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Lecture 26: Circular domains

(*Compiled 3 March 2014*)

In this lecture we consider the solution of Laplace's equations on domains that have a boundary that has at least one boundary segment that comprises a circular arc.

Key Concepts: Laplace's equation; Circular domains; Pizza Slice-shaped regions, Dirichlet and Mixed BC.

Reference Section: Boyce and Di Prima Section 10.8

26 General Analysis of Laplace's Equation on Circular Domains:

26.1 Laplacian in Polar Coordinates

For domains whose boundary comprises part of a circle, it is convenient to transform to polar coordinates. For this purpose the Laplacian is transformed from cartesian coordinates (x, y) to polar coordinates (r, θ) as follows:

$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating with respect to x and y we obtain:

$$\begin{aligned}
2rr_x &= 2x \quad r_x = \frac{x}{r} \quad r_y = \frac{y}{r}\\
\theta_x &= \frac{(-\frac{y}{x^2})}{\left(1+(\frac{y}{x})^2\right)} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} \quad \theta_y = \frac{x}{r^2}\\
u(x,y) &= U(r,\theta)\\
u_x &= U_r r_x + U_\theta \theta_x = U_r\left(\frac{x}{r}\right) + U_\theta\left(-\frac{y}{r^2}\right)\\
u_y &= U_r r_y + U_\theta \theta_y = U_r\left(\frac{y}{r}\right) + U_\theta\left(\frac{x}{r^2}\right)\\
u_{xx} &= (U_r)_x r_x + U_r r_{xx} + (U_\theta)_x \theta_x + U_\theta \theta_{xx}\\
&= U_{rr} r_x^2 + U_{r\theta}\theta_x r_x + U_r r_{xx} + U_{r\theta} r_x \theta_x + U_{\theta\theta}\theta_x^2 + U_\theta \theta_{xx}\\
r_{xx} &= \frac{r - \left(\frac{x^2}{r}\right)}{r^2} = \frac{y^2}{r^3} \quad \theta_{xx} = \frac{2yx}{r^4}\\
u_{xx} &= U_{rr}\left(\frac{x^2}{r^2}\right) + 2U_{r\theta}\left(-\frac{y}{r^2}\right)\left(\frac{x}{r}\right) + U_{\theta\theta}\frac{y^2}{r^4} + U_r\left(\frac{y^2}{r^3}\right) + U_\theta\left(\frac{2xy}{r^4}\right)\\
u_{yy} &= U_{rr} r_y^2 + U_{r\theta} r_y \theta_y + U_r r_{yy} + U_{\theta r} r_y \theta_y + U_{\theta\theta}\theta_y^2 + U_\theta \theta_{yy}\\
&= U_{rr}\left(\frac{y^2}{r^2}\right) + 2U_{r\theta}\left(\frac{x}{r^2}\right)\left(\frac{y}{r}\right) + U_{\theta\theta}\left(\frac{x^2}{r^4}\right) + U_r\left(\frac{x^2}{r^3}\right) + U_\theta\left(-\frac{2xy}{r^4}\right)
\end{aligned}$$

$$\boxed{u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}}$$

26.2 Introductory remarks about circular domains

Recall the Laplacian in polar coordinates:

$$0 = \Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \qquad \begin{array}{l} r = (x^2+y^2)^{1/2} \\ \theta = \tan^{-1}(y/x) \end{array} . \tag{26.1}$$

Let

$$u(r,\theta) = R(r)\Theta(\theta) \tag{26.2}$$

$$\frac{r^2R'' + rR'}{R(r)} = -\frac{\Theta''}{\Theta(\theta)} = \lambda^2 \tag{26.3}$$

which leads to $r^2R'' + rR' - \lambda^2 R = 0$ and $\Theta'' + \lambda^2\Theta = 0$.

The R Equation: $r^2R'' + rR' - \lambda^2R = 0$:

$\lambda = \mathbf{0}$: $r^2R'' + rR' = 0$, $R = r^\gamma \Rightarrow \gamma(\gamma-1) + \gamma = \gamma^2 = 0 \Rightarrow R(r) = C + D\ln r$

$\lambda \neq \mathbf{0}$: $r^2R'' + rR' - \lambda^2 = 0$, $R = r^\gamma \Rightarrow \gamma(\gamma-1) + \gamma - \lambda^2 = \gamma^2 - \lambda^2 = 0 \Rightarrow R(r) = Cr^\lambda + Dr^{-\lambda}$.

The Θ Equation $\Theta'' + \lambda^2\Theta = 0$:

$\Theta'' + \lambda^2\Theta = 0$, $\Theta = A\cos\lambda\theta + B\sin\lambda\theta$, $\Theta' = -A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta$

Different Boundary Conditions and corresponding eigenfunctions:

(I) $\Theta(0) = 0 = \Theta(\alpha)$ $\lambda_n = n\pi/\alpha$, $n = 1,2,\ldots,$ $\Theta_n(\theta) = \sin\lambda_n\theta$

(II) $\Theta'(0) = 0 = \Theta'(\alpha)$, $\lambda_n = n\pi/\alpha$ $n = 0,1,2,\ldots,$ $\Theta_n(\theta) \in \{1, \cos\lambda_n\theta\}$

(III) $\Theta(0) = 0 = \Theta'(\alpha)$, $\lambda_n = (2n-1)\pi/2\alpha$ $n = 1,2,\ldots,$ $\Theta_n(\theta) = \sin\lambda_n\theta$

(IV) $\Theta'(0) = 0 = \Theta(\alpha)$, $\lambda_n = (2n-1)\pi/2\alpha$, $n = 1,2,\ldots,$ $\Theta_n(\theta) = \cos\lambda_n\theta$

(V) $\left.\begin{array}{rcl} \Theta(-\pi) & = & \Theta(\pi) \\ \Theta'(-\pi) & = & \Theta'(\pi) \end{array}\right\}$ $\lambda_n = n$, $n = 0,1,2,\ldots,$ $\Theta_n(\theta) \in \{1, \cos\lambda_n\theta, \sin\lambda_n\theta\}$.

The most general solution is thus of the form

$$u(r,\theta) = \{A_0 + \alpha_0 \ln r\}\cdot 1 + \sum_{n=1}^{\infty}\left\{A_n r^{\lambda_n} + \alpha_n r^{-\lambda_n}\right\}\cos\lambda_n\theta \tag{26.4}$$

$$+ \sum_{n=1}^{\infty}\left\{B_n r^{\lambda_n} + \beta_n r^{-\lambda_n}\right\}\sin\lambda_n\theta. \tag{26.5}$$

Observations:

- For problems that include the origin, the condition $|u| < \infty$ as $r \to 0$ dictates that $\alpha_0 = 0$, $\alpha_n = 0$ and $\beta_n = 0$.
- For problems that involve infinite domains the condition $|u| < \infty$ as $r \to \infty$ dictates that $A_n = 0$ and $B_n = 0$.
- The values of λ_n and the corresponding eigenfunctions depend on the boundary conditions (I)–(V) that apply.

26.3 Wedge Problems

Example 26.1 *Wedge with homogeneous BC on $\theta = 0$, $\theta = \alpha < 2\pi$*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 0 < r < a, \quad 0 < \theta < \alpha \tag{26.6}$$

$$u(r,0) = 0 \quad u(r,\alpha) = 0,\ u(r,\theta) \textit{ bounded as } r \to 0,\ u(a,\theta) = f(\theta) \tag{26.7}$$

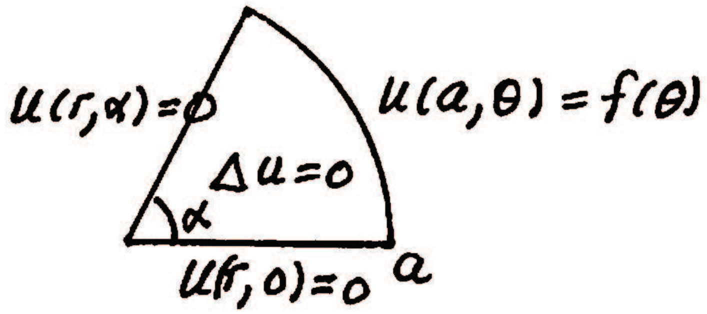

FIGURE 1. Homogeneous Dirichlet Boundary conditions on a wedge shaped domain (26.7)

Let $u(r,\theta) = R(r)\cdot\Theta(\theta)$.

$$r^2\frac{(R''+\frac{1}{r}R')}{R} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \Rightarrow \begin{array}{rcl} r^2R''+rR'-\lambda^2 R &=& 0 \text{ Euler Eq.} \\ \Theta''+\lambda^2\Theta &=& 0 \end{array}$$

$u(r,0) = R(r)\Theta(0) = 0 \Rightarrow \Theta(0) = 0$; $u(r,\alpha) = R(r)\Theta(\alpha) = 0 \Rightarrow \Theta(\alpha) = 0$

$$\begin{array}{l}\text{Eigenvalue} \\ \text{Problem}\end{array} \left\{ \begin{array}{ll} \Theta''+\lambda^2\Theta = 0 & \Theta = A\cos\lambda\theta + B\sin(\lambda\theta) \\ \Theta(0) = 0 = \Theta(\alpha) & \Theta(0) = A = 0 \quad \Theta(\alpha) = B\sin(\lambda\alpha) = 0 \end{array}\right. \tag{26.8}$$

Therefore

$$\lambda_n = (n\pi/\alpha) \quad n = 1,2,\ldots \quad \Theta_n = \sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{26.9}$$

To solve the Euler Eq. let $R(r) = r^\gamma$, $R' = \gamma r^{\gamma-1}$, $R'' = \gamma(\gamma-1)r^{\gamma-2}$. Therefore

$$\gamma(\gamma-1)+\gamma-\lambda^2 = \gamma^2-\lambda^2 = 0 \Rightarrow \gamma = \pm\lambda. \tag{26.10}$$

Therefore

$$R(r) = c_1 r^\lambda + c_2 r^{-\lambda}. \tag{26.11}$$

Now since $u(r,\theta) < \infty$ as $r \to 0$ we require $c_2 = 0$. Therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \sin\left(\frac{n\pi\theta}{\alpha}\right) \tag{26.12}$$

$$u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} \left\{c_n a^{\left(\frac{n\pi}{\alpha}\right)}\right\} \sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{26.13}$$

This is just a Fourier Sine Series for $f(\theta)$: Therefore

$$c_n a^{\left(\frac{n\pi}{\alpha}\right)} = \frac{2}{\alpha}\int_0^\alpha f(\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)\,d\theta \tag{26.14}$$

$$c_n = \frac{2}{\alpha}a^{-\left(\frac{n\pi}{\alpha}\right)}\int_0^\alpha f(\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)\,d\theta. \tag{26.15}$$

Therefore

$$u(x,\theta) = \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{26.16}$$

Example 26.2 *A wedge with Inhomogeneous BC*

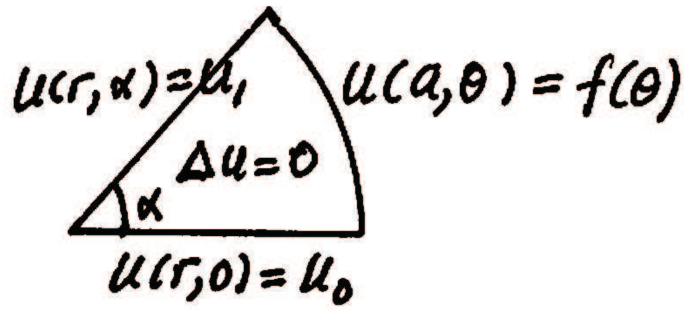

FIGURE 2. Inhomogeneous Dirichlet Boundary conditions on a wedge shaped domain (26.18)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 0 < r < a, \quad 0 < \theta < \alpha \tag{26.17}$$

$$u(r,0) = u_0, \quad u(r,\alpha) = u_1, \quad u(r,\theta) < \infty \text{ as } r \to 0, \quad u(a,\theta) = f(\theta) \tag{26.18}$$

Let us look for the simplest function of θ only that satisfies the inhomogeneous BC of the from: $w(\theta) = (u_1 - u_0)\frac{\theta}{\alpha} + u_0$. Note that $w_{\theta\theta} = 0$ and that $w(0) = u_0$ and $w(\alpha) = u_1$. Then let $u(r,\theta) = w(\theta) + v(r,\theta)$.

$$\left.\begin{array}{c} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 \\ v(r,0) = 0 \quad v(r,\alpha) = 0 \\ v(a,\theta) = f(\theta) - w(\theta) \end{array}\right\} \begin{array}{l}\text{Essentially the problem} \\ \text{solved in Example 26.1}\end{array} \tag{26.19}$$

The solution is

$$u(r,\theta) = (u_1 - u_0)\frac{\theta}{\alpha} + u_0 + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \sin\left(\frac{n\pi\theta}{\alpha}\right) \tag{26.20}$$

where

$$c_n = \frac{2}{\alpha} a^{-\left(\frac{n\pi}{\alpha}\right)} \int_0^{\infty} \left[f(\theta) - w(\theta)\right] \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta. \tag{26.21}$$

Example 26.3 *A wedge with insulating BC on $\theta = 0$ and $\theta = \alpha < 2\pi$.*

$$\begin{array}{l} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u_\theta(r,0) = 0, \quad u_\theta(r,\alpha) = 0, \; u(a,\theta) = f(\theta) \end{array} \tag{26.22}$$

Let

$$u(r,\theta) = R(r)\Theta(\theta) \Rightarrow r^2\left(R'' + \frac{1}{r}R'\right)/R(r) = -\Theta''/\Theta = \lambda^2 \tag{26.23}$$

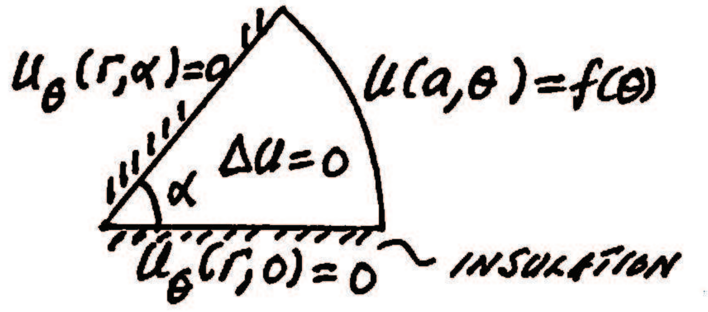

FIGURE 3. Mixed Boundary conditions on a wedge shaped domain (26.22)

Θ equation⟩

$$\left.\begin{array}{l}\Theta'' + \lambda^2\Theta = 0\\ \Theta'(0) = 0 = \Theta'(\alpha)\end{array}\right\} \begin{array}{l}\Theta(\theta) = A\cos\lambda\theta + B\sin(\lambda\theta)\\ \Theta'(0) = B\lambda = 0 \;\lambda = 0 \text{ or } B = 0;\end{array} \tag{26.24}$$

$$\begin{array}{l}\Theta'(\theta) = -A\lambda\sin(\lambda\theta) + B\lambda\cos(\lambda\theta)\\ \Theta'(\alpha) = -A\lambda\sin(\lambda\alpha) = 0 \;\lambda_n = \frac{n\pi}{\alpha};\; n = 0, 1, \ldots\end{array} \tag{26.25}$$

R equation⟩ $r^2R_n'' + rR_n' - \lambda_n^2 R_n = 0$.

$\mathbf{n = 0}$: $rR_0'' + R_0' = (rR_0')' = 0 \Rightarrow rR_0' = d_0 \Rightarrow R_0(r) = c_0 + d_0\ln r$.

$\mathbf{n \geq 1}$: $r^2R_n'' + rR_n' - \lambda^2 R_n = 0 \Rightarrow R_n = c_n r^{\lambda_n} + D_n r^{-\lambda_n}$.

Since $u(r,\theta) < \infty$ (i.e. must be bounded) as $r \to 0$ we require $d_0 = 0 = D_n$. Therefore

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right) \tag{26.26}$$

$$f(\theta) = u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n a^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right) \tag{26.27}$$

$$c_0 = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)d\theta \quad c_n = \frac{2}{\alpha}a^{-\left(\frac{n\pi}{\alpha}\right)}\int_0^{\alpha} f(\theta)\cos\left(\frac{n\pi\theta}{\alpha}\right)d\theta \tag{26.28}$$

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{26.29}$$

Example 26.4 *Mixed BC - a 'crack like' problem.*

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{26.30}$$

subject to

$$u(r,0) = 0 \quad \frac{\partial u}{\partial\theta}(r,\pi) = 0 \tag{26.31}$$

$$u(a,\theta) = f(\theta). \tag{26.32}$$

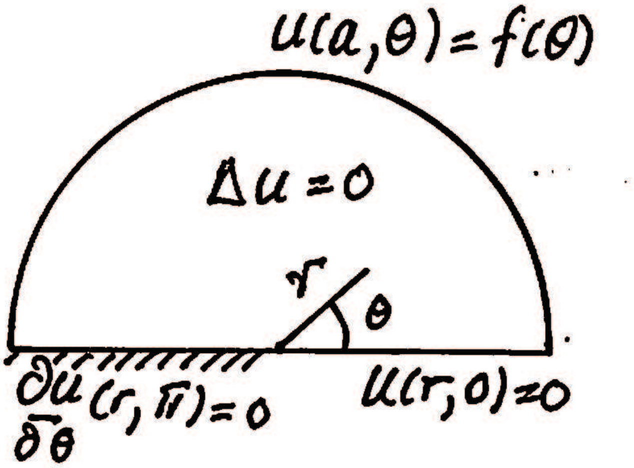

FIGURE 4. Inhomogeneous Neumann Boundary conditions on a rectangular domain as prescribed in (??)

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$r^2 \frac{\left(R'' + \frac{1}{r}R'\right)}{R} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \tag{26.33}$$

Θ equation⟩

$$\begin{array}{ll} \Theta'' + \lambda^2\Theta = 0 & \Theta = A\cos\lambda\theta + B\sin\lambda\theta \quad \Theta' = -A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta \\ \Theta(0) = 0 \quad \Theta'(\pi) = 0 & \Theta(0) = A = 0 \quad \Theta'(\pi) = B\lambda\cos(\lambda\pi) = 0 \Rightarrow \pi\lambda_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \end{array} \tag{26.34}$$

or $\lambda_n = (2n+1)\frac{1}{2}$ $n = 0, 1, \ldots$ $\lambda \neq 0$ as this would be trivial.
R equation⟩ $r^2R'' + rR' - \lambda^2 R = 0$ $R(r) = r^\gamma \Rightarrow \gamma^2 - \lambda^2 = 0$ $\gamma = \pm\lambda$. Therefore

$$u_n(r,\theta) = \left(c_n r^{\lambda_n} + d_n r^{-\lambda_n}\right)\sin\lambda_n\theta. \tag{26.35}$$

Since u should be bounded as $r \to 0$ we conclude that $d_n = 0$. The general solution is thus

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^{(2n+1)/2} \sin\left(\frac{(2n+1)}{2}\theta\right) \tag{26.36}$$

$$f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} c_n a^{(2n+1)/2} \sin\left(\left(\frac{2n+1}{2}\right)\theta\right). \tag{26.37}$$

Check orthogonality

$$\int_0^\pi \sin\left(\left(\frac{2m+1}{2}\right)\theta\right)\sin\left(\left(\frac{2n+1}{2}\right)\theta\right)d\theta = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \end{cases}. \tag{26.38}$$

Therefore

$$c_n = \frac{2a^{-\left(n+\frac{1}{2}\right)}}{\pi}\int_0^\pi f(\theta)\sin\left(\left(n+\frac{1}{2}\right)\theta\right)d\theta \tag{26.39}$$

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^{\left(n+\frac{1}{2}\right)}\sin\left(\left(n+\frac{1}{2}\right)\theta\right) \tag{26.40}$$