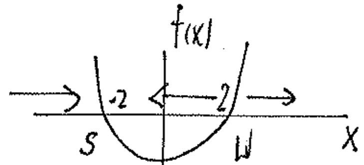
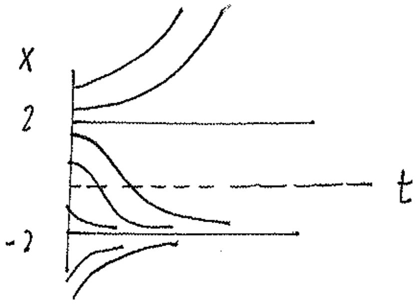
MATH 345 HW # 1

PROBLEM 2.2.1

$\dot{x} = f(x) = 4x^2 - 16$





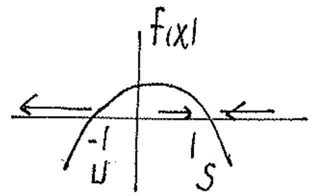
$f(x) = 0$ AT $x = \pm 2$

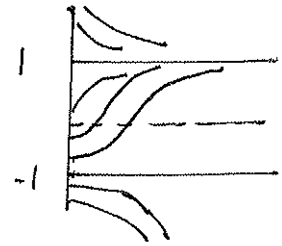
$x = 2$ ~~U~~UNSTABLE $x = -2$ STABLE.

PROBLEM 2.2.2

$\dot{x} = 1 - x^{14} = f(x)$

NOW $f = 0$ AT $x = \pm 1$

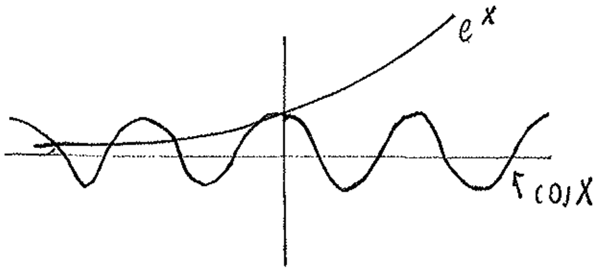


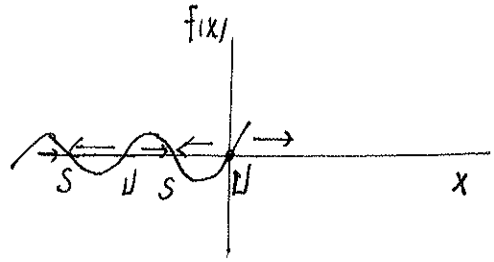


$x = 1$ IS STABLE, $x = -1$ UNSTABLE.

* PROBLEM 2.2.7

$\dot{x} = f(x) = e^x - \cos x$



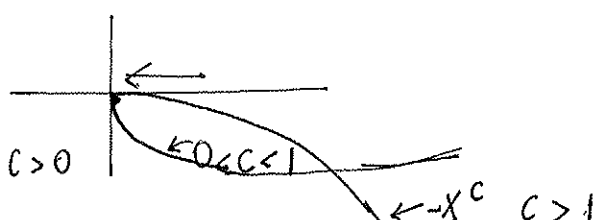


$\exists$ INFINITE # OF EQUILIBRIAIN REGION $x \leq 0$. THEY ALTERNATE FROM UNSTABLE TO STABLE. LET x_k BE THE EQUILIBRIA. THEN FOR $|k| >> 1$ WE HAVE $x_k \approx \frac{\pi}{2}(2k+1)$ FOR k large Negative integer.

PROBLEM 2.5.1

CONSIDER $\dot{x} = -x^c$ $x \geq 0$ c CONSTANT.

(i) THEN WE PLOT $f(x) = -x^c$ FOR $x \geq 0$



SO STABLE ON $x \geq 0$ FOR ANY $c > 0$

i.e. $\lim_{t \to \infty} x(t) = 0$ FOR ANY $x(0) > 0$ WHEN $c > 0$

(ii) $x^{-c}\, dx = -dt$ WITH $x(0) = x_0$. WE INTEGRATE $\frac{x^{1-c}}{1-c} - \frac{x_0^{1-c}}{1-c} = t$

NOW $\lim_{x \to 0} x^{1-c} = 0$ when $0 < c < 1$ $\longrightarrow$ $x(T) = 0$ AT $T = \frac{x_0^{1-c}}{1-c} < \infty$.

$\longrightarrow$ FINITE-TIME TOUCHDOWN WHEN $0 < c < 1$

PROBLEM 2.7.1

$$\dot{X} = X(1-X) = f(X)$$

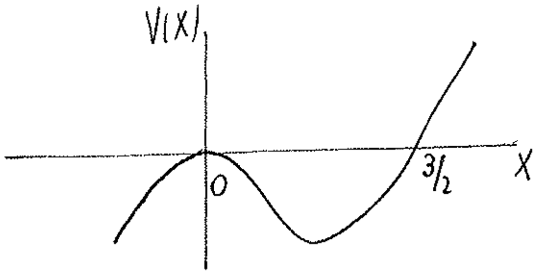
NOW $V'(X) = -X(1-X) = -f(X)$

SO $V(X) = -\dfrac{X^2}{2} + \dfrac{X^3}{3} = -\dfrac{X^2}{2}\left(1 - 2X/3\right)$

NOW $V' = 0$ AT $X = 0, 1 \longrightarrow$ EQUILIBRIA

$V''(0) = -1 < 0 \rightarrow$ MAX (UNSTABLE) $\longrightarrow$ $V(0) = 0$

$V''(1) = 1 > 0 \rightarrow$ MINIMUM (STABLE) $\longrightarrow$ $V(1) = -1/6$



* PROBLEM 2.7.6

$$\dot{X} = r + X - X^3 = f(X) \qquad V'(X) = -f(X)$$

THE EQUILIBRIUM POINTS ARE $r = -X + X^3$

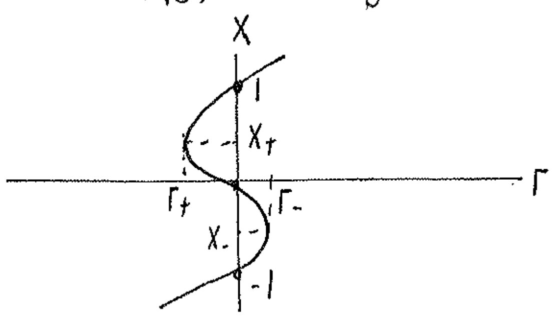
SO $\dfrac{dr}{dx} = -1 + 3X^2$ $\qquad \dfrac{dr}{dX} = 0$ WHEN $X_\pm = \pm 1/\sqrt{3}$

$r_\pm = \mp 1/\sqrt{3} \pm 1/3^{3/2}$

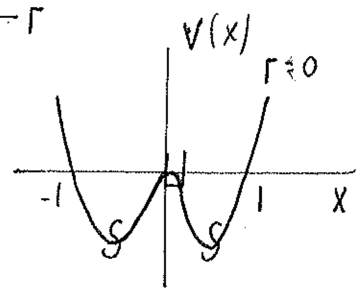
$r_\pm = \mp 2/3^{3/2}$

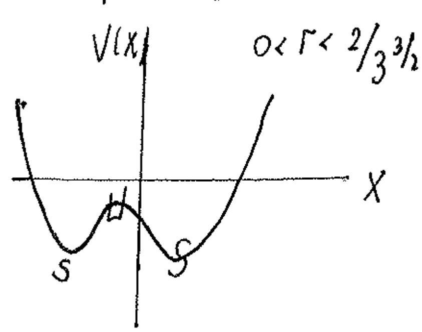
NOW $X_+ = 1/\sqrt{3}$, $r_+ = -2/3^{3/2}$

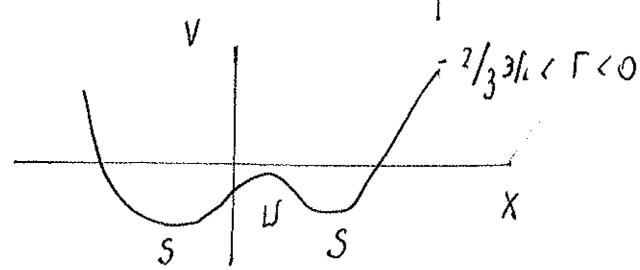
$X_- = -1/\sqrt{3}$, $r_- = 2/3^{3/2}$



NOW $V = X^4/4 - X^2/2 - rX$







* PROBLEM 2.2.13

(i) $m\dot{V} = mg - KV^2 \qquad V(0)=0$

let $V_e^2 = mg/K$ THEN $\dfrac{m}{K}\dot{V} = V_e^2 - V^2$

$$\frac{dV}{V_e^2 - V^2} = \frac{K}{m}\,dt \longrightarrow \frac{1}{2V_e}\left(\frac{1}{V_e - V} + \frac{1}{V_e + V}\right)dV = \frac{K}{m}\,dt$$

THEN $\dfrac{1}{2V_e}\Big(\log(V_e + V) - \log(V_e - V)\Big) = \dfrac{K}{m}t$ which satisfies $V(0)=0$

NOW $\log\left(\dfrac{V_e + V}{V_e - V}\right) = \dfrac{2V_e K}{m}t$

SO $\dfrac{V_e + V}{V_e - V} = e^{Bt} \qquad B \equiv \dfrac{2V_e K}{m} = 2\sqrt{\dfrac{m}{K}}\sqrt{g}\,\dfrac{K}{m} = 2\sqrt{\dfrac{Kg}{m}}.$

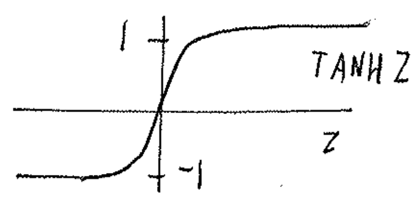
THEN $(V_e + V) = e^{Bt}(V_e - V) \longrightarrow V(1 + e^{Bt}) = V_e(e^{Bt} - 1)$

SO $V = V_e\dfrac{(e^{Bt/2} - e^{-Bt/2})}{(e^{Bt/2} + e^{-Bt/2})} = V_e\,\mathrm{TANH}(B\,t/2)$

$B/2 = \sqrt{gK/m}$

THUS, $V = V_e\,\mathrm{TANH}\left(\sqrt{\dfrac{Kg}{m}}\,t\right)$

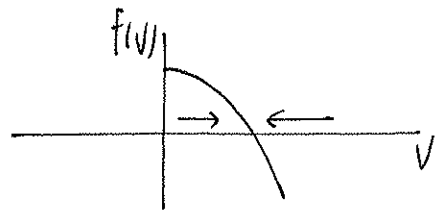
SO $V = V_e\,\mathrm{TANH}\left(\sqrt{\dfrac{Kg}{m}}\,t\right)$



(ii) NOW $\lim\limits_{t\to\infty} V(t) = V_e = \sqrt{mg/K}$ terminal velocity

(iii) $m\dot{V} = mg - KV^2 = f(V)$

$f = 0$ when $V_e = \sqrt{mg/K}$



SO $\lim\limits_{t\to\infty} V(t) = V_e$ FOR ANY $V(0) > 0$

(iv) $V_{age} = \dfrac{31{,}400 - 2{,}100}{116}$ (ft/SEC) $\approx$ 252.6 ft/SEC

(V) NOW $V = V_e \tanh(\beta t)$ $\quad \beta = \sqrt{\dfrac{kg}{m}}$

NOW $s' = V_e \tanh(\beta t)$

SO $$s(t) = V_e \int_0^t \tanh(\beta \lambda)\, d\lambda = V_e \int_0^t \frac{\sinh(\beta\lambda)}{\cosh(\beta\lambda)}\, d\lambda$$

$$s(t) = \frac{V_e}{\beta} \log\left[\cosh(\beta t)\right]$$

NOW $$\frac{V_e}{\beta} = \frac{\sqrt{mg/k}}{\sqrt{kg/m}} = \frac{\sqrt{mg/k}}{g\sqrt{\frac{k}{mg}}} = \frac{V_e^2}{g}$$

SO $$s(t) = \frac{V_e^2}{g} \log\left[\cosh\left(gt/V_e\right)\right]$$

WE WANT TO FIND V_e given $s = 29{,}300$, $t = 116$ AND $g = 32.2$

ASSUMING $V = V_{avge}$. NOW $$\frac{gt}{V_{ave}} \sim \frac{(32.2)(116)}{252.6} \approx 14.787 \gg 1$$

SO $\cosh(z) = \dfrac{e^z + e^{-z}}{2} \sim e^z/2$ FOR $z \gg 1$ YIELDS THAT

$$s \sim \frac{V_e^2}{2} \log\left[\cosh\left(gt/V_e\right)\right] \sim \frac{V_e^2}{g}\left(\frac{gt}{V_e} - \log 2\right)$$

so $$s \sim V_e t - \frac{V_e^2}{g} \ln 2 .$$

USING $s = 29{,}300$, $t = 116$, $g = 32.2$ we get quadratic equation FOR V_e which has the solution

$$V_e \approx 265.7$$

then $\sqrt{\dfrac{mg}{k}} = V_e \longrightarrow k = .12$ IN ft-lb.

* PROBLEM 2.3.3

$\dot{N} = -aN\log(bN) \qquad a>0,\ b>0$

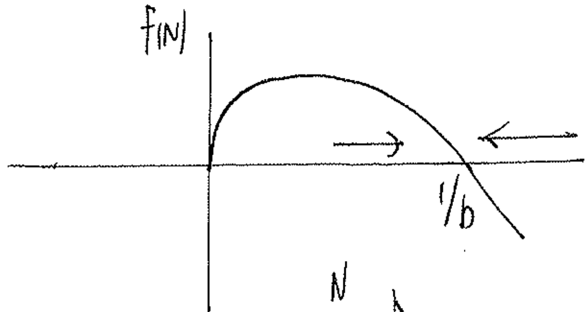
a) a has units of $1/\text{TIME}$. SO ITS A characteristic Rate OF TUMOUR GROWTH WITH $1/a$ BEING A characteristic time scale. Also b has units of $1/\#\text{ cells}$, so $1/b$ is a characteristic size of the tumour.

b) let $\dot{N} = f(N) \qquad f(N) := -aN\log(bN)$

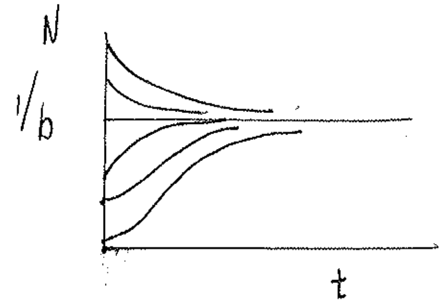
NOW SINCE $\lim_{x\to 0} x\log x = 0$ THEN $\lim_{x\to 0^+} f(x) = 0$

TOGETHER WITH $f(1/b) = 0$. ALSO $f>0$ ON $0<N<1/b$.

THEREFORE, WE GET THE PICTURE



NOTICE $f'(N) \to +\infty$ AS $N\to 0$.



PROBLEM 2.4.2

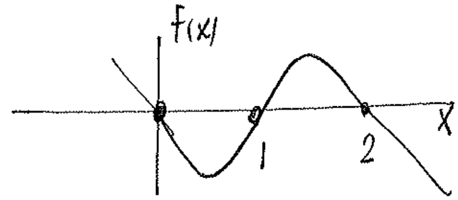
$$\mathring{X} = X(1-X)(2-X) = f(X)$$

EQUILIBRIA AT $X=0,1,2$. NOW $f'(X) = (1-X)(2-X) - X(2-X) - X(1-X)$

SO $f'(0) = 2 > 0 \rightarrow$ unstable

$f'(1) = -1 < 0 \rightarrow$ stable

$f'(2) = 2 > 0 \rightarrow$ unstable



* PROBLEM 2.3.4

(i) $\frac{\mathring{N}}{N} = r - a(N-b)^2 = g(N)$

$g'(N) = -2a(N-b) \rightarrow g' = 0$ when $N = b$.

NOW WANT $g(0) = r - ab^2 < 0$

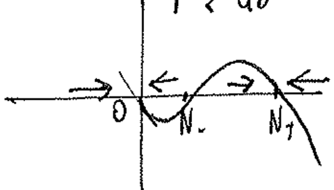
$\Longrightarrow r < ab^2 \Longrightarrow$ stability of zero solution (NO Allee effect).

WE EXPECT THE CONDITION IS $r > ab^2$.

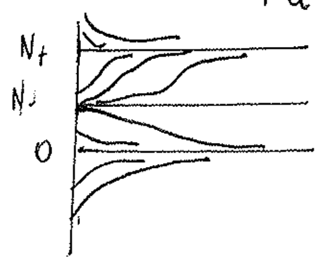
(ii), (iii) WE LET $\mathring{N} = f(N)$ $\quad f(N) = N(r - a(N-b)^2)$.

THEN $f = 0$ when $N = 0$, $N_\pm = b \pm \sqrt{\frac{r}{a}}$

$r < ab^2$

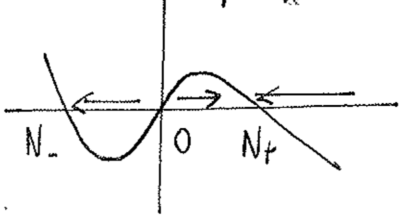


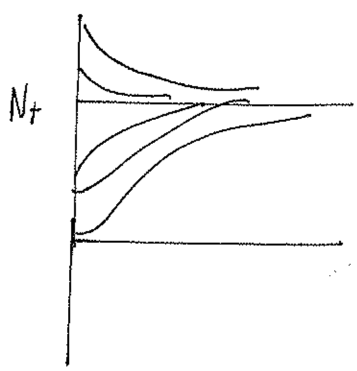
$\Longrightarrow$



$r < ab^2$

$r > ab^2$

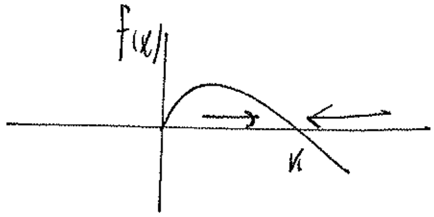




$N_- < 0 \longrightarrow$ unphysical since N = population density.

(iv) THE LOGISTIC EQUATION IS

$$x' = rx(1 - x/K)$$



IT IS similar to THE DYNAMICS ABOVE ONLY when $r > ab^2$.