MATH 257/316: MIDTERM 1: February 22st 2016

Closed Book and Notes. 55 minutes. Total 50 points

PROBLEM 1: (20 Points) Consider the differential equation

$$(2+x^2)y'' - xy' - 3y = 0\,.$$

i) Find two linearly independent series solutions of this differential equation in powers of x.

ii) What is the radius of convergence of each of the series from i).

iii) Find the specific solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 2$.

PROBLEM 2: (15 Points) Consider the differential equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0\,, \quad \text{for} \quad x > 0\,.$$

i) Show that $x = 0$ is a regular singular point.

ii) Find a Frobenius series solution in powers of x that satisfies

$$\lim_{x\to 0^+} y(x) = 0\,, \qquad \lim_{x\to 0^+} x^{1/2} y'(x) = 2\,.$$

iii) Determine a more explicit solution by summing the series in ii).

PROBLEM 3: (15 Points) Consider a metal bar $0 < x < L$ with insulated ends, where there is heat loss proportional to the temperature. The temperature distribution $u(x,t)$ satisfies

$$u_t = u_{xx} - u\,, \quad 0 \le x \le L\,, \quad t \ge 0\,,$$
$$u_x(0,t) = 0 \qquad u_x(L,t) = 0\,, \quad t > 0\,,$$
$$u(x,0) = x\,, \quad 0 \le x \le L\,.$$

i) By using the method of separation of variables find an infinite series solution to this problem, and calculate explicitly the Fourier coefficients.

ii) What is $\lim_{t\to\infty} u(x,t)$?

iii) Define $M(t) \equiv \int_0^L u(x,t)\,dx$. Show that $M(t) = M(0)e^{-t}$, where $M(0) = L^2/2$.

MIDTERM 1 MATH 316/257
SOLUTIONS

PROBLEM 1

1a) FIND TWO LINEARLY INDEPENDENT SERIES SOLUTIONS IN POWERS OF X FOR

$$(2+x^2)y'' - xy' - 3y = 0$$

LET $y = \sum_{n=0}^{\infty} a_n x^n$ AND SUBSTITUTE TO GET

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\Big(2a_{n+2}(n+2)(n+1) + a_n\big(n(n-1) - n - 3\big)\Big)x^n + (4a_2 - 3a_0) + (12a_3 - 4a_1)x = 0$$

SO $a_2 = 3/4\, a_0$ $\quad a_3 = a_1/3$ $\quad a_{n+2} = -\dfrac{(n-3)}{2(n+2)} a_n \quad n = 2,3,4,5,\ldots$

THE RECURSION RELATION ALSO HOLDS FOR $n = 0, 1$ SO THAT

$$a_{n+2} = -\frac{(n-3)}{2(n+2)} a_n \quad (5 \text{ POINTS})$$

TO FIND $y_2(x)$ SET $a_0 = 0$ AND $a_1 = 1$. THEN $a_{2n} = 0$ FOR $n = 1,2,3,\ldots$

AND $a_3 = 1/3$ $\quad a_5 = -\dfrac{(3-3)a_3}{10} = 0$ SO $a_{2n+1} = 0$ FOR $n = 2,3,\ldots$

THUS WE GET (1) $\boxed{y_2(x) = a_1\left(x + x^3/3\right)_{a_1=1}}$ WHICH IS AN EXACT POLYNOMIAL SOLUTION!

NOW TO FIND $y_1(x)$ SET $a_0 = 1$ AND $a_1 = 0$. THEN $a_{2n+1} = 0$ FOR $n = 0,1,2,\ldots$

THEN $a_2 = 1\cdot 3/4$ $\quad a_4 = \dfrac{1}{8}a_2 = \dfrac{1\cdot 3}{4\cdot 8} = \dfrac{1\cdot 3}{2^4(1\cdot 2)}$ $\quad a_6 = \dfrac{-1}{12}a_4 = \dfrac{-1\cdot 3}{4\cdot 8\cdot 12} = \dfrac{-1\cdot 3}{2^6(1\cdot 2\cdot 3)}$

THUS $a_{2n} = (-1)^n \dfrac{[(-3)\cdot(-1)\cdot(1)\cdot(3)\cdots(2n-5)]}{2^{2n}(n!)}$ $\quad n = 1,2,3,\ldots$

(2) $$\boxed{y_1(x) = a_0\left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{[(-3)\cdot(-1)\cdot(1)\cdots(2n-5)]}{2^{2n}\, n!} x^{2n}\right)}$$ WITH $a_0 = 1$

WRITING OUT A FEW TERMS

$$y_1(x) = 1 + 3x^2/4 + 1\cdot 3x^4/2^4\cdot(1\cdot 2) - 1\cdot 3x^6/2^6(1\cdot 2\cdot 3) + 1\cdot 3\cdot 3x^8/2^8(1\cdot 2\cdot 3\cdot$$

$+\ldots$

ii) THE THEOREM IN BOYCE AND DIPRIMA ASSERTS THAT y_1 AND y_2 HAVE RADII OF CONVERGENCE AT LEAST AS LARGE AS THE DISTANCE FROM $x=0$ TO THE ZERO OF $x^2+2=0$, i.e. $x=\pm i\sqrt{2}$.

THUS THE RADIUS OF CONVERGENCE FOR y_1 IS $\geq \sqrt{2}$.

THE RADIUS OF CONVERGENCE FOR y_2 IS INFINITE SINCE IT IS A POLYNOMIAL.

NOW FOR y_1 WE USE THE RATIO TEST. WE CALCULATE

NEED
$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |x|^2 \lim_{n\to\infty}\left|\frac{-3(-1)\cdots(2n-5)(2n-3)}{2^{2(n+1)}(n+1)!}\right|\left|\frac{n!\,2^{2n}}{1\cdot 3\cdots(2n-5)}\right|$$

$$= |x|^2 \lim_{n\to\infty}\frac{(2n-3)}{(n+1)4} = \frac{2|x|^2}{4} < 1.$$

THUS $|x|<\sqrt{2}$. SO RADIUS OF CONVERGENCE FOR y_1 IS $\sqrt{2}$.

(iii) WE WANT $y(0)=1$ AND $y'(0)=2$.

THUS $y = c_1 y_1 + c_2 y_2$ WITH $y_1(0)=1,\ y_1'(0)=0$

$y_2(0)=0,\ y_2'(0)=1$.

THUS $c_1=1,\ c_2=2$.

PROBLEM 2

$$x^2 y'' + x y' + (x^2 - 1/4) y = 0$$

(i) $P = x^2$, $Q = x$, $R = x^2 - 1/4$. SO $\lim_{x\to 0} \frac{xQ}{P} = 1$, $\lim_{x\to 0} \frac{x^2 R}{P} = -1/4$

SINCE $P(0) = 0$, $R(0) \neq 0$ AND THE TWO LIMITS ARE FINITE THEN $x = 0$ IS A REGULAR SINGULAR POINT.

(ii) FROM (i) THE INDICIAL EQUATION IS $r(r-1) + r - 1/4 = r^2 - 1/4 = 0$ SO $r = \pm 1/2$.

THUS $\exists$ SOLUTION OF FORM

$$y = x^{1/2} A_1 [1 + b_1 x + ..] + x^{-1/2} A_2 [1 + c_1 x + ..].$$

NOW $\lim_{x\to 0^+} y = 0 \to A_2 = 0.$

NOW $\lim_{x\to 0^+} x^{1/2} y' = \lim_{x\to 0^+} x^{1/2} [A_1 \frac{1}{2} x^{-1/2} + ..] = A_1/2 = 2.$ THUS $A_1 = 4.$

IN CONCLUSION, ALL WE NEED TO DO IS SET $r = 1/2$ AND LOOK FOR A SOLUTION IN THE FORM $y = x^{1/2}[a_0 + a_1 x + ..] = \sum_{n=0}^{\infty} a_n x^{n+r}$ WITH $r = 1/2$,

AND $a_0 = 4$.

NOW SUBSTITUTE $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ INTO $x^2 y'' + x y' + (x^2 - 1/4) y = 0.$

THIS GIVES

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \frac{1}{4} \sum_{n=0}^{\infty} x^{n+r} a_n = 0.$$

THUS,
$$\sum_{n=0}^{\infty} \left(a_n (n+r)^2 x^{n+r} - \frac{1}{4} a_n x^{n+r} \right) + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

COMBINING:
$$x^r [a_0 (r^2 - 1/4)] + x^{r+1} \left[a_1 \left((r+1)^2 - \frac{1}{4} \right) \right]$$
$$+ \sum_{n=2}^{\infty} \left[\left((n+r)^2 - \frac{1}{4} \right) a_n + a_{n-2} \right] x^{n+r} = 0.$$

WE GET $r = \pm 1/2 \rightarrow$ WANT $r = 1/2$.

ALSO $a_1 = 0$ AND $$a_n = \frac{-a_{n-2}}{\left[(n+r)^2 - 1/4\right]}, \quad n \geq 2.$$

FOR $r = 1/2$, WE HAVE $(n + 1/2)^2 - 1/4 = n^2 + n = n(n+1)$

$$a_n = \frac{-a_{n-2}}{n(n+1)}, \quad n \geq 2.$$

WITH $a_0 = 4$, $a_1 = 0$.

SO $a_3 = a_5 = \dots = 0$.

$$a_2 = -\frac{a_0}{2\cdot 3}, \quad a_4 = -\frac{a_2}{4\cdot 5} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \quad a_6 = -\frac{a_4}{6\cdot 7} = -\frac{a_0}{7!}$$

THUS $a_{2m} = \dfrac{(-1)^m a_0}{(2m+1)!}$. $\quad y = x^{1/2} \displaystyle\sum_{m=0}^{\infty} \frac{a_0 x^{2m}}{(2m+1)!}(-1)^m$

THIS GIVES WITH $a_0 = 4$,

$$y = 4x^{1/2} \sum_{m=0}^{\infty} \frac{x^{2m}(-1)^m}{(2m+1)!}$$

(iii) WE CAN WRITE ABOVE AS $y = \dfrac{4x^{1/2}}{x} \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$.

WE RECOGNIZE THE SUM AS SIN X SO

$$y = \frac{4}{x^{1/2}} \sin x.$$

PROBLEM 3

$$u_t = u_{xx} - u, \quad 0 \le x \le L, \quad t > 0$$

$$u_x = 0 \text{ AT } x = 0, L \ ; \qquad u(x,0) = x \quad \text{ON} \quad 0 \le x \le L.$$

(i) $\quad u = T(t)\,\Phi(x)$ SO $\qquad T'\Phi = T\Phi'' - T\Phi \longrightarrow \dfrac{T' + T}{T} = \dfrac{\Phi''}{\Phi} = -\lambda.$

THUS $\quad \Phi'' + \lambda\Phi = 0 \qquad\qquad T' = -(1+\lambda)T \longrightarrow T = e^{-(1+\lambda)t}.$

$$\Phi'(0) = \Phi'(L) = 0$$

NOW IF $\quad \lambda_0 = 0, \quad \Phi_0 = 1 \qquad\qquad T_0(t) = e^{-t}$

$$\lambda_n = n^2\pi^2/L^2, \quad \Phi_n(x) = \cos\left(\frac{n\pi x}{L}\right). \qquad T_n(t) = e^{-(1+n^2\pi^2/L^2)t}, \; n = 1, 2, \ldots$$

THUS
$$u(x,t) = e^{-t}\left[c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)\right].$$

NOW $\quad u(x,0) = x = c_0 + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)c_n. \qquad c_0 = \dfrac{1}{L}\displaystyle\int_0^L x\,dx = L/2.$

AND
$$c_n\, L/2 = \int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{FOR } n \ge 1.$$

$$u = x \quad du = dx \qquad dv = \cos\left(\frac{n\pi x}{L}\right)dx \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$c_n\, L/2 = \underbrace{\frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L}_{=0} - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

$$c_n\, L/2 = +\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = \left(\frac{L}{n\pi}\right)^2\left[(-1)^n - 1\right] = \begin{cases} \dfrac{-2L^2}{n^2\pi^2} & n \text{ ODD} \\ 0 & n \text{ EVEN} \end{cases}$$

$$u(x,t) = e^{-t}\left[\frac{L}{2} + \sum_{\substack{n=1 \\ n \text{ ODD}}}^{\infty}\left(\frac{-4L}{n^2\pi^2}\right)e^{-n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)\right]$$

SO
$$u(x,t) = e^{-t}\left[\frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}e^{-(2m-1)^2\pi^2 t/L^2}\cos\left(\frac{(2m-1)\pi x}{L}\right)\right].$$

$(*)$

(ii) CLEARLY $u \to 0$ AS $t \to \infty$ FROM (*)

(iii) INTEGRATE THE PDE

$$\int_0^L u_t \, dx = \int_0^L u_{xx} \, dx - \int_0^L u \, dx$$

$$\frac{d}{dt}\left(\int_0^L u \, dx\right) = \underbrace{u_x \Big|_0^L}_{=0} - \left(\int_0^L u \, dx\right)$$

DEFINE $M(t) = \int_0^L u \, dx$ SO $M'(t) = -M(t)$

$$\to M(t) = M(0) e^{-t}$$

WITH $M(0) = \int_0^L u(x,0) \, dx = \int_0^L x \, dx = \frac{L^2}{2}$.

THUS $M(t) = L^2/_2 \, e^{-t}$.