MATH 345 HW #5 SOLUTIONS

PROBLEM 6.5.10 WE HAVE $H = \dot{r}^2/2 + V(r) = E$

where $E =$ CONSTANT AND $V(r) = h^2/2r^2 - K/r$ $(K>0)$.

(i) NOW $V \to +\infty$ AS $r \to 0^{\pm}$

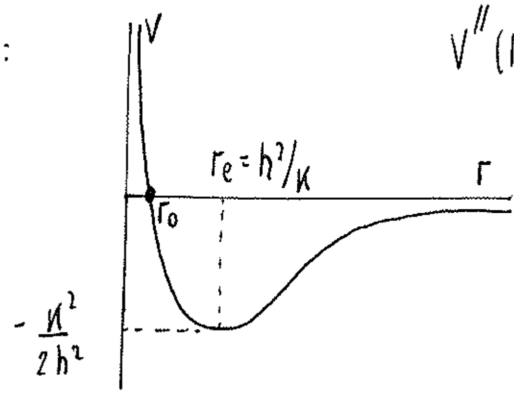
$V \to 0^-$ AS $r \to +\infty$, $V \to 0^+$ AS $r \to -\infty$

HOWEVER, SINCE $r > 0$ WE ONLY PLOT $V(r)$ FOR $r \geq 0$.

NOW $V = 0$ WHEN $r_0 = h^2/2K$ AND $V' = -\dfrac{h^2}{r^3} + \dfrac{K}{r^2} = 0$ WHEN $r_e = h^2/K > 0$

NOW $V(r_e) = \dfrac{h^2}{2}\left[\dfrac{K^2}{h^4}\right] - \dfrac{K^2}{h^2} = -\dfrac{1}{2}\dfrac{K^2}{h^2}$. SINCE $K > 0$

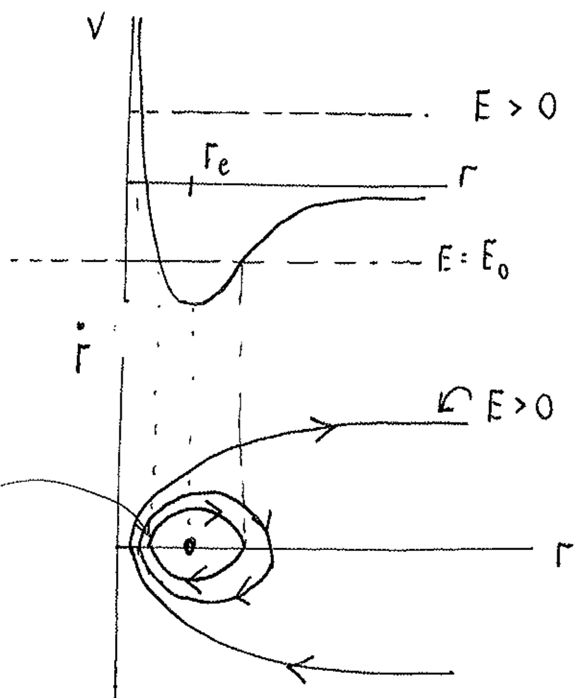
THE PLOT IS AS FOLLOWS:



$V''(r_e) > 0$

r_e IS NEUTRALLY STABLE EQUILIBRIUM

(ii) NOW WE PLOT THE CONTOUR CURVES IN THE PHASE PLANE WHERE $H = E$.

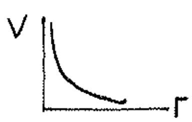


$-\dfrac{1}{2}\dfrac{K^2}{h^2} < E_0 < 0$

THUS, WE GET PERIODIC ORBITS WHENEVER THE INITIAL CONDITIONS ARE SUCH THAT

$$\frac{-K^2}{2h^2} < E < 0.$$

(iii) IF $K < 0$ THEN $V' < 0$ FOR ALL $r > 0$



$\Rightarrow$ NO PERIODIC SOLUTION).

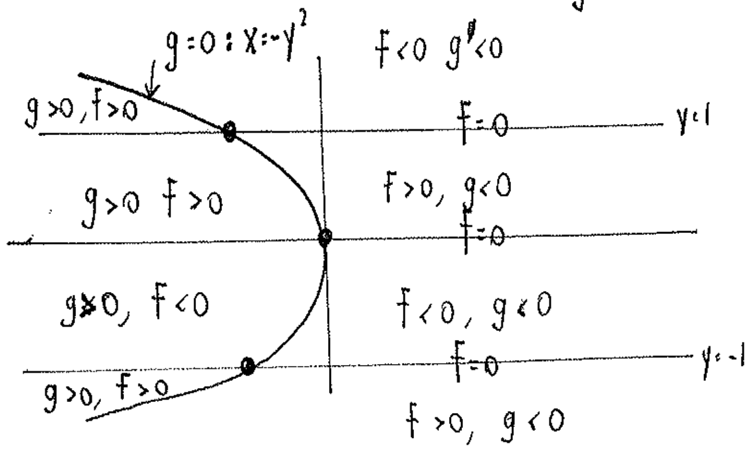
PROBLEM 6.6.6

$$(*)\begin{cases} x' = y - y^3 = f(x,y) \\ y' = -x - y^2 = g(x,y) \end{cases}$$

NOTICE THIS IS REVERSIBLE SINCE f IS ODD IN y AND g EVEN IN y. THUS $(*)$ IS INVARIANT UNDER $y \mapsto -y$ AND $t \mapsto -t$.

(i),(ii) TO PLOT NULLCLINES we have $f=0$ when $y = 0, \pm 1$

$g = 0$ when $x = -y^2$.



• = EQUILIBRIUM POINTS AT $(x,y) = (0,0), (-1,1), (-1,-1)$.

(iii) NOW AT $x=-1$, $y=\pm 1$ WE CALCULATE $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$

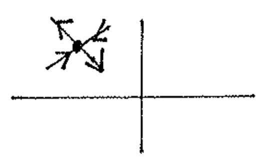
• SO AT $(-1,1) \longrightarrow J = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$ trace $J = -2 < 0$, det $J = -2 < 0$ $\longrightarrow$ saddle point:

NOW $-\lambda(-2-\lambda) - 2 = 0$ $\quad \lambda^2 + 2\lambda - 2 = 0 \rightarrow (\lambda^2 + 2\lambda + 1) = 3$

$\lambda + 1 = \pm\sqrt{3}$ so $\lambda_\pm = -1 \pm \sqrt{3}$

$\lambda_+^1 = -1 + \sqrt{3} > 0 \quad \underline{V}_+ = \begin{pmatrix} 2 \\ -\lambda_+ \end{pmatrix}$

$\lambda_-^2 = -1 - \sqrt{3} < 0 \quad \underline{V}_- = \begin{pmatrix} 2 \\ -\lambda_- \end{pmatrix}$



• SO AT $(-1,-1) \longrightarrow J = \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}$

$-\lambda(2-\lambda) - 2 = 0$

$\lambda^2 - 2\lambda - 2 = 0$

trace $J = 2$

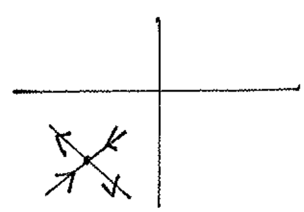
det $J = -2$.

so $(\lambda^2 - 2\lambda + 1) = 3$ so $\lambda - 1 = \pm\sqrt{3}$

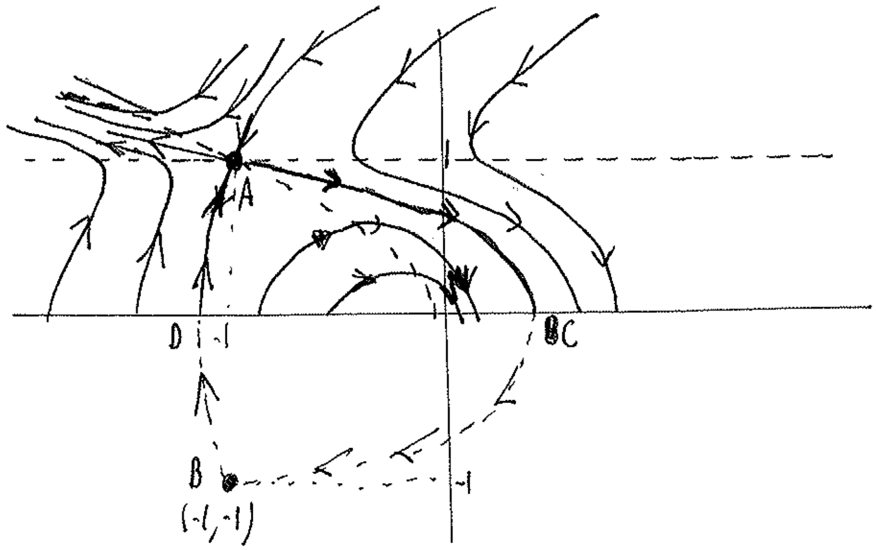
so $\lambda_\pm = 1 \pm \sqrt{3}$

NOW $\lambda_+^1 = 1 + \sqrt{3} > 0 \longrightarrow \underline{V}_+ = \begin{pmatrix} 2 \\ -\lambda_+ \end{pmatrix}$

$\lambda_+^2 = 1 - \sqrt{3} < 0 \longrightarrow \underline{V}_- = \begin{pmatrix} 2 \\ -\lambda_- \end{pmatrix}$



(iv),(v) WE SKETCH THE PHASE PLANE IN THE FIRST AND SECOND QUADRANT TO OBTAIN

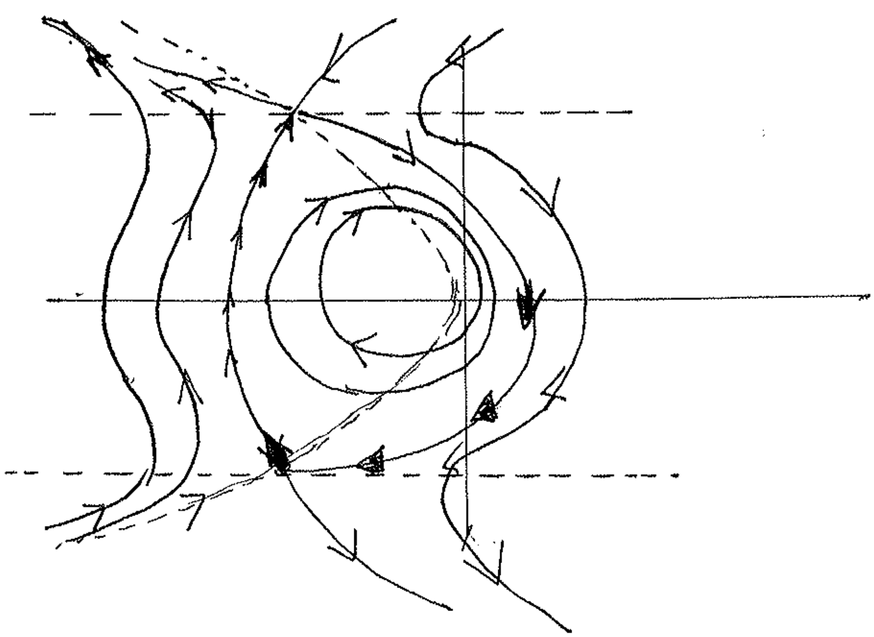


NOW BY REVERSIBILITY THERE EXISTS DOTTED TRAJECTORIES AS SHOWN

BDA $\rightarrow$ heteroclinic trajectory

ACB $\rightarrow$ heteroclinic trajectory

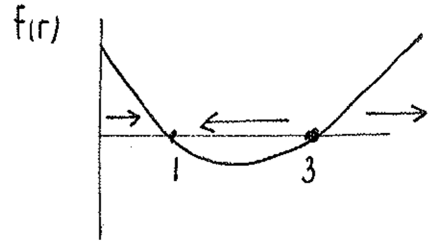
THUS THE FINAL PHASE PLANE IS



PROBLEM 7.1.2

$\dot{r} = r(1-r^2)(9-r^2)$, $\dot{\varphi} = 1$ WITH $r > 0$.

THE PLOT OF $f(r) \equiv r(1-r^2)(9-r^2)$ IS AS FOLLOWS:



NOTICE $f(1) = f(3) = 0$

HENCE $r = 3$ UNSTABLE LIMIT CYCLE

$r = 1$ STABLE LIMIT CYCLE

PROBLEM 7.1.5

$$x' = x - y - x(x^2 + y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

NOW LET $r^2 = x^2 + y^2$, $\varphi = \tan^{-1}(y/x)$

SO $r r' = x x' + y y'$, $\longrightarrow$ $r' = \frac{x}{r} x' + \frac{y}{r} y'$

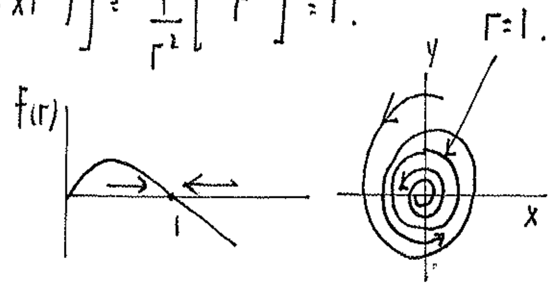
$\varphi = \tan^{-1}(y/x)$ $\longrightarrow$ $\varphi' = \frac{x y' - x' y}{r^2}$

THUS $r' = \frac{x}{r}\left[x - y - x r^2\right] + \frac{y}{r}\left[x + y - y r^2\right] = \frac{1}{r}\left[r^2 - r^4\right]$

$\varphi' = \frac{1}{r^2}\left[x(x + y - y r^2) - y(x - y - x r^2)\right] = \frac{1}{r^2}\left[r^2\right] = 1.$

HENCE $r' = r - r^3 = f(r)$

$\varphi' = 1$ (COUNTER CLOCKWISE)



THUS $r = 1$ IS STABLE LIMIT CYCLE

PROBLEM 7.1.8

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0 \qquad a > 0$$

(i) LET $\dot{x} = \ddot{x} = 0$. THEN $x = 0$. HENCE $x = 0$ IS ONLY EQUILIBRIUM POINT.

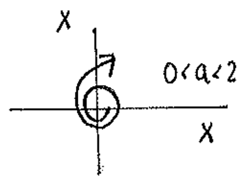
THE LINEARIZATION IS $\ddot{x} - a\dot{x} + x = 0$.

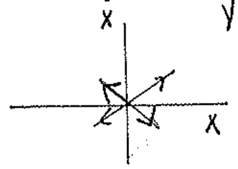
LET $x = e^{\lambda t}$, THEN $\lambda^2 - a\lambda + 1 = 0$ SO $\lambda_\pm = \dfrac{a \pm \sqrt{a^2 - 4}}{2}$

NOW IF $a > 2$, THEN $\lambda_\pm > 0 \rightarrow$ UNSTABLE NODE

IF $0 < a < 2$ THEN $\lambda_\pm = \dfrac{a}{2} \pm \dfrac{i}{2}\sqrt{4 - a^2}$ unstable spiral

$\ddot{x} - a\dot{x} + x = 0$. LET $\dot{x} = y$, $\dot{y} = ay - x$ IF $(x,y) = (b,0)$ $b > 0$ $\rightarrow$ $\dot{y} < 0$ clockwise





$a > 2$

(ii) NOW WE NOTE THAT $\dot{x}^2 + x^2 = 1$ AND $\ddot{x} + x = 0$ IS SIMULTANEOUSLY SATISFIED BY $x = \cos(t + \phi)$ FOR ANY ϕ CONSTANT.

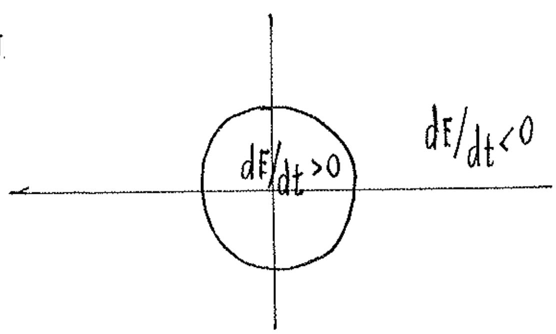
HENCE LIMIT CYCLE IS $x = \cos(t + \phi)$

THE AMPLITUDE IS ONE AND PERIOD IS 2π.

(iii) MULTIPLY BY $\dot{x}$ TO GET $\dot{x}\ddot{x} + \dot{x}x = -a\dot{x}^2(x^2 + \dot{x}^2 - 1)$

HENCE, $\dfrac{dE}{dt} = -a\dot{x}^2(x^2 + \dot{x}^2 - 1)$, WITH $E = \frac{1}{2}(x^2 + \dot{x}^2)$

THUS

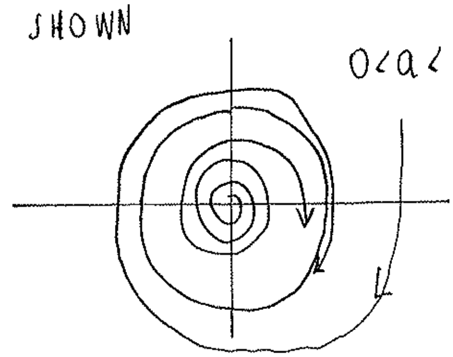


$E \downarrow$ as $t \uparrow$ FOR $x^2 + \dot{x}^2 > 1$

$E \uparrow$ as $t \downarrow$ FOR $x^2 + \dot{x}^2 < 1$

THEREFORE, WE GET A STABLE LIMIT CYCLE OF THE FORM AS SHOWN

$0 < a < 2$



PROBLEM 7.2.7

$$x' = y + 2xy = f$$
$$y' = x + x^2 - y^2 = g$$

(i) $\partial f/\partial y = 1 + 2x \qquad \partial g/\partial x = 1 + 2x.$ SO $\partial f/\partial y = \partial g/\partial x \rightarrow$ GRADIENT SYSTEM.

(ii) HENCE THERE IS A $V(x,y)$ WITH

$x' = -V_x \qquad y' = -V_y,$ AND SO THERE ARE NO PERIODIC SOLUTIONS.

NOW $\quad V_x = -y - 2xy \quad$ SO $\quad V = -xy - x^2 y + h(y)$

BUT $\quad V_y = -x^2 - x + h'(y) = -g = -x - x^2 + y^2 \quad$ SO $\quad h(y) = y^3/3.$

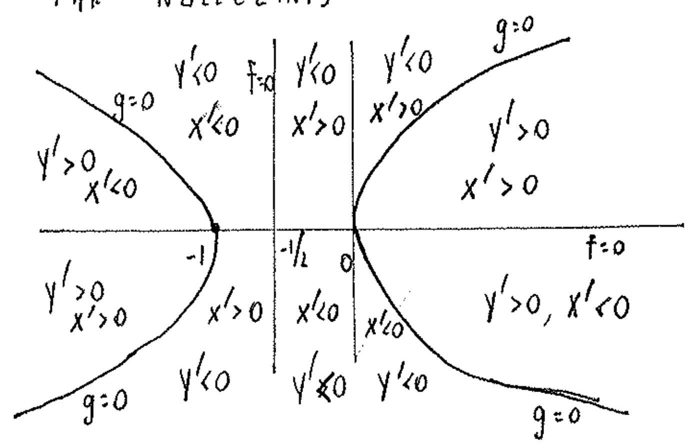
HENCE $\quad V(x,y) = -xy - x^2 y + y^3/3 = y\left[-x - x^2 + y^2/3\right].$

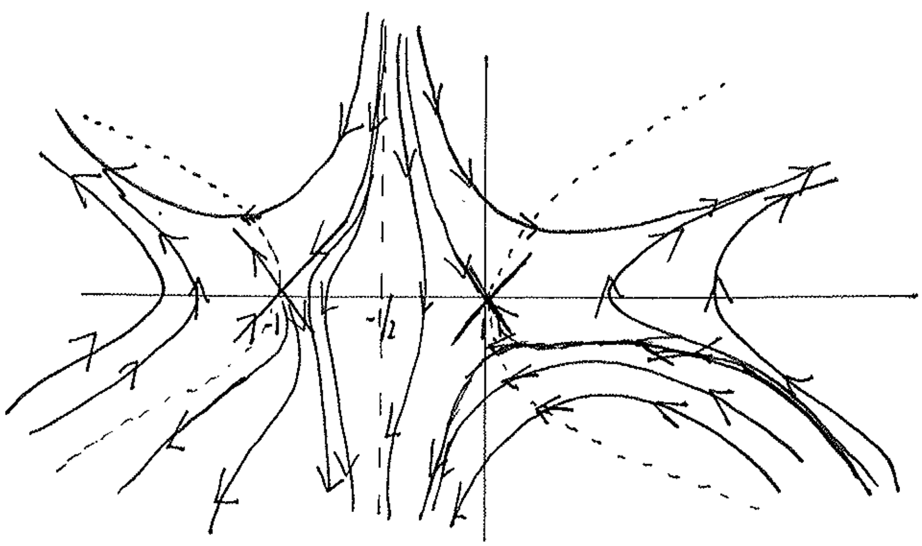
(iii) SKETCH PHASE-PLANE. RECALL FOR A GRADIENT SYSTEM THAT TRAJECTORIES ARE ORTHOGONAL TO LEVEL CURVES $V(x,y) =$ CONSTANT. HOWEVER, IT IS EASIER TO PLOT THE NULLCLINES

$$x' = y(1+2x) = f$$
$$y' = x + x^2 - y^2 = g$$

$f = 0$ ON $y = 0,\ x = -1/2$

$g = 0$ ON $\quad y = \pm\sqrt{x(1+x)}$





SADDLE POINTS AT EQUILIBRIA AT $x = \pm 1, y = 0$.

PROBLEM 7.2.10

LET $x' = -x^3 + y$ SHOW THERE ARE NO CLOSED ORBITS.
$y' = -x - y^3$

NOTICE THAT $(0,0)$ IS AN EQUILIBRIUM SOLUTION.

TRY A LIAPUNOV FUNCTION OF THE FORM $V = ax^2 + by^2$ WITH $a > 0$, $b > 0$.

NOW $V(0,0) = 0$, AND $V(x,y) > 0$ FOR $(x,y) \neq (0,0)$.

THUS TO SHOW THAT V IS A LYAPUNOV FUNCTION WE NEED ONLY CHOOSE $a>0$, $b>0$ IF WE CAN SO THAT $\frac{dV}{dt} = \nabla V \cdot \underline{x}' < 0$, FOR $(x,y) \neq (0,0)$

NOW $\frac{dV}{dt} = (V_x, V_y) \cdot (x', y') = (2ax, 2by) \cdot (-x^3 + y, -x - y^3)$

SO $\frac{dV}{dt} = -2ax^4 + 2axy - 2bxy - 2by^4 = -2(ax^4 + by^4) + 2xy(a-b)$

THUS $dV/dt < 0$ FOR $(x,y) \neq (0,0)$ WHEN $a = b > 0$.

HENCE $V = \frac{1}{2}x^2 + \frac{1}{2}y^2$ IS A LYAPUNOV FUNCTION $\longrightarrow$ NO PERIODIC SOLUTIONS ∎

PROBLEM 7.2.13 CONSIDER THE COMPETITION MODEL

$$N_1' = r_1 N_1 \left(1 - N_1/K_1\right) - b_1 N_1 N_2 \qquad N_2' = r_2 N_2 \left(1 - N_2/K_2\right) - b_2 N_1 N_2$$

WITH $N_1 > 0$, $N_2 > 0$ AND $b_1, b_2, r_1, r_2, K_1, K_2 > 0$. PROVE THAT THERE ARE NO PERIODIC SOLUTIONS IN $N_1 > 0$, $N_2 > 0$.

TO DO SO WE USE DULAC'S CRITERION WITH $g(N_1,N_2) = \frac{1}{N_1 N_2}$.

MUST SHOW $\nabla \cdot [g \underline{N}']$ IS OF ONE SIGN IN $N_1 > 0$, $N_2 > 0$.

WE CALCULATE

$$\nabla \cdot [g\underline{N}'] = \frac{\partial}{\partial N_1}\left[\frac{1}{N_1 N_2} N_1'\right] + \frac{\partial}{\partial N_2}\left[\frac{1}{N_1 N_2} N_2'\right] = \frac{\partial}{\partial N_1}\left[\frac{r_1}{N_2}\left(1 - \frac{N_1}{K_1}\right) - b_1\right] + \frac{\partial}{\partial N_2}\left[\frac{r_2}{N_1}\left(1 - \frac{N_2}{K_2}\right) - b_2\right]$$

THUS $\nabla \cdot [g\underline{N}'] = -\frac{r_1}{K_1 N_2} - \frac{r_2}{K_2 N_1} < 0$ IN $N_1 > 0, N_2 > 0$ $\longrightarrow$ DULAC'S CRITERION PROVES NO PERIODIC SOLUTION IN $N_1 > 0, N_2 > 0$.