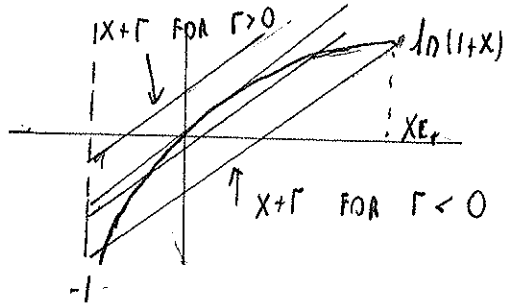
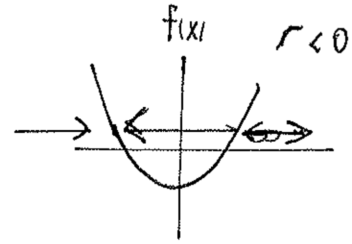
MATH 345 HW #2

PROBLEM 3.13

$x' = r + x - \ln(1+x) = f(x)$

NOW WE SET $f = 0$ TO GET $x + r = \ln(1+x)$





NOTICE THAT FOR $r<0 \Longrightarrow \exists$ TWO ROOTS x_-, x_+
FOR $r>0 \Longrightarrow$ there are NO ROOTS. $\Big\}$ SADDLE- NODE

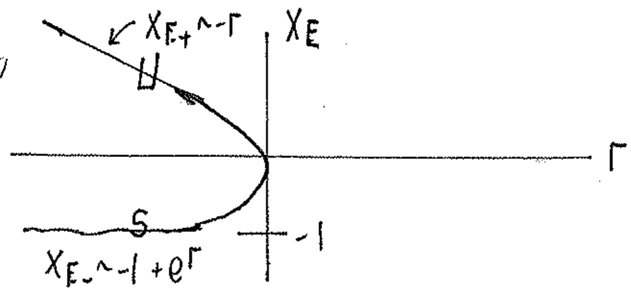
SET $f = f_x = 0$ TO GET BIFURCATION POINT

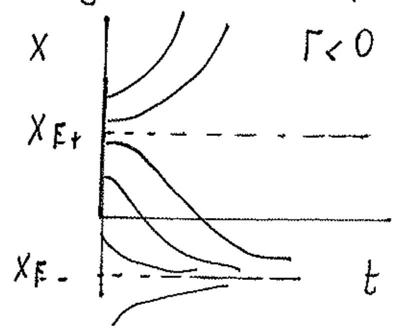
$$\left.\begin{array}{l} r + x - \ln(1+x) = 0 \\ 1 - \dfrac{1}{x+1} = 0 \end{array}\right\} \Longrightarrow x = 0,\ r = 0.$$

NOW FOR $r \to -\infty$, $x_{E_+} \sim -r$

$r \to -\infty$ $\quad x_{E_-} \sim -1 + e^{r}$ (SOLVING $r \sim \ln(1+x)$)

THERFFORE,



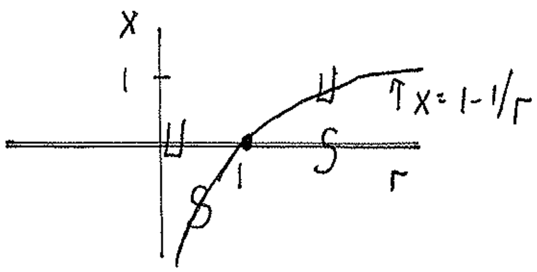


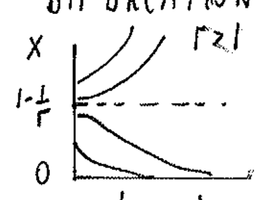
PROBLEM 3.2.3

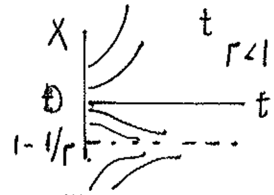
$\dot{x} = x - rx(1-x) = x\left[1 - r(1-x)\right] = f(x,r).$

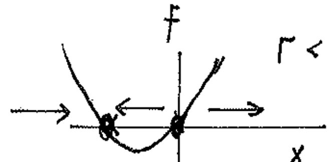
NOW $x=0$ IS AN EQUILIBRIA AND $1 - x = 1/r$ OR $x = 1 - 1/r$.

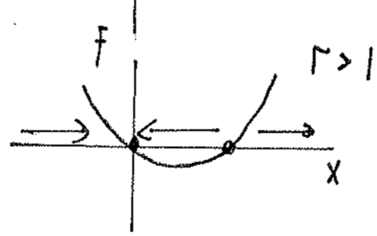
THUS $r = 1$ IS A TRANSCRITICAL BIFURCATION.





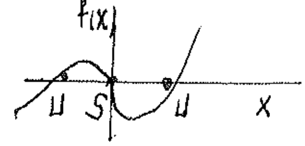






PROBLEM 3.4.4

$$x' = x + \frac{r x}{1+x^2} = f(x)$$

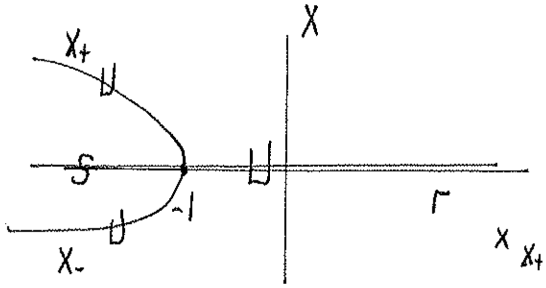


$r < -1$

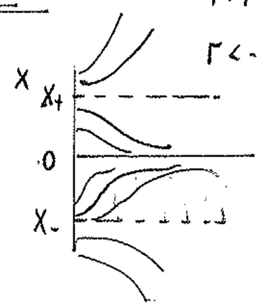
NOW $f = 0$ when $x = 0$ OR $1 + \frac{r}{1+x^2} = 0 \rightarrow 1 + x^2 = -r$

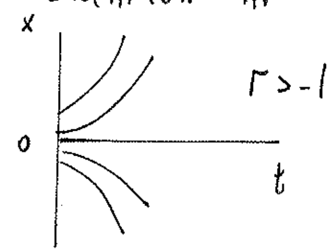
OR $x^2 = -r - 1$

OR $x_{\pm} = \pm\sqrt{-r-1}$ FOR $r < -1$



PITCHFORK BIFURCATION AT $r = -1$, $x = 0$



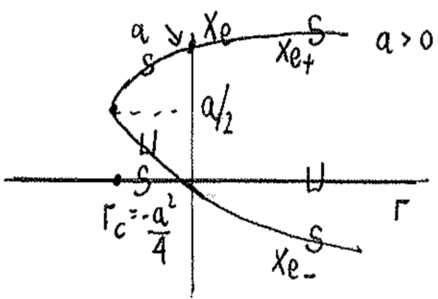


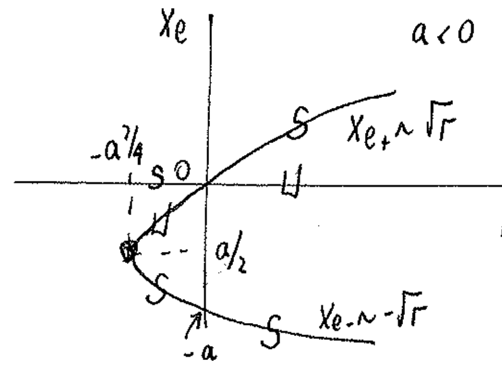
PROBLEM 3.6.3

$$x' = x(r + ax - x^2) = f(x), \quad -\infty < a < \infty, \ |r| < \infty.$$

EQUILIBRIA AT $x_e = 0$ OR $x^2 - ax - r = 0 \rightarrow x_{e\pm} = \frac{a \pm \sqrt{a^2 + 4r}}{2}$

WE WILL PLOT x_e VERSUS r. WE GET TWO SIMPLE PLOTS





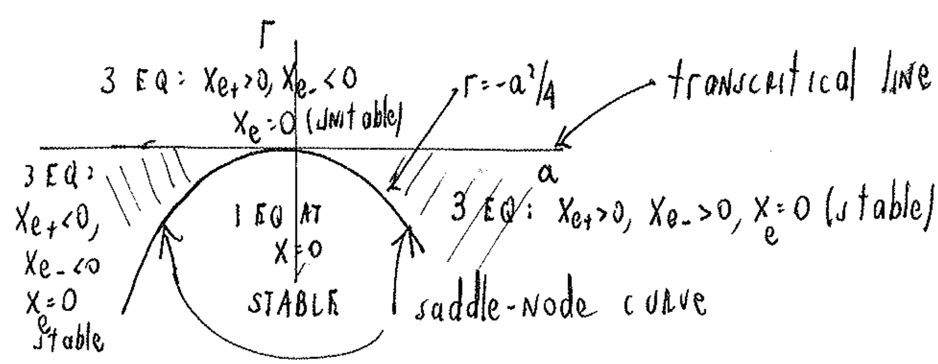
NOTICE $x_{e+} = a$ when $r = 0$

$x_{e-} = 0$ when $r = 0$

$x_{e\pm} \sim \pm\sqrt{r}$, $r \rightarrow \infty$. $r_c = -a^2/4$

SADDLE-NODE BIF AT $r_c = -\frac{a^2}{4}$ FOR ANY $a > 0$ OR $a < 0$

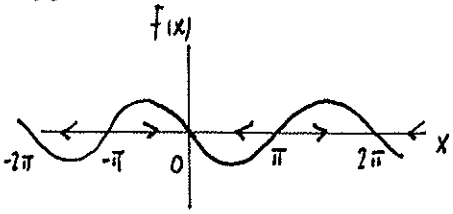
TRANSCRITICAL BIFURCATION AT $r = 0$ FOR ANY a.



PROBLEM 3.4.11

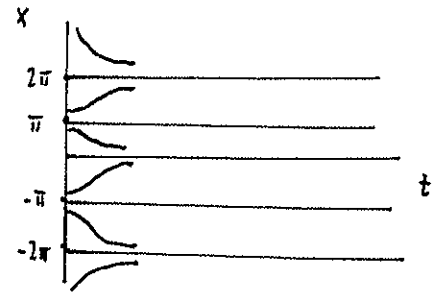
$$\frac{dx}{dt} = rx - \sin x = f(x)$$

(i) IF $r = 0$

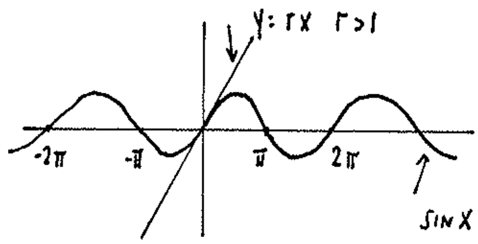


$x_E = 2k\pi$ stable k integer

$x_E = (2k+1)\pi$ unstable



(ii) WE graph rx VERSUS $\sin x$ WITH $r > 1$.



SINCE $rx > \sin x$ ON $x > 0$ WHEN $r > 1$

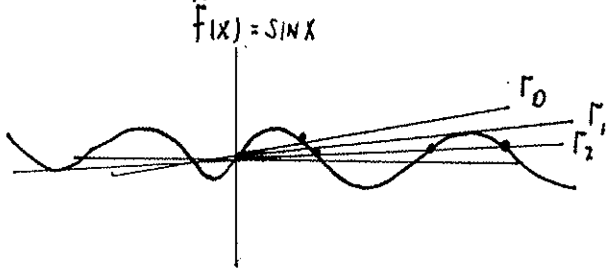
THERE IS ONLY ONE equilibrium at $x = 0$

NEAR $x = 0$, $f(x) \approx rx - (x - x^3/3!) + \ldots$

so $f(x) \approx (r-1)x + \ldots$

THUS Near $x = 0$ $\frac{dx}{dt} \approx (r-1)x$ SO IF $r > 1$, $x = 0$ IS unstable.

(iii)+iv)+v)+vi) THERE ARE NO BIFURCATIONS FOR $|r| > 1$. AS r DECREASES we get MORE ROOTS AS shown below of $f(x)$.



$r_2 < r_1 < r_0 < 1$

NOTICE AT $r = 1$ THERE IS A PITCHFORK BIFURCATION AS NEW ZEROES OF $f(x)$ ARE CREATED.

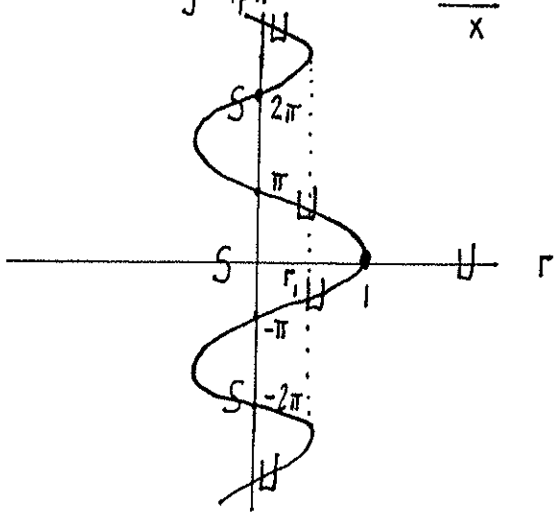
NOTICE AS r DECREASES BELOW r_1 TWO NEW ROOTS APPEAR → SADDLE NODE BIFURCATIONS

NEAR $X=0$ we get

$$\frac{dx}{dt} \simeq rx - \sin x \approx rx - \left(x - x^3/3! + \dots\right) \approx (r-1)x + x^3/6 + \dots$$

THIS IS THE NORMAL FORM FOR A SUBCRITICAL PITCHFORK BIFURCATION AT $r=1$

NOW IF we graph $r = \frac{\sin x}{x}$ we get



THUS $r=1$ PITCHFORK BIFURCATION

$r = r_K$ $K=1,2,\dots$ SADDLE NODE BIFURCATIONS.

NOW THE LOCAL MAXIMA OF $\sin X$ OCCUR AT $X_n \approx (4n+1)\frac{\pi}{2}$. $n = 0,1,2,\dots$ FOR $X \geq 0$

HENCE FOR K large the saddle node bifurcations occur at

$$r_K \approx \frac{\sin(X_K)}{X_K} \approx \frac{1}{[4K+1]\pi/2} \quad K \text{ large.}$$

FOR stability: IF $r > 0$, $f(x) \approx rx > 0$ when X is large. THUS largest equilibria is unstable and they alternate.

PROBLEM 3.7.4

$$\frac{dN}{dt} = rN\left(1 - N/K\right) - \frac{HN}{A+N}$$

A IS A SATURATION PARAMETER THAT MEASURES A TYPICAL SIZE OF THE POPULATION WHERE THE SATURATION OF HARVESTING OCCURS.

(ii) Let $N = KX$, $t = T\tau$

$$\frac{K}{T}\frac{dx}{d\tau} = rKX(1-X) - \frac{HKX}{K(a+X)} \qquad a = A/K.$$

$$\frac{1}{rT}\frac{dx}{d\tau} = X(1-X) - \frac{hX}{a+X} \qquad h = H/rK$$

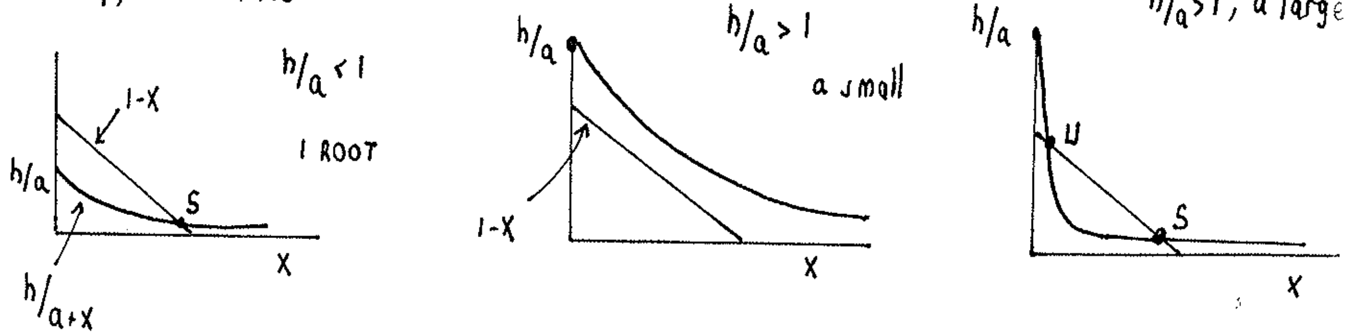
CHOOSE $T = 1/r$.

THUS $N = KX$, $t = \tau/r$ REDUCES MODEL TO

$$\frac{dx}{d\tau} = X(1-X) - \frac{hX}{a+X} \qquad h, a > 0, \quad X \geq 0.$$

(iii) THE EQ. POINTS SATISFY $X=0$ AND $1-X = h/_{a+X}$.

GRAPHically, we have



QUALITATIVely: there can either be 0, 1, OR 2 ROOTS. Exactly ONE ROOT IF $h/_a < 1$. FOR stability: $f(x) \approx -x^2 < 0$ AS $X \to \infty$. HENCE largest ROOT is always stable. Thus stability is on the diagram.

NOW NOTICE THAT SADDLE NODE bifurcation occurs at point when

$$f(X)=0 \qquad f'(X)=0$$

where $f(X) = 1 - X - \dfrac{h}{a+X}$.

THIS GIVES, $\quad -1 = -\dfrac{h}{(a+X)^2} \longrightarrow h = (a+X)^2 \quad$ AND $\quad 1-X = \dfrac{h}{a+X} = a+X$

THIS gives $X = \dfrac{(1-a)}{2}$ AND THUS $h = \left(a + \dfrac{(1-a)}{2}\right)^2 = \dfrac{1}{4}(a+1)^2$.

THUS $h = h_c = \dfrac{1}{4}(a+1)^2$ is saddle node bifurcation value.

NOTICE THAT FOR $X>0$ we need $a<1$.

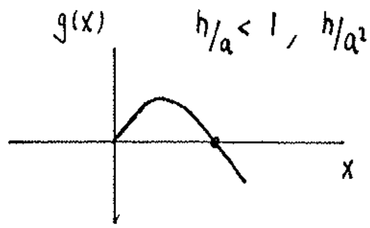
THUS we only need the RANGE $a<1$ FOR saddle-node bifurcations WITH $X>0$. WHEN $h < h_c$ AND $h>a$ we have $X_{EQ} = \dfrac{1-a \pm \sqrt{(a+1)^2 - 4h}}{2} \geq 0$.

(iv) NEAR $X=0$ WE HAVE BY LINEarization.

$$\frac{dx}{dt} = x - x^2 - \frac{hx}{a}(1 - x/_a) + \dots \approx x\left(1 - \frac{h}{a}\right) + x^2\left(\frac{h}{a^2} - 1\right) + \dots = g(x)$$

HENCE FOR $\quad \dfrac{h}{a} < 1 \longrightarrow X=0$ is UNstable

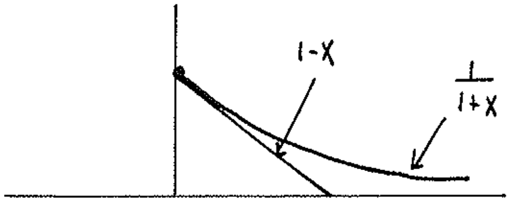
$\dfrac{h}{a} > 1 \longrightarrow X=0$ is stable.



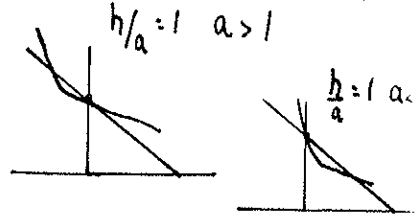
$h/_a < 1$, $h/_{a^2} < 1$: we get a NEW ROOT emerging when $h/_a < 1$ AND $h/_{a^2} < 1$. $\longrightarrow$ ~~SADDLE NODE~~ transcritical bifurcation

when $\dfrac{h}{a} = 1$ AND $\dfrac{h}{a^2} = 1 \longrightarrow a = 1$.

NOTICE THAT WHEN $\frac{h}{a} = 1$ AND $a = 1$ WE HAVE



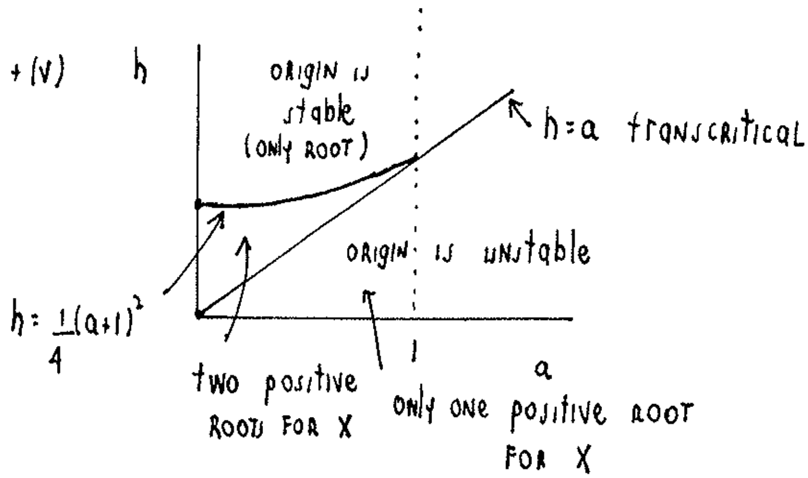
tangency at $X = 0$.



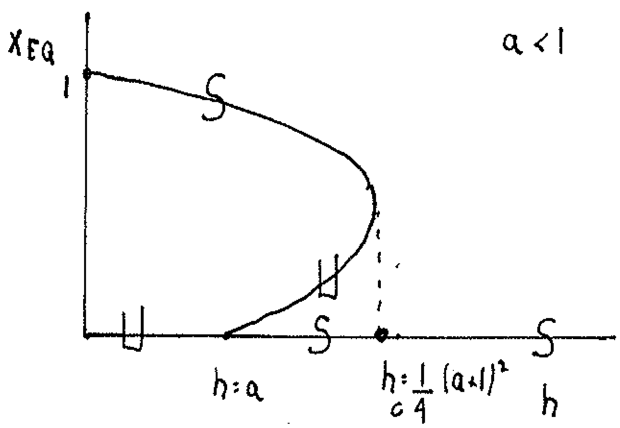
IF WE INCREASE $a > 1$ AND Keep $h/a = 1$ we have a ROOT at $X = 0$ BUT Another one at $X < 0$ which is unphysical. transcritical at $h = a = 1$

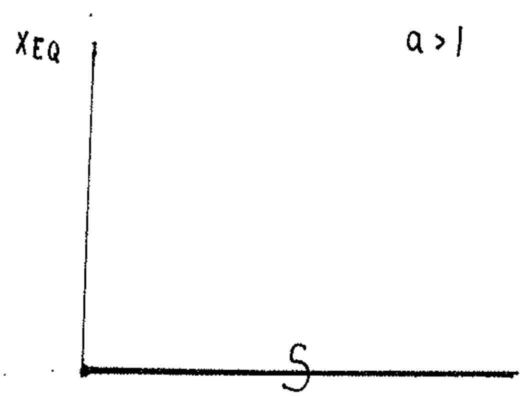
(iii) AS FAR AS THE OTHER BIFURCATION EXPLAINED ABOVE, WE HAVE A SADDLE NODE BIFURCATION AT $h = h_c = \frac{1}{4}(a+1)^2$ AND we need $a < 1$ AS EXPLAINED above for there to be ROOTS WITH $X > 0$.

(iv) + (v)

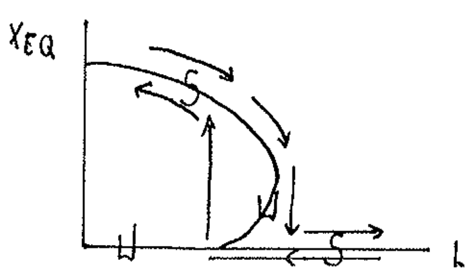


FOR $a > 1$ there are NO positive ROOTS FOR X so we do not draw anything except $X = 0$





NOW FOR STABILITY: $\frac{dx}{dt} = f(x)$ $f(x) \approx -x^2$ AS $X \to \infty$. THIS IMPLIES that the largest ROOT will be stable. HERE would be a hysteresis loop as h is decreased



PROBLEM 3.4.16

$$\dot{x} = r x + x^3 - x^5 = f(x)$$

NOW $V'(x) = -r x - x^3 + x^5$.

WE integrate $V(x) = -\frac{r x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6}$

NOTICE THAT $V(x)$ HAS CRITICAL POINTS AT $x=0$ AND WHERE $x^4 - x^2 - r = 0$. THIS YIELDS $x_{1\pm} = \left(\frac{1 \pm \sqrt{1+4r}}{2}\right)^{1/2}$ AND $x_{2\pm} = -\left(\frac{1 \pm \sqrt{1+4r}}{2}\right)^{1/2}$.

SUPPOSE $r < -1/4$. THEN $x_{1\pm}, x_{2\pm}$ COMPLEX

$-1/4 < r < 0$ THEN $x_{1\pm}, x_{2\pm}$ REAL

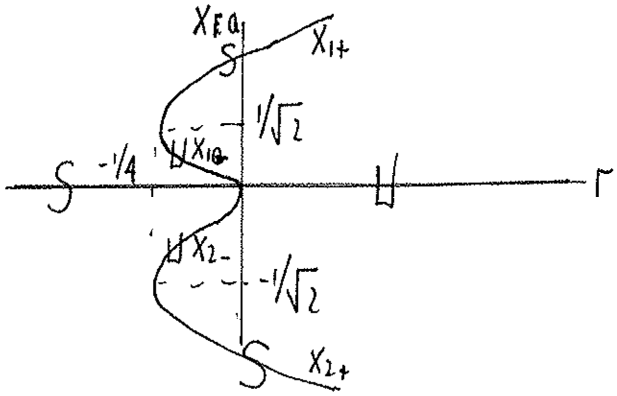
$r > 0$ ONLY x_{1+} AND x_{2+} ARE REAL.

IF $r = -1/4$ THEN

$x_{1\pm} = 1/\sqrt{2}$

$x_{2\pm} = -1/\sqrt{2}$.

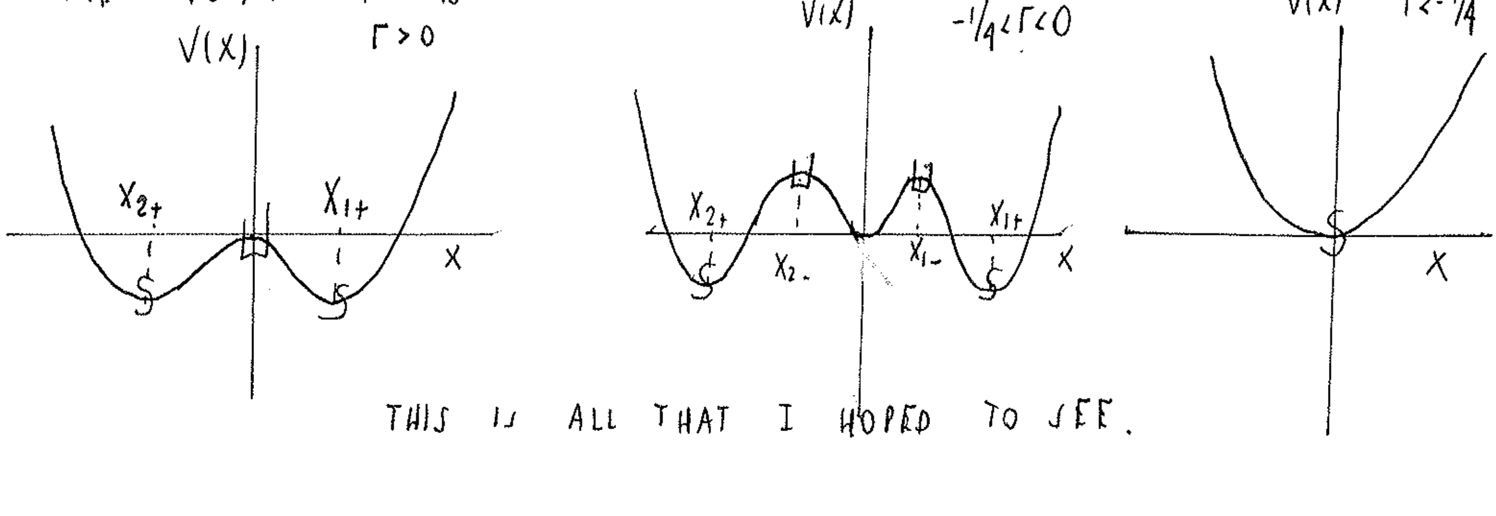
THIS LEADS TO THE BIFURCATION DIAGRAM



- STABILITY FOLLOWS EASILY BY LOOKING AT $f(x)$ FOR $|x| >> 1$ SO THAT $f(x) < 0$ FOR $x \to +\infty$
- PITCHFORK BIFURCATION AT $x=0$, $r=0$

  SADDLE-NODE BIFURCATION AT $r = -1/4$, $x = \pm 1/\sqrt{2}$.

THE POTENTIAL IS

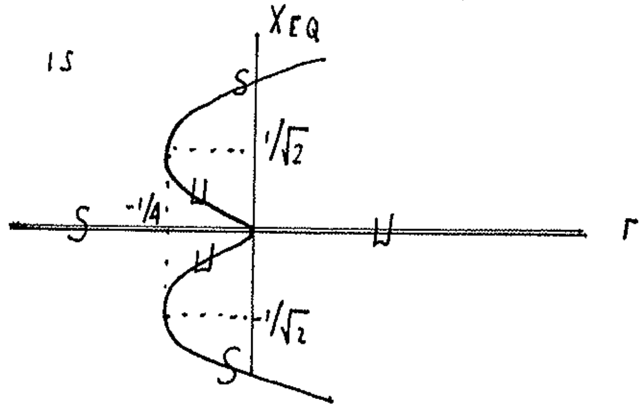


THIS IS ALL THAT I HOPED TO SEE.

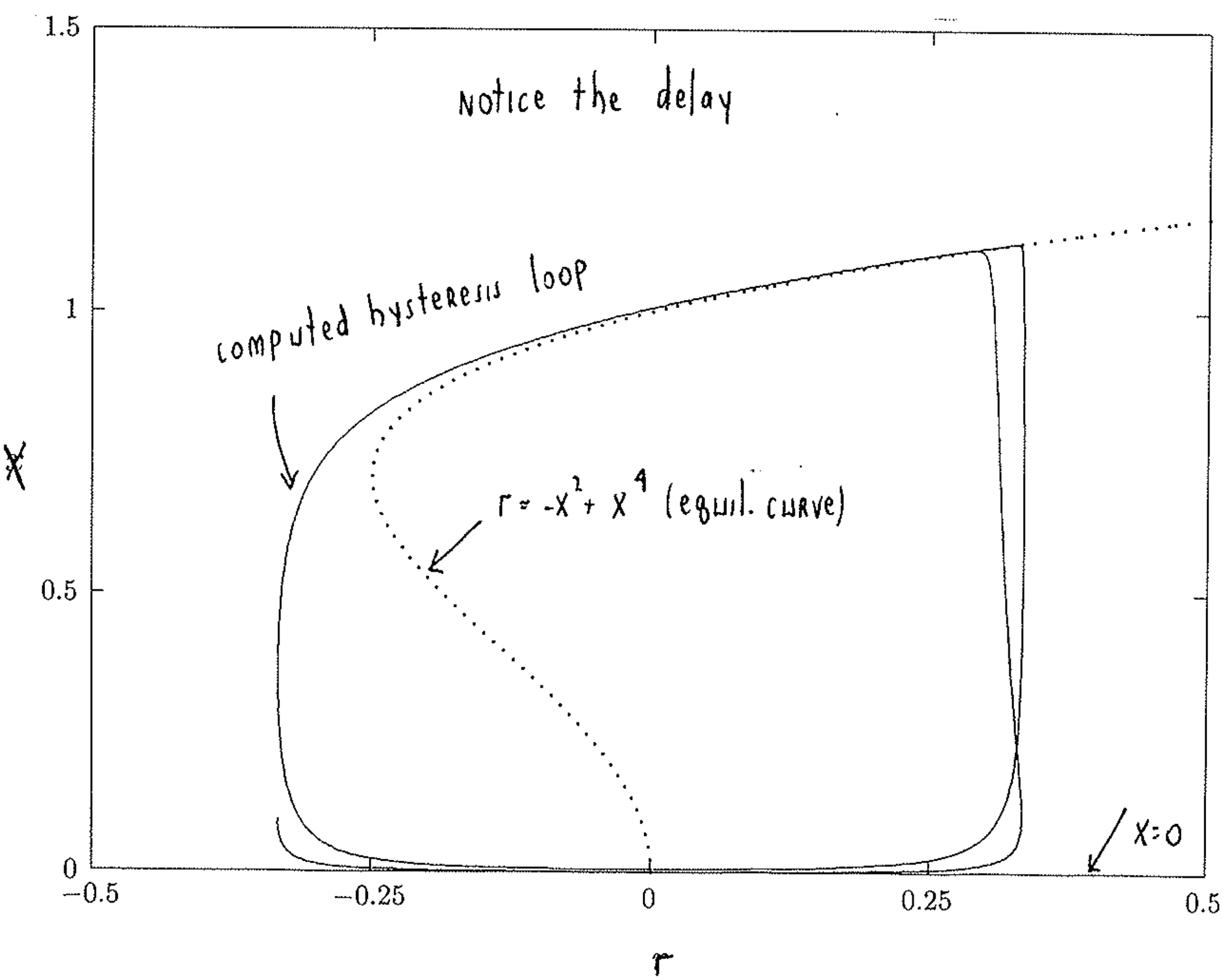
$$\frac{dx}{dt} = rx + x^3 - x^5$$ AND WE WILL NOW SHOW A HYSTERESIS CYCLE WITH $x(0) = 0.1$, $r = -\frac{1}{3}\cos(\varepsilon t)$ $\varepsilon = .075$.

The steady-state bifurcation diagram was explained above.

The picture is

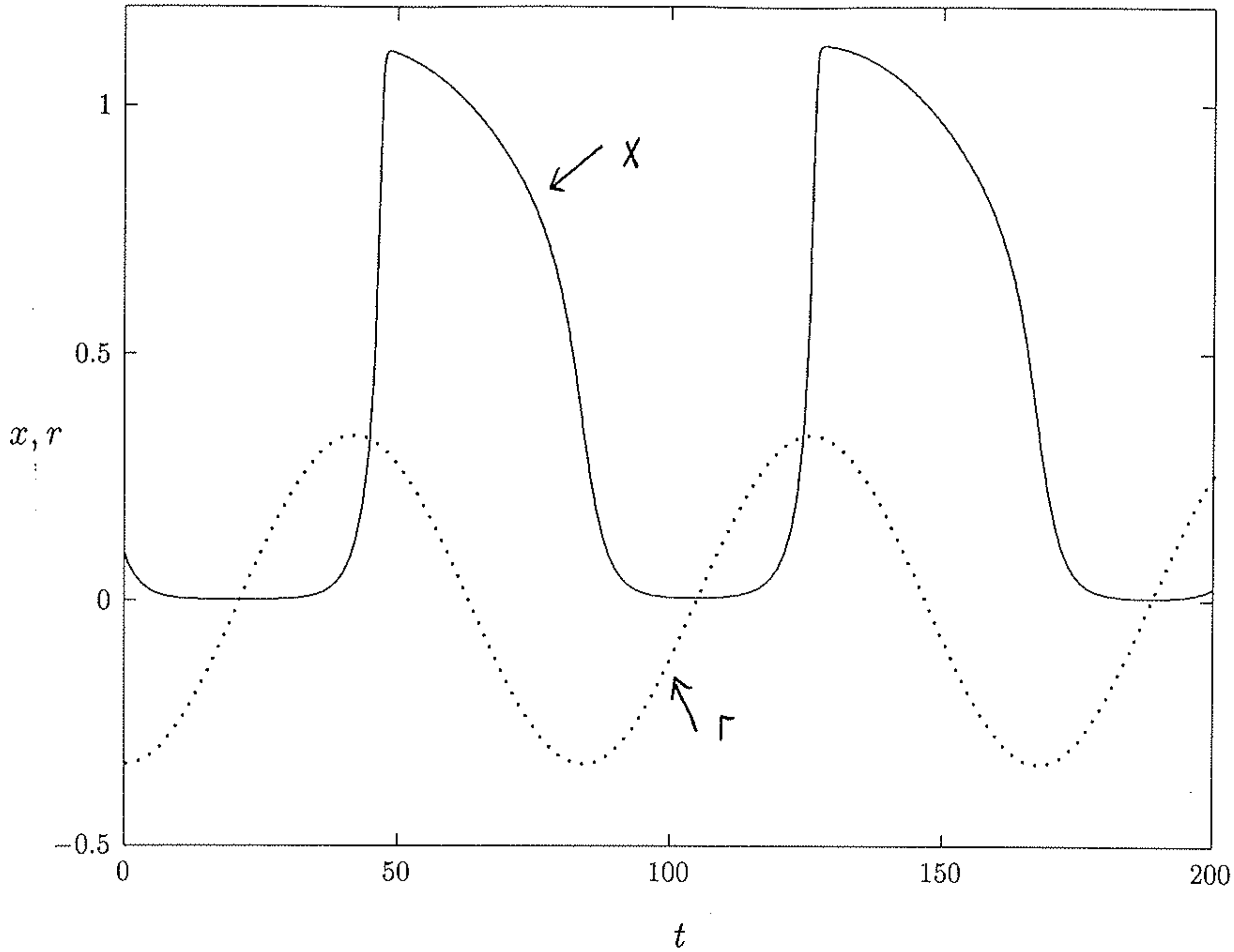


WE WILL GET A HYSTERESIS LOOP AS r IS INCREASED. IN FIG. 1 WE PLOT x VS t AND r VS t. IN FIG. 2 WE PLOT THE HYSTERESIS LOOP. THIS IS NOT FOR MARKS ONLY TO SHOW SOMETHING INTERESTING HERE.



Hysteresis cycle $\quad dx/dt = rx + x^3 - x^5$

$x(0) = 0.1 \quad r = -\frac{1}{3}\cos(\varepsilon t) \quad \varepsilon = .075$



PROBLEM 3.7.5 THE ORIGINAL SYSTEM IS

$$\frac{dg}{dt} = K_1 S_0 - K_2 g + \frac{K_3 g^2}{K_4^2 + g^2}$$

(i) WE NON-DIMENSIONALIZE BY $g = K_4 X$ AND $t = T\tau$. WE OBTAIN,

$$\frac{K_4}{T} dX/d\tau = K_1 S_0 - K_2 K_4 X + K_3 \frac{X^2}{1+X^2}$$

divide by K_3 $\quad \frac{K_4}{K_3 T} dX/d\tau = \frac{K_1 S_0}{K_3} - \frac{K_2 K_4}{K_3} X + \frac{X^2}{1+X^2}.$

CHOOSE $T = K_4/K_3$ AND IDENTIFY $S \equiv K_1 S_0/K_3$, $r = K_2 K_4/K_3$.

THEN WE OBTAIN

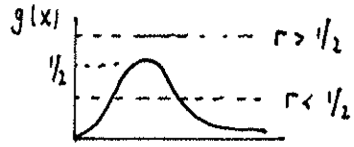
$$\frac{dX}{d\tau} = S - rX + \frac{X^2}{1+X^2}.$$

(ii) IF $S=0$ THE EQUILIBRIA ARE AT $X=0$ AND

$$r = \frac{X}{1+X^2} = g(X)$$

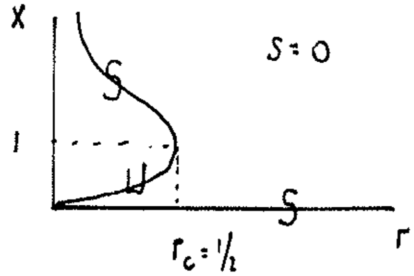
NOW $g'(X) = \dfrac{1-X^2}{(1+X^2)^2}$ $\quad g'(X)=0$ when $X = \pm 1$

AT $X=1$ $\quad r = r_c = \dfrac{1}{2}$



THUS $r = r_c$ IS A saddle-node bifurcation value.

FOR $r < r_c$ we solve for X to get $\quad X_\pm = \dfrac{1}{2r} \pm \dfrac{1}{2}\sqrt{\dfrac{1}{r^2} - 4} > 0.$



SINCE $f(X) \approx -rX$ AS $X \to \infty$

THEN $f(X) < 0$ FOR X large

$\longrightarrow$ largest ROOT is stable.

(iii) NOW SUPPOSE $S > 0$. THE EQUILIBRIA NOW SATISFY

$$S = rX - \frac{X^2}{1+X^2}$$

WE DRAW 2 PLOTS. ONE X VS r FOR $S > 0$ AND THE OTHER X VS S FOR $r > 0$.

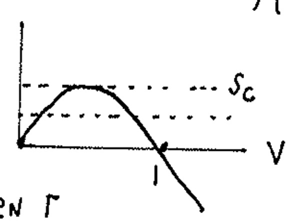
<u>X VERSUS r</u> $\qquad r = \dfrac{S}{X} + \dfrac{X}{1+X^2}$ $\qquad$ NOTICE: $r \to +\infty$ AS $X \to 0^+$

$\qquad r \approx \dfrac{S+1}{X}$ AS $X \to +\infty$

NOW $\quad \dfrac{dr}{dX} = -\dfrac{S}{X^2} + \dfrac{(1-X^2)}{(1+X^2)^2}$

$h(V) = (1-V)V/(1+V)^2$

SET $\dfrac{dr}{dx} = 0 \longrightarrow S = \dfrac{(1-V)V}{(1+V)^2}$ $\qquad V = X^2$



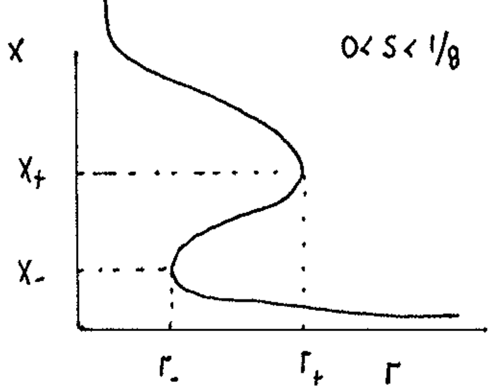
THUS FOR $0 < S < S_c$ we have multiple X for given r

$S > S_c$ we have one X for given r

SOLVE FOR V: $\qquad (1+V)^2 S = V(1-V)$

IMPLIES $\quad \left(1+\dfrac{1}{S}\right)V^2 + \left(2-\dfrac{1}{S}\right)V + 1 = 0 \longrightarrow V_\pm = \dfrac{\left(2-\frac{1}{S}\right) \pm \sqrt{\frac{1}{S^2} - \frac{8}{S}}}{2(1+1/S)}$ $\quad 0 < S < \dfrac{1}{8}$

THUS $\quad X_\pm = \sqrt{\dfrac{\left(2-\frac{1}{S}\right) \pm \sqrt{\frac{1}{S^2} - 8/S}}{2(1+1/S)}}$ $\quad$ IN $X > 0$ ARE ROOTS OF $\dfrac{dr}{dx} = 0$



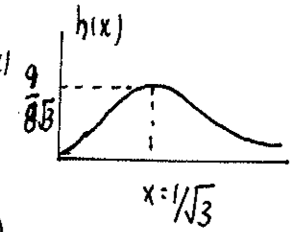
HERE $r_{\pm} = \dfrac{s}{x_{\pm}} + \dfrac{x_{\pm}}{1+x_{\pm}^2}$

WHERE $x_{\pm}$ ON PREVIOUS page.

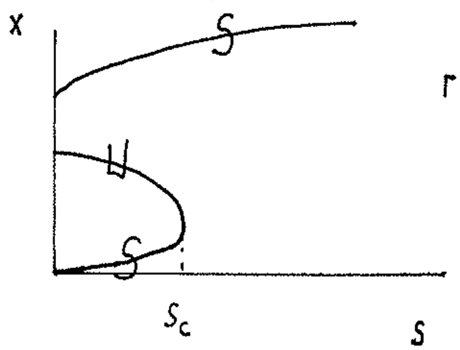
X VERSUS S $\qquad s = rx - \dfrac{x^2}{1+x^2}$ $\qquad s \approx rx$ AS $x \to 0$

$s \approx rx - 1$ AS $x \to \infty$.

NOW $\dfrac{ds}{dx} = r - \dfrac{2x}{(1+x^2)^2}$. SET $\dfrac{ds}{dx} = 0 \longrightarrow r = \dfrac{2x}{(1+x^2)^2} = h(x)$



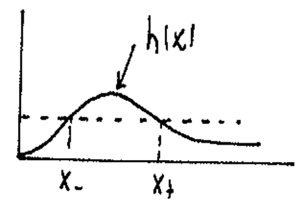
THUS we get multiple solutions IF $0 < r < 9/8\sqrt{3}$ (plot only on $x > 0$)



$r < 9/8\sqrt{3}$

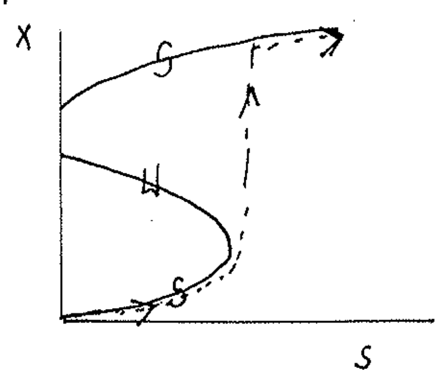
$s_c = r x_- - \dfrac{x_-^2}{1+x_-^2}$

where x_- is the smallest ROOT shown below



NOW FOR $x \to \infty$ $\quad \dfrac{dx}{dt} = f(x)$ HAS $f(x) < 0$. THUS implies largest root is stable.

NOW imagine that $s = s(\varepsilon t)$ is slowly increased FROM zero when $r < 1/2$. we will get a quick jump to a large value of x as s crosses past s_c as shown



(iv) stability boundary for multiple solutions.

RECALL FROM PREVIOUS page that $\frac{ds}{dx} \neq 0$ IF $r > \text{MAX}\left[\frac{2x}{(1+x^2)^2}\right] = r_c(x)$

NOW $s = \cancel{x}\, r_c - \frac{x^2}{1+x^2} = \frac{x^2(1-x^2)}{(1+x^2)^2}$.

THUS WE MUST PLOT THE BOUNDARY CURVE

$r = \frac{2x}{(1+x^2)^2}$ AND $s = \frac{x^2(1-x^2)}{(1+x^2)^2}$ IN REGION $0 < x < 1$.

THIS IS DONE WITH A computer and the picture is below.

(THIS PART IS OPTIONAL, AS IT WAS NOT ASSIGNED TO DO PART e)