Lecture 25: More Rectangular Domains: Neumann Problems, mixed BC, and semi-infinite strip problems

(*Compiled 6 November 2013*)

In this lecture we Proceed with the solution of Laplace's equations on rectangular domains with Neumann, mixed boundary conditions, and on regions which comprise a semi-infinite strip.

Key Concepts: Laplace's equation; Rectangular domains; The Neumann Problem; Mixed BC and semi-infinite strip problems.

Reference Section: Boyce and Di Prima Section 10.8

25 More Rectangular Domains with mixed BC and semi-infinite strip problems

25.1 The Neumann Problem on a rectangle - only flux boundary conditions

Example 25.1 *The Neumann Problem:*

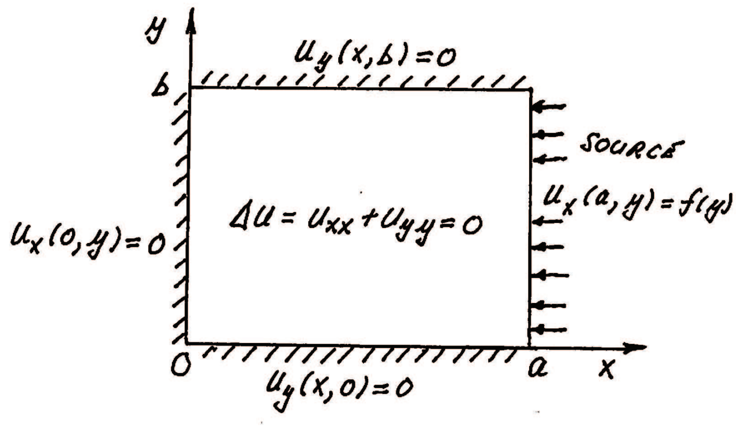

FIGURE 1. Inhomogeneous Neumann Boundary conditions on a rectangular domain as prescribed in (??)

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a \quad 0 < y < b \tag{25.1}$$

$$u_x(0, y) = 0 \quad u_x(a, y) = f(y) \tag{25.2}$$

$$u_y(x, 0) = 0 = u_y(x, b). \tag{25.3}$$

Let $u(x, y) = X(x)Y(y)$.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda^2 \tag{25.4}$$

$$\left.\begin{array}{l} Y''(y)+\lambda^2 Y(y)=0 \\ Y'(0)=0=Y'(b) \end{array}\right\} \quad \begin{array}{rcl} Y &=& A\cos\lambda y + B\sin\lambda y \\ Y' &=& -A\lambda\sin\lambda y + B\lambda\cos\lambda y \end{array} \tag{25.5}$$

$$Y'(0)=\lambda B=0 \quad \lambda=0 \text{ or } B=0. \tag{25.6}$$

$$Y'(b)=-A\lambda\sin\lambda b=0 \quad \begin{array}{rcl} \lambda_n &=& (n\pi/b) \quad n=0,1,\ldots \\ Y_n &=& \cos\left(\frac{n\pi y}{b}\right), \quad Y_0=1 \end{array} \tag{25.7}$$

$$X_n''-\lambda^2 X_n=0 \tag{25.8}$$

$$X_n'(0)=0 \tag{25.9}$$

$n=0$: $X_0''=0$, $X_0=c_0x+D_0 \Rightarrow X_0'=c_0 \Rightarrow X_0'(0)=c_0=0$.

Choose $D_0=1$: $X_0=1$

$$\begin{array}{rcl} n\geq 1 \quad X_n &=& c_n\cosh(\lambda_n x)+D_n\sinh(\lambda_n x) \\ X_n' &=& c_n\lambda\sinh(\lambda_n x)+D_n\lambda\cosh(\lambda_n x) \\ X_n'(0) &=& \lambda_n D_n=0 \end{array} \tag{25.10}$$

Choose $c_n=1$: $X_n=\cosh(\lambda_n x)$.

Thus

$$\left.\begin{array}{rcccl} u_n(x,y) &=& X_nY_n &=& \cosh(\lambda_n x)\cos(\lambda_n y) \\ u_0(x,y) &=& X_0Y_0 &=& 1 \end{array}\right\} \text{ satisfy homog. BC.} \tag{25.11}$$

Therefore

$$u(x,y)=A_0+\sum_{n=1}^{\infty}A_n\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right). \tag{25.12}$$

Now $f(y)=u_x(a,y)$.

$$u_x(x,y)=\sum_{n=1}^{\infty}A_n\left(\frac{n\pi}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{25.13}$$

$$u_x(a,y)=\sum_{n=1}^{\infty}\left\{A_n\left(\frac{n\pi}{b}\right)\sinh\left(\frac{n\pi a}{b}\right)\right\}\cos\left(\frac{n\pi y}{b}\right)=f(y)\ldots \tag{25.14}$$

This is like a Fourier Cosine Series for $f(y)$ but without the constant term a_0.

Recall

$$f(y)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi y}{b}\right), \; a_n=\frac{2}{b}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)\,dy. \tag{25.15}$$

Thus the expansion (25.14) is consistent only if $a_0=0$. For this to be true we require that

$$\int_0^b f(y)\,dy=0 \tag{25.16}$$

if $\int_0^b f(y)\,dy\neq 0$ then there is no solution to the boundary value problem 1.

Note

(1) If $\int_0^b f(y)\,dy \neq 0$ there is a **net flux** into the domain through the right hand boundary and, since the other boundaries are insulated, there can be no steady solution – the temperature will continually change with time.

(2) If $\int_0^b f(y)\,dy = 0$ there is no net flux through the boundary and a steady state can exist. i.e. It is possible that $u_{xx} + u_{yy} = u_t = 0$. If $\int_0^b f(y)\,dy = 0$ then

$$A_n \left(\frac{n\pi}{b}\right) \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)\,dy. \tag{25.17}$$

Therefore

$$A_n = \frac{2}{n\pi\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)\,dy \quad n \geq 1 \tag{25.18}$$

and

$$u_\infty(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{25.19}$$

where A_0 is undetermined. $u(x,y)$ is said to be known up to an arbitrary constant.

(3) If $u_\infty(x,y)$ is the steady state of a 2D Heat Equation $u_t = u_{xx} + u_{yy}$ with $u(x,y,0) = u_0(x,y)$ then

$$\int_D u_t\,dx\,dy = \int_D \nabla\cdot\nabla u\,dx\,dy = \int_{\partial D}\frac{\partial u}{\partial n}\,ds = 0. \tag{25.20}$$

Therefore

$$\frac{\partial}{\partial t}\left(\int_D u\,dx\,dy\right) = 0 \Rightarrow \int_D u\,dx\,dy = \text{ const for all time } = \int_D u_0(x,y)\,dx\,dy. \tag{25.21}$$

Now

$$\int_D u_\infty(x,y)dxdy = A_0 \times \text{ area}(D) = \int_D u_0(x,y)\,dx \tag{25.22}$$

Which the condition that determines A_0.

25.2 Rectangular domains with mixed BC

Example 25.2 *Insulating BC along two sides and specified temperatures on the others:*

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{25.23}$$

$$0 = u_x(0,y) = u_x(a,y) = u(x,0) \tag{25.24}$$

$$u(x,b) = f(x). \tag{25.25}$$

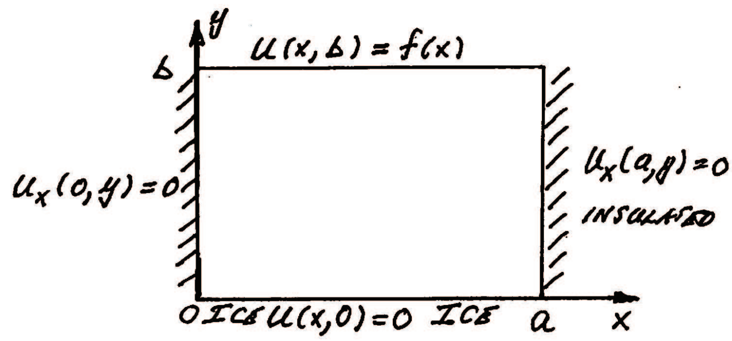

FIGURE 2. Mixed Boundary conditions on a rectangular domain as prescribed in (25.24)

Let $u(x,y) = X(x)Y(y)$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm\lambda^2. \tag{25.26}$$

Since we have homogeneous BC on $X'(0) = 0 = X'(a)$ *choose* $-\lambda^2$.

(1) $X'' + \lambda^2 X = 0 \quad X'(0) = 0 = X'(a)$.

$$\begin{array}{rclrcl} X(x) & = & A\cos\lambda x + B\sin\lambda x & X'(x) & = & -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x) \\ X'(0) & = & B\lambda = 0 \Rightarrow B = 0 & X'(a) & = & -A\lambda\sin(\lambda a) = 0 \end{array} \tag{25.27}$$

Therefore

$$\lambda_n = (n\pi/a) \quad n = 0, 1, 2, \dots \quad X_n(x) = \cos\left(\frac{n\pi y}{a}\right) \tag{25.28}$$

are eigenfunctions and eigenvalues.

(2) $\lambda_n \neq 0$*:* $Y'' - \lambda^2 Y = 0$ *and* $Y(0) = 0 \Rightarrow Y_n(y) = A\sinh\left(\frac{n\pi y}{a}\right)$ $n \neq 0$. *Thus*

$$u_n(x,y) = \cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \tag{25.29}$$

satisfy homogeneous BC.

$\lambda_0 = 0$*: In this case the ODE for* Y_0 *is:*

$$Y_0'' = 0 \Rightarrow Y(y) = c_1 y + c_2 \tag{25.30}$$

$$Y_0(0) = c_2 = 0 \Rightarrow Y_0(y) = y \tag{25.31}$$

and $u_0(x,y) = y \cdot 1$ *satisfies the homogeneous BC.*

Therefore

$$u(x,y) = c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right) \tag{25.32}$$

$$u(x,b) = \frac{(2c_0 b)}{2} + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right)\cos\left(\frac{n\pi x}{a}\right) = f(x) \tag{25.33}$$

$$(2c_0 b) = \frac{2}{a}\int_0^a f(x)\,dx; \quad c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx \tag{25.34}$$

$$c_0 = \frac{1}{ab}\int_0^a f(x)\,dx; \quad c_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx \tag{25.35}$$

$$u(x,y) = c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right). \tag{25.36}$$

25.3 Semi-infinite strip problems

Example 25.3 *A Semi-infinite strip with specified temperatures:*

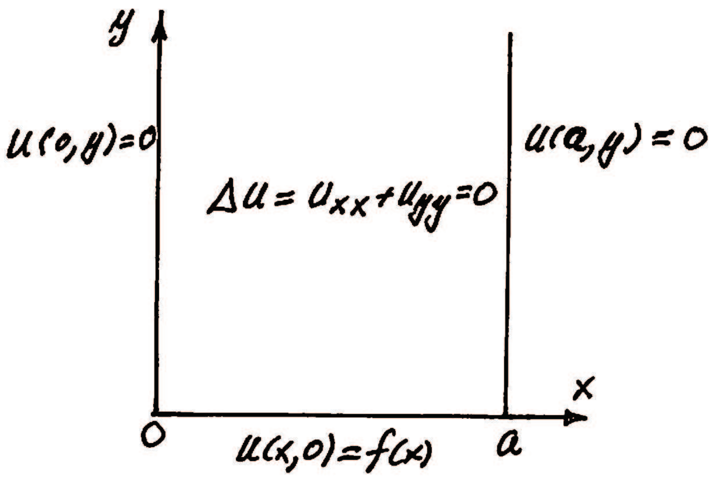

Figure 3. Diriclet Boundary conditions on a semi-infinite strip as prescribed in (25.39)

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \quad 0 < y < \infty \tag{25.37}$$

$$u(0,y) = 0 = u(a,y) \tag{25.38}$$

$$u(x,0) = f(x) \qquad u(x,y) \to 0 \text{ as } y \to \infty \tag{25.39}$$

Let $u(x,t) = X(x)T(t)$ *and plug into (1a?):*

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2 \text{ since we have homogeneous BC on } X. \tag{25.40}$$

(1)

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(a) \end{array}\right\} \quad \begin{array}{lcl} \lambda_n & = & n\pi/a \quad n = 1, 2, \ldots \\ X_n & = & \sin\left(\frac{n\pi x}{a}\right) \end{array} \tag{25.41}$$

(2) $Y'' - \lambda^2 Y = 0 \quad Y(y) = A\mathrm{e}^{-\lambda y} + B\mathrm{e}^{\lambda y}$. *Since* $u(x,y) \to 0$ *as* $y \to \infty$ *we require* $B = 0$. *Therefore*

$$u_n(x,y) = \mathrm{e}^{-\lambda_n y} \sin\left(\frac{n\pi x}{a}\right) \tag{25.42}$$

satisfy the homogeneous BC and the BC at ∞. *Thus*

$$u(x,y) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right). \tag{25.43}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \Rightarrow c_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx. \tag{25.44}$$

Example 25.4 *Semi-infinite strip with inhomogeneous BC:*

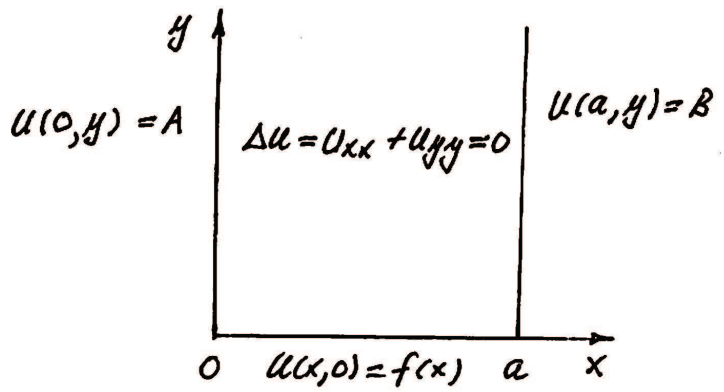

FIGURE 4. Diriclet Boundary conditions on a semi-infinite strip as prescribed in (25.47)

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \quad 0 < y < \infty \tag{25.45}$$

$$u(0,y) = A, \quad B = u(a,y) \tag{25.46}$$

$$u(x,0) = f(x) \qquad u(x,y) \to 0 \ \text{as} \ y \to \infty \tag{25.47}$$

Look for a function $v(x)$ *for which* $v'' = 0$ *and which satisfies the inhomogeneous BC.*
$v = \alpha x + \beta \quad v(0) = A = \beta \quad v(a) = \alpha a + A = B$
Therefore $v(x) = \left(\frac{B - A}{a}\right) x + A$.

Now let $u(x,y) = v(x) + w(x,y)$.

$$0 = u_{xx} + u_{yy} = v_{xx} + w_{xx} + v_{yy} + w_{yy} \Rightarrow \Delta w = 0 \tag{25.48}$$

$$A = u(0,y) = v(0) + w(0,y) \Rightarrow w(0,y) = 0 \tag{25.49}$$

$$B = u(a,y) = v(a) + w(a,y) \Rightarrow w(a,y) = 0 \tag{25.50}$$

$$f(x) = u(x,0) = v(x) + w(x,0) \Rightarrow w(x,0) = f(x) - v(x). \tag{25.51}$$

Thus w *satisfies the same BVP as does* u *in Eg. 3 above.*

Therefore

$$u(x,y) = (B-A)(x/a) + A + \sum_{n=1}^{\infty} d_n \mathrm{e}^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right) \tag{25.52}$$

where

$$d_n = \frac{2}{a} \int_0^a \{f(x) - v(x)\} \sin\left(\frac{n\pi x}{a}\right)\, dx. \tag{25.53}$$