a maximal period $2L$. As a result the function $S(x)$ is also periodic.

How does this relate to $f(x)$ which may not be periodic?

The function $S(x)$ represented by the series is known as the periodic extension of f on $[-L, L]$.

(2) If f (or its periodic extension) is discontinuous at a point x_0 then $S(x)$ converges to the average value of f across the discontinuity.

$$S(x_0) = \frac{1}{2}\left\{f(x_0^+) + f(x_0^-)\right\} \tag{13.3}$$

Example 13.1

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \qquad L = \pi \\ x & 0 \le x \le \pi \end{cases} \tag{13.4}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{0}^{\pi} x\, dx = \frac{\pi}{2} \tag{13.5}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} x \cos(nx)\, dx \\ &= \frac{1}{\pi} \left\{ x \frac{\sin(nx)}{n} \bigg|_0^{\pi} - \frac{1}{n} \int_0^{\pi} 1. \sin(nx)\, dx \right\} \\ &= \frac{1}{\pi} \left\{ \frac{\pi \sin}{n} (\cancel{n\pi}) + \frac{1}{n^2} \cos(nx) \bigg|_0^{\pi} \right\} \end{aligned}$$

$$= \frac{1}{\pi n^2} \left[(-1)^n - 1 \right] \qquad \begin{array}{ccccc} n & 1 & 2 & 3 & 4 \\ (-1)^n - 1 & -2 & 0 & -2 & 0 \end{array} \tag{13.6}$$

$$a_{2m+1} = -\frac{2}{\pi (2m+1)^2} \qquad m = 0, 1, 2, \ldots \tag{13.7}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} x \sin(nx)\, dx \\ &= \frac{1}{\pi} \left\{ -x \frac{\cos(nx)}{n} \bigg|_0^{\pi} + \frac{1}{n} \int_0^{\pi} 1. \cos(nx)\, dx \right\} \\ &= \frac{1}{\pi} \left\{ -\pi \frac{\cos(n\pi)}{n} + \frac{0. \cos 0}{n} + \frac{1}{n^2} \sin(nx) \bigg|_0^{\pi} \right\} \end{aligned}$$

$$= (-1)^{n+1}/n \tag{13.8}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos\left[(2m+1)x\right]}{(2m+1)^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n} \tag{13.9}$$

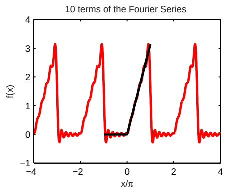

FIGURE 1. Truncated Fourier Series approximation to $f(x)$ using 10 terms. Notice the periodic extension of the function that was sampled on $[-\pi, \pi]$ and the oscillations in the Fourier Series near the points of discontinuity. Also note that at the point of discontinuity $x = \pi$, $S(\pi) = \frac{1}{2}\left\{f(\pi^+) + f(\pi^-)\right\}$

13.2 It can be useful to shift the interval of integration from $[-L, L]$ to $[c, c+2L]$

Since the periodic extension $f_e(x)$ is periodic with period $2L$ (as are the basis functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$).

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{c}^{c+2L} f_e(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{13.10}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{c}^{c+2L} f_e(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{13.11}$$

Example 13.2 *Previous Example:*

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \le x \le \pi \end{cases} \tag{13.12}$$

On $[\pi, 3\pi]$

$$f_e(x) = \begin{cases} 0 & \pi < x < 2\pi \\ x - 2\pi & 2\pi \le x \le 3\pi \end{cases} \tag{13.13}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{\pi}^{3\pi} f_e(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{2\pi}^{3\pi} (x-2\pi)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)\,dt. \end{aligned} \qquad \begin{array}{ll} t = x - 2\pi & dx = dt \\ x = t + 2\pi & x = \pi \Rightarrow t = -\pi \\ & x = 3\pi \Rightarrow t = \pi \\ \textit{since} & \cos n(t+2\pi) = \cos t \end{array} \tag{13.14}$$

13.3 Complex Form of Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\left(\frac{n\pi x}{L}\right)} + e^{-i\left(\frac{n\pi x}{L}\right)}}{2}; \quad \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\left(\frac{n\pi x}{L}\right)} - e^{-i\left(\frac{n\pi x}{L}\right)}}{2i}$$

$$\text{Therefore } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left\{e^{i\left(\frac{n\pi x}{L}\right)} + e^{-i\left(\frac{n\pi x}{L}\right)}\right\} + \frac{b_n}{2i}\left\{e^{i\left(\frac{n\pi x}{L}\right)} - e^{-i\left(\frac{n\pi x}{L}\right)}\right\}$$

$$= \underset{\substack{\uparrow \\ c_0}}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underset{\substack{\uparrow \\ c_n}}{\left(\frac{a_n - ib_n}{2}\right)} e^{i\left(\frac{n\pi x}{L}\right)} + \underset{\substack{\uparrow \\ c_{-n}}}{\left(\frac{a_n + ib_n}{2}\right)} e^{-i\left(\frac{n\pi x}{L}\right)} \tag{13.15}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)}$$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\left\{\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right\} dx \tag{13.16}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)} dx \qquad b_{-n} = -b_n \tag{13.17}$$

Therefore

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)} \tag{13.18}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)} dx. \tag{13.19}$$

Example 13.3

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 < x < \pi \end{cases} \qquad L = \pi \tag{13.20}$$

$$c_n = \frac{1}{2\pi}\left\{-\int_{-L}^{0} e^{-inx}\, dx + \int_{0}^{\pi} e^{-inx}\, dx\right\} \tag{13.21}$$

$$= \frac{1}{2\pi}\left\{-\frac{e^{-inx}\big|_{-\pi}^{0}}{(-in)} + \frac{e^{-inx}\big|_{0}^{\pi}}{(-in)}\right\} \tag{13.22}$$

$$= \frac{i}{2\pi n}\left\{-2 + e^{+in\pi} + e^{-in\pi}\right\} = \begin{cases} 0 & n \text{ even} \\ (2/i\pi n) & n \text{ odd} \end{cases} \tag{13.23}$$

Therefore

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi i(2n+1)} e^{i\left((2n+1)x\right)}. \tag{13.24}$$