MATH 345 HW 4 SOLUTIONS

PROBLEM 6.3.4

$$\dot{x} = y + x - x^3 = f(x,y)$$

$$\dot{y} = -y = g(x,y)$$

EQ. POINTS solve $y = x^3 - x$ AND $y = 0$ simultaneously

$\rightarrow$ $(0,0)$, $(1,0)$, $(-1,0)$ are eq. points.

NOW THE JACOBIAN $J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{pmatrix}$

AT $(0,0)$ $J = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\det(J - \lambda I) = 0$ $(1-\lambda)(-1-\lambda) = 0$

$\lambda = \pm 1$ $(0,0)$ saddle point

FOR $\lambda_1 = 1 \longrightarrow (J - I)\underline{v} = 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

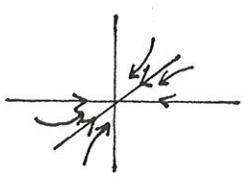
FOR $\lambda_2 = -1 \longrightarrow (J + I)\underline{v} = 0 \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

THUS $\underline{v} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \rightarrow C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}$ AS $t \to \infty$.

AT $(\pm 1, 0)$ $J = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$ $(\lambda + 2)(\lambda + 1) = 0 \rightarrow \lambda = -1, -2$

FOR $\lambda_1 = -2 \longrightarrow (J + 2I)\underline{v} = 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

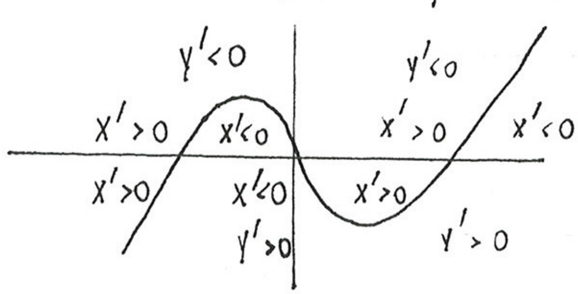
FOR $\lambda_2 = -1 \longrightarrow (J + I)\underline{v} = 0 \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \underline{v} = 0 \rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



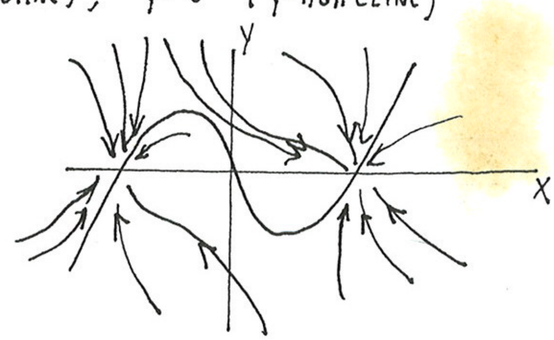
THUS $\underline{v} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \longrightarrow C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ AS $t \to \infty$

$(\pm 1, 0)$ are stable nodes

NOW PLOT THE NULLCLINES $y = -x + x^3$ (x-nullcline), $y = 0$ (y-nullcline)



$\Longrightarrow$



PROBLEM 6.3.13

$\dot{x} = -y - x^3 \quad \dot{y} = x$

linearization is $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues are $\lambda^2 + 1 = 0$

$\lambda = \pm i \rightarrow$ center.

Now convert to polar coordinates,

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$\dot{\varphi} = \frac{1}{r^2}(x\dot{y} - \dot{x}y)$$

the system becomes,

$$\dot{r} = -r^3 \cos^4\varphi$$

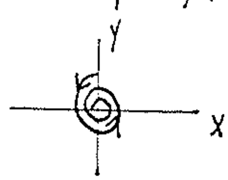
$$\dot{\varphi} = 1 + r^2 \cos^3\varphi \sin\varphi$$

Notice $\dot{r} \leq 0$ for all r (with equality only at $\varphi = \pi/2, 3\pi/2$;

$\Longrightarrow \quad r(t) \rightarrow 0$ as $t \rightarrow \infty$.

(spiral).

small r



(spiral not a center)

PROBLEM 6.4.5

$$\dot{N}_1 = r_1 N_1 \left(1 - N_1/K_1 \right) - b_1 N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2 .$$

let $\quad x = N_1 b_2 / r_1 \qquad y = N_2 b_1 / r_1 \qquad t = T \tau$

$$\frac{r_1}{b_2 T} \frac{dx}{d\tau} = \frac{r_1^2}{b_2} x \left(1 - \frac{r_1}{b_2 K_1} x \right) - \frac{b_1 r_1^2}{b_1 b_2} xy$$

$$\frac{r_1}{b_1 T} \frac{dy}{d\tau} = \frac{r_2 r_1}{b_1} y - \frac{b_2 r_1^2}{b_1 b_2} xy$$

NOW
$$\frac{r_1^{-1}}{T} \frac{dx}{d\tau} = x \left(1 - x/a \right) - xy$$

$$\frac{1}{r_2 T} \frac{dy}{d\tau} = y - \frac{r_1}{r_2} xy$$

let $\quad T = 1/r_1$

EQ. POINTS $\quad y = 0 \rightarrow x = 0, a$

$\qquad x = \mu \rightarrow y = 1 - \mu/a$

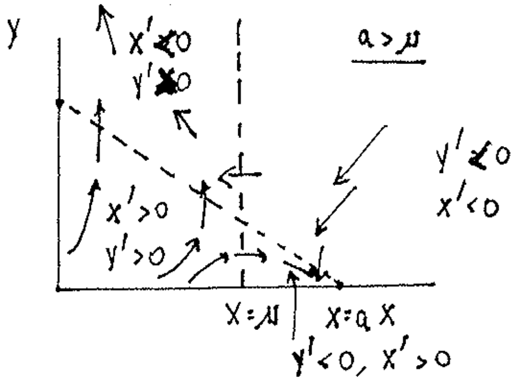
THEN
$$\boxed{\begin{aligned} x' &= x(1 - x/a) - xy \\ y' &= \mu y - xy \end{aligned}}$$

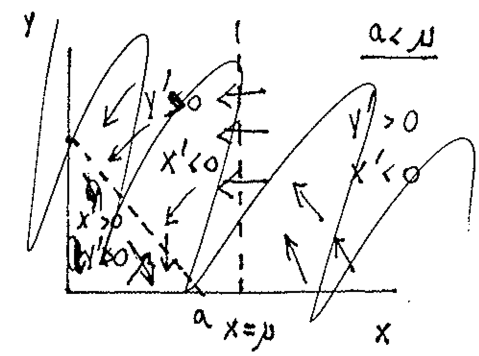
$$a = \frac{b_2 K_1}{r_1} > 0$$

$$\mu = \frac{r_2}{r_1} > 0$$

nullclines are $\quad y = 1 - x/a$

AND $\quad x = \mu$





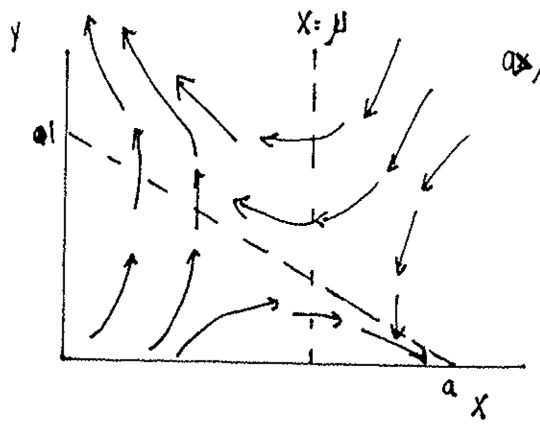


Fig. 1

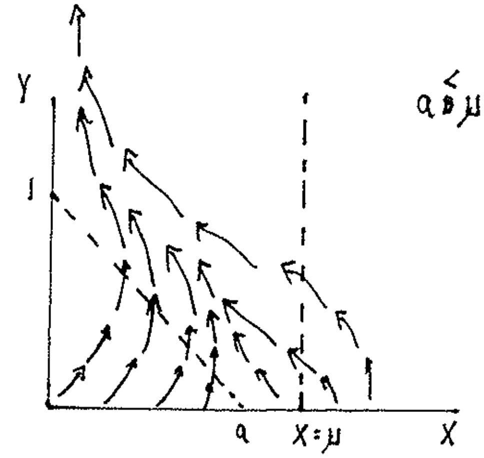


Fig. 2

For Fig. 1 $\quad a > \mu \longrightarrow b_2 K_1 > r_2 \Rightarrow K_1 > r_2/b_2$

" Fig. 2 $\quad a < \mu \longrightarrow b_2 K_1 < r_2 \Rightarrow K_1 < r_2/b_1$

Now linearized problem is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$f_x = 1 - 2x/a - y \qquad f_y = -x$$

$$g_x = -y \qquad g_y = \mu - x$$

For $(a, 0) \longrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1 & -a \\ 0 & \mu - a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \lambda_1 = -1$, $\lambda_2 = \mu - a$

$\longrightarrow$ stable if $\mu < a$

unstable if $\mu > a$

For $(\mu, 1 - \mu/a) \longrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -\mu/a & -\mu \\ \mu/a - 1 & 0 \end{pmatrix}$

eigenvalues $\quad -\lambda(-\lambda - \mu/a) + \mu^2/a - \mu = 0$

$\lambda^2 + \lambda\mu/a + \mu^2/a - \mu = 0$ $\quad$ unstable if $\mu^2/a - \mu <$

$\rightarrow \mu < a$

PROBLEM 6.4.6

$$\dot{N}_1 = r_1 N_1 \left(1 - N_1/K_1\right) - b_1 N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 \left(1 - N_2/K_2\right) - b_2 N_1 N_2$$

i) NOW NON-DIMENSIONALIZE $N_1 = K_1 X$ $\quad t = \tau / r_1$

$N_2 = \dfrac{r_1}{b_1} y$

EQ. POINTS ON boundary $(0,0)$, $(0, 1/$ $(1,0)$

THIS gives

$$x' = x(1-x) - xy \qquad a = r_2/r_1$$

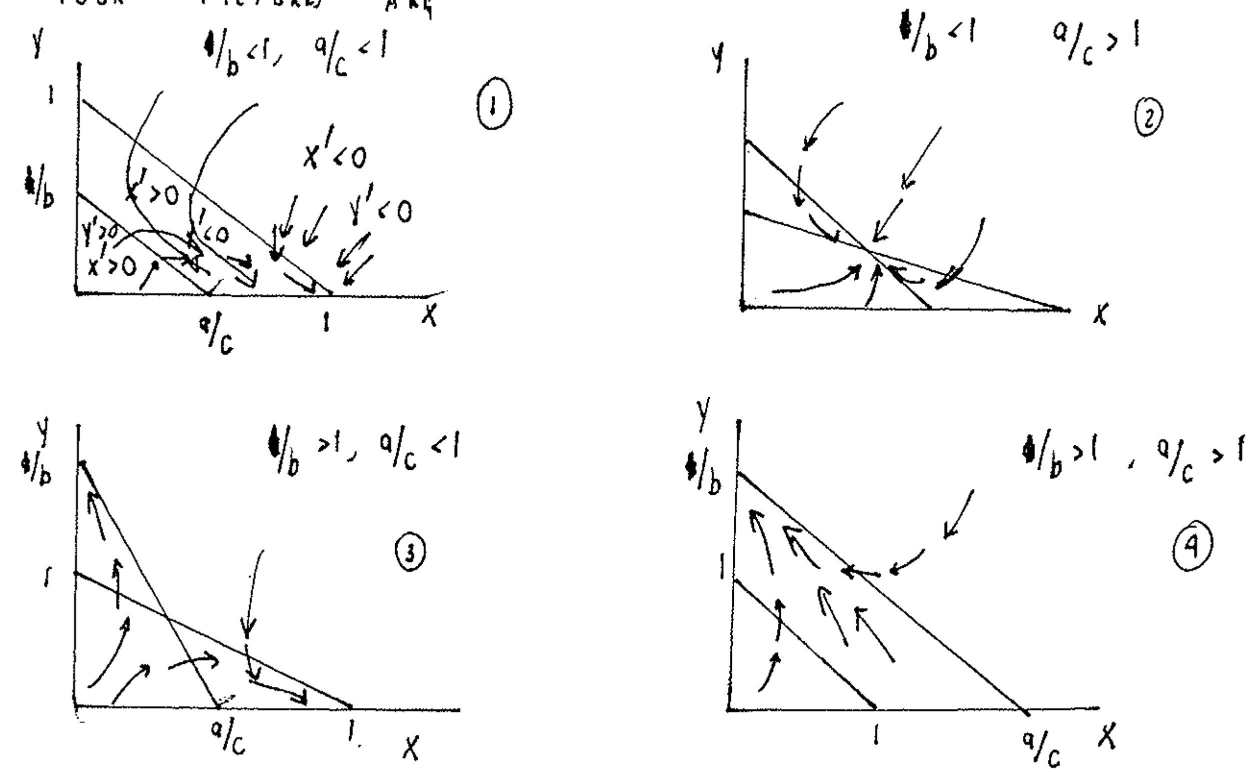
$$y' = a y (1 - b y) - c x y \qquad b = r_1/b_1 K_2$$

$$c = K_1 b_2 / r_1$$

nullclines are $y = 1 - x$

$$y = \frac{1}{b} - \frac{c}{b} x$$

THE FOUR PICTURES ARE



(1) long-term behavior is $(1,0)$

(2) long-term behavior is intersection of $y = 1 - x$ and $y = \frac{1}{b} - \frac{c}{b} x$

(3) long-term behavior is either $(0, 1/b)$ or $(1,0)$

(4) long-term behavior is $(0, 1/b)$

can get stable coexistence if $1/b < 1$ $a/c > 1$

$$\frac{r_1}{b_1 K_2} > 1 \qquad \frac{r_2}{K_1 b_2} > 1$$

thus need that growth rates r_1/b_1, r_2/b_2 exceeds carrying capacity of system.

PROBLEM 6.4.2

$x' = x(3-2x-y) = f(x,y)$

$y' = y(2-x-y) = g(x,y)$

EQUILIBRIA ARE AT $(0,0)$, $(0,2)$, $(\frac{3}{2},0)$, $(1,1)$.

NOW THE JACOBIAN J IS $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-4x-y & -x \\ -y & 2-2y-x \end{pmatrix}$

NOW AT $(1,1) \longrightarrow J = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$ $\det(J-\lambda I)=0 \longrightarrow (-2-\lambda)(-1-\lambda)-1=0$

$\lambda^2+3\lambda+1=0$ trace $J<0$

$\lambda_\pm = \dfrac{-3\pm\sqrt{9-4}}{2}$ $\det J>0$

$\lambda_\pm<0 \rightarrow$ stable node.

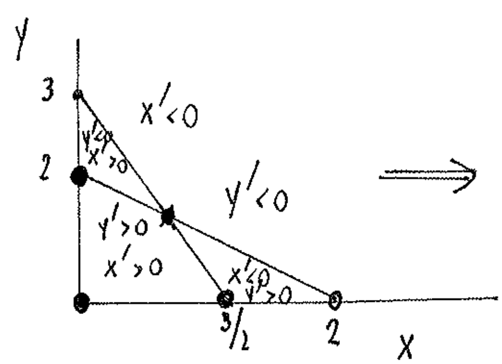
NOW AT $(0,0) \longrightarrow J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \lambda_\pm>0$ $\lambda_+=3$, $\lambda_-=2$.

unstable node

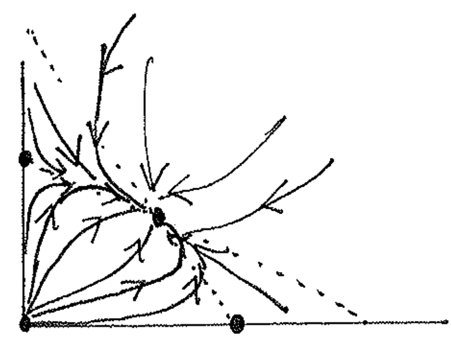
NOW AT $(0,2) \longrightarrow J = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \rightarrow \lambda_+=1$, $\lambda_-=-2$ saddle point.

FINALLY AT $(\frac{3}{2},0) \longrightarrow J = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix} \longrightarrow \lambda_+ = 1/2 > 0$, $\lambda_- = -3 < 0$

saddle point



$\Longrightarrow$ global phase portrait



this is a stable competition

for $x(0)>0, y(0)>0$ then $\lim_{t\to\infty}(x,y) = (1,1)$.

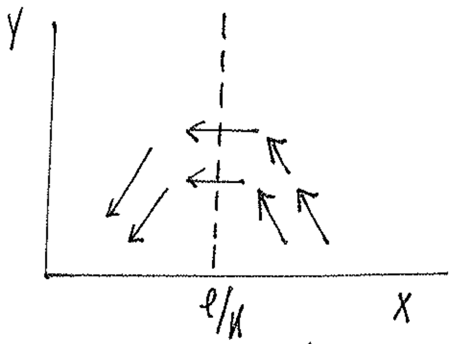
PROBLEM 6.5.6

$\dot{x} = -kxy, \quad \dot{y} = kxy - ly \qquad k, l > 0$

(i) FIXED POINTS $y = 0$ FOR ANY x. SO THERE IS A WHOLE LINE OF FIXED POINTS.

(ii) $\dot{x} = 0$ ON line $x = 0$ AND $y = 0$

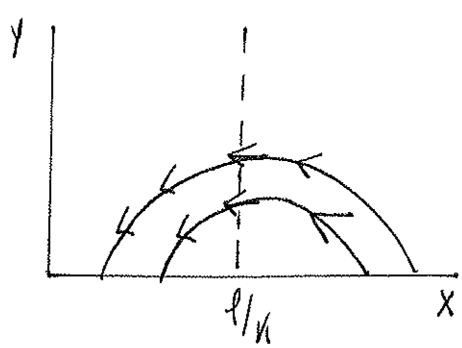
$\dot{y} = 0$ when $x = l/k$ OR $y = 0$



(iii) NOW $\dfrac{dy}{dx} = \dfrac{kx - l}{-kx} = -1 + \dfrac{l}{kx} \longrightarrow y = -x + \dfrac{l}{k} \ln x + C$

NOW $E = x + y - \dfrac{l}{k} \ln x$ IS A CONSERVED QUANTITY.

(iv) AS $t \to \infty$ WE SEE THAT $y \to 0$ (sick population dies out) AND x IS reduced FROM its initial value to some value $< l/k$

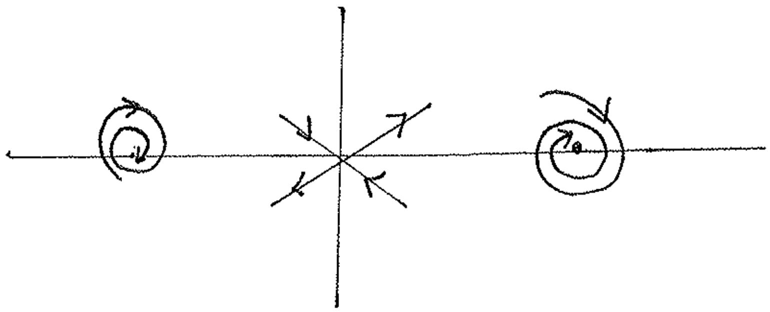


(v) EPIDEMIC IFF $y_0 > 0$ AND $x_0 > l/k$. THEN $y(t)$ INITIALLY INCREASES (MORE SICK) BEFORE y dies out leaving total Number of healthy people far Reduced.

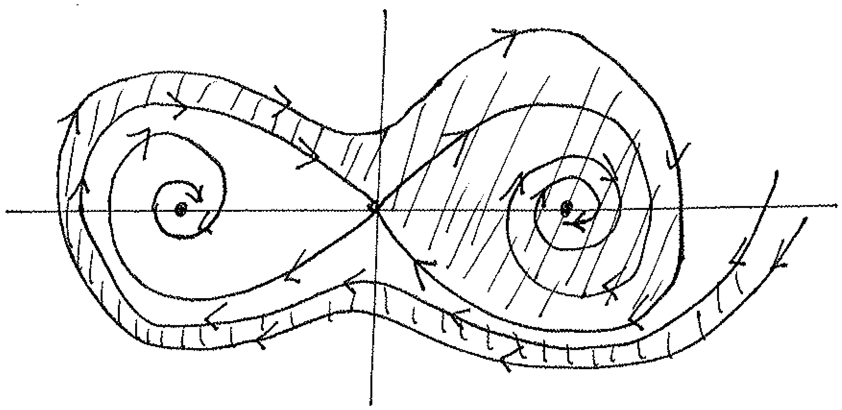
PROBLEM 6.5.11

$$\dot{x} = y$$
$$\dot{y} = -by + x - x^3$$

FIXED POINTS AT $(\pm 1, 0)$ stable spirals AND $(0,0)$ SADDLE



VECTOR FIELD AND FIXED POINTS



NOW TAKE THE POINT $(1,0)$. THE BASIN OF $(1,0)$ IS BOUNDED BY THE STABLE MANIFOLD OF THE SADDLE AND IT LOOKS AS SHOWN WITH THE SHADED REGION