MATH 345 HW 6 SOLUTIONS

PROBLEM 7.3.3

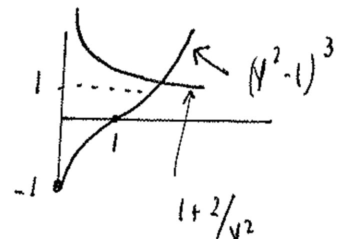
$$x' = x - y - x^3 = f(x,y)$$
$$y' = x + y - y^3 = g(x,y)$$

EQUILIBRIA SATISFY $\quad x = y + x^3, \quad x = -y + y^3$

THUS $\quad -y + y^3 = y + (-y + y^3)^3 \quad$ OR $\quad y^3 + 2y = y^3(y^2-1)^3.$

CLEARLY $(0,0)$ IS AN EQUILIBRIUM POINT. THE OTHERS WOULD SATISFY

$$y^2 + 2 = y^2(y^2-1)^3 \quad \text{OR} \quad 1 + \frac{2}{y^2} = (y^2-1)^3$$



THE OTHER ROOTS SATISFY $y^2 > 2$ CLEARLY OR $|y| > \sqrt{2}$.

THEREFORE IN A NEIGHBORHOOD OF THE ORIGIN THERE SHOULD ONLY BE ONE ROOT.

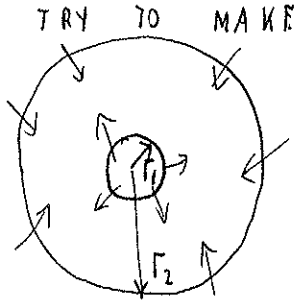
NOW CONVERT TO POLAR COORDINATES

$$rr' = xx' + yy'$$
$$\varphi' = \frac{xy' - x'y}{r^2}.$$

WE OBTAIN $\quad rr' = x(x - y - x^3) + y(x + y - y^3) = x^2 + y^2 - x^4 - y^4.$

HENCE $\quad r' = r - r^3 f(\varphi) \qquad f(\varphi) = \cos^4\varphi + \sin^4\varphi.$

WE TRY TO MAKE A TIGHT TRAPPING REGION OF THE FORM



NOTICE $\quad \max_{[0,2\pi]} f(\varphi) \leq 1.$

TO COMPUTE: $\quad f'(\varphi) = 4\sin\varphi\cos\varphi[\sin^2\varphi - \cos^2\varphi]$

SO $\quad f' = 0$ AT $\varphi = \pi/4 \quad \rightarrow \quad f(\pi/4) = \left(\frac{\sqrt{2}}{2}\right)^4 \cdot 2 = \frac{1}{2}$

THUS $\quad \min[f(\varphi)] = \frac{1}{2}$

NOW ON r_1: WANT $r' > 0 \longrightarrow -r_1^3 f(\varphi) + r_1 \geq r_1 - r_1^3 \geq 0 \Longrightarrow r_1 \leq 1$

NOW ON r_2: WANT $r' < 0 \longrightarrow r_2 - r_2^3 f(\varphi) \leq r_2 - r_2^3/2 \leq 0 \Rightarrow r_2 \geq \sqrt{2}$

THUS BY THE PB THEOREM, THERE IS A PERIODIC SOLUTION INSIDE THE REGION $1 - \varepsilon \leq r \leq \sqrt{2} + \varepsilon$ FOR ANY $\varepsilon > 0, \varepsilon \ll 1$.

PROBLEM 7.3.1 $\quad x' = x - y - x(x^2+5y^2)$

$\qquad\qquad\qquad\quad y' = x + y - y(x^2+y^2)$

(i) AT THE ORIGIN we have the linearization,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow (1-\lambda)^2+1=0 \rightarrow \lambda = 1 \pm i \quad \text{unstable spiral.}$$

FIND THE EQUILIBRIUM POINTS:

(1) $x - y = x(x^2+5y^2)$

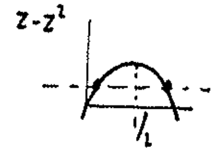
(2) $x + y = y(x^2+y^2)$

clearly $(0,0)$ is an equilibrium.

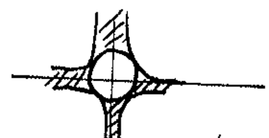
Are there other equilibria? multiply (1) by x and (2) by y AND ADD:

$$x^2+y^2 = x^2(x^2+y^2) + 4x^2y^2 + y^2(x^2+y^2).$$

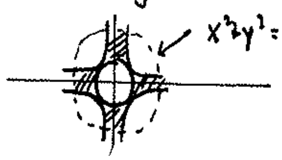
THUS $\quad z - z^2 = 4x^2y^2 \geq 0. \qquad z = x^2+y^2.$



HENCE THERE ARE NO ROOTS IF $x^2+y^2>1$ AND NO ROOTS IF $x^2y^2 > \frac{1}{16}$.
NOW ALSO WE WOULD HAVE ONE ROOT WITH $x^2+y^2 < 1/2$ AND $x^2y^2 < \frac{1}{16}$. THE INTERSECTION OF THIS REGION IS $x^2+y^2 < \frac{1}{2}$. ALSO we could have a ROOT WITH $x^2+y^2 > 1/2$ WITH $x^2y^2 < 1/16$. THE PICTURE OF THAT IS



Thus the only place where we have NOT Ruled out equilibria when $x^2+y^2 \geq 1/2$ is in shaded Region below



suppose $\frac{1}{2} < x^2+y^2 < 1$

AND $x^2y^2 < 1/16$

it is tedious to prove that there are NO ROOTS in that region but if you use a computer you can verify it.

So $(0,0)$ is only equilibria.

(ii) NOW $\quad r' = \dfrac{xx'}{r} + \dfrac{yy'}{r} \qquad \theta' = \dfrac{xy' - yx'}{x^2+y^2}$

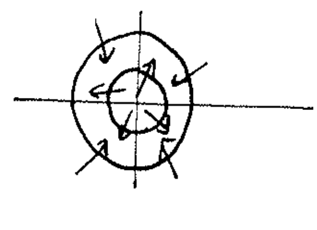
THIS GIVES $\quad r' = r - r^3 - r^3\sin^2(2\theta) \qquad \theta' = 1 + 4r^2 \sin^3\theta\cos\theta$

(iii) NOW $\quad r - 2r^3 \leq r' \leq r - r^3$

(iv)

THUS $\quad r' < 0 \quad$ IF $\quad r - r^3 < 0 \longrightarrow r > 1$

$\qquad\quad r' > 0 \quad$ IF $\quad r(1-2r^2) > 0 \longrightarrow r < 1/\sqrt{2}$.



PB Theorem: THUS THE REGION

$\frac{1}{\sqrt{2}} - \varepsilon < r < 1 + \varepsilon$ FOR ANY $\varepsilon > 0$ HAS A PERIODIC SOLUTION

PROBLEM 7.3.10

WE HAVE $\underline{X}' = A\underline{X} - r^2 \underline{X}$ WITH $r = \|X\|$. eigenvalues of A are $\alpha \pm i\omega$.

NOW THE LINEARIZATION GIVES $\underline{X}' = A\underline{X}$. IF trace $A > 0$, det $A > 0$ then $\underline{X} = 0$ is a repellor. trace $A = 2\alpha$ det $A = \alpha^2 + \omega^2 > 0$.

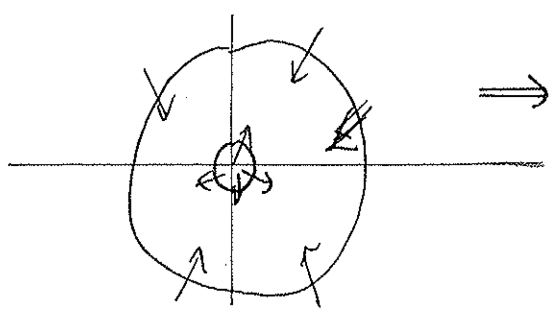
- IF $\alpha > 0$ THEN $\underline{X} = 0$ is a repellor
- NOW FOR $\|\underline{X}\| >> 1$ we NOTE $r^2 = \underline{X}^T \underline{X}$ AND $2rr' = 2\underline{X}^T X' = 2X_1X_1' + 2X_2X_2'$.

(HERE t denotes transpose). THEN

$$\underline{X}^T \underline{X}' = \underline{X}^T A\underline{X} - r^2 \underline{X}^T\underline{X} = \underline{X}^T A\underline{X} - r^4 \quad \text{SINCE } r^2 = \underline{X}^T\underline{X}.$$

THUS $$r r' = \underline{X}^T A \underline{X} - r^4 \quad \text{OR} \quad r' = \frac{1}{r} \underline{X}^T A \underline{X} - r^3.$$

NOW FOR $r >> 1$ WE ESTIMATE $\underline{X}^T A \underline{X} = O(r^2)$ SO THAT $r' < 0$ FOR r LARGE ENOUGH. HENCE when $\alpha > 0$ WE HAVE A TRAPPING REGION OF THE FORM AS SHOWN



$\Longrightarrow$ PB theorem yields a periodic solution.

WHEN $\alpha < 0$ THEN WE HAVE

$$r' = \frac{1}{r} \underline{X}^T A \underline{X} - r^3.$$

NOW LET $A q_1 = \lambda_1 q_1 \qquad A q_2 = \lambda_2 q_2$.

BUT $q_2 = q_1^*$ $*$ = conjugate since $\lambda_1 = \alpha + i\omega$, $\lambda_2 = \alpha - i\omega$.

NOW $AQ = Q\Lambda \longrightarrow A = Q\Lambda Q^{-1}$ $\quad Q = \begin{pmatrix} | & | \\ q_1 & q_2 \\ | & | \end{pmatrix}$

$$\Lambda = \begin{pmatrix} \alpha + i\omega & 0 \\ 0 & \alpha - i\omega \end{pmatrix}$$

THIS GIVES THAT

$$r' = \frac{1}{r}\left[\, \underline{x}^T Q \Lambda Q^{-1} \underline{x} \,\right] - r^3. \quad \text{WITH} \quad \Lambda = \begin{pmatrix} \alpha + i\omega & 0 \\ 0 & \alpha - i\omega \end{pmatrix}$$

when $\alpha < 0$, $\det \Lambda > 0$, $\operatorname{trace} \Lambda < 0$. THE ORIGIN IS NOW A SINK FOR r SMALL AND THE FLOW IS ALWAYS POINTING IN SINCE $r' < 0$ ALWAYS. $\longrightarrow$ NO PERIODIC SOLUTIONS.

PROBLEM 8.2.9

$$x' = x\left(b - x - \frac{y}{1+x}\right) = f \qquad a, b > 0 \qquad x, y \geq 0.$$
$$y' = y\left(\frac{x}{1+x} - ay\right) = g$$

(ii) THE EQUILIBRIUM POINTS ARE $(0,0)$, $(b,0)$ AND WHERE

$$b - x = \frac{y}{1+x} \quad \text{TOGETHER WITH} \quad \frac{x}{1+x} = ay.$$

THUS
$$(*)\left\{\begin{array}{l} a(1+x)(b-x) = ay \\ \dfrac{x}{1+x} = ay \end{array}\right\} \Longrightarrow a(1+x)^2(b-x) = x$$

HENCE x_e SATISFIES THE TRANSCENDENTAL EQUATION $\hat{f}(x_e) = a(1+x)^2(b-x) - x = 0$.

NOTICE THAT $\hat{f}(x) < 0$ FOR $x > b$ AND $\hat{f}(0) > 0$. THEREFORE there IS A ROOT IN $0 < x < b$.

$$f_y = -x/(1+x)$$
$$f_x = b - 2x - \frac{y}{(1+x)^2}$$
$$g_x = \frac{y}{(1+x)^2} \qquad g_y = \frac{x}{1+x} - 2ay$$

(iii) NOW THE JACOBIAN J IS

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$J = \begin{pmatrix} b - 2x - \dfrac{y}{(1+x)^2} & -\dfrac{x}{1+x} \\ \dfrac{y}{(1+x)^2} & \dfrac{x}{1+x} - 2ay \end{pmatrix}$$

USE EQUILIBRIUM SOLUTION TO OBTAIN

$$J = \begin{pmatrix} b - 2x - \dfrac{(b-x)}{1+x} & \dfrac{-x}{1+x} \\ \dfrac{b-x}{1+x} & \dfrac{-x}{1+x} \end{pmatrix} = \begin{pmatrix} -x + \dfrac{(b-x)x}{1+x} & \dfrac{-x}{1+x} \\ \dfrac{b-x}{1+x} & \dfrac{-x}{1+x} \end{pmatrix}$$

$$\det J = \frac{x^2}{1+x} - \frac{x^2(b-x)}{(1+x)^2} + \frac{x(b-x)}{(1+x)^2} = \frac{x\left[x + b + 2x^2 - xb\right]}{(1+x)^2} > 0 \quad \text{ON } 0 < x < b.$$

$$\text{trace } J = \frac{x}{(1+x)}\left[b - 2(x+1)\right]$$

THEREFORE, trace $J = 0$ when $b = 2(X+1)$ WHERE X IS THE EQUILIBRIUM SOLUTION. HOWEVER FROM (*)

$$a = \frac{X}{(1+X)^2(b-X)} = \frac{b/2 - 1}{b^2/4\left[\frac{b}{2}+1\right]} = \frac{4b-8}{b^2(b+2)} = \frac{4(b-2)}{b^2(b+2)}$$

THEREFORE, THERE IS A HOPF BIFURCATION WHEN

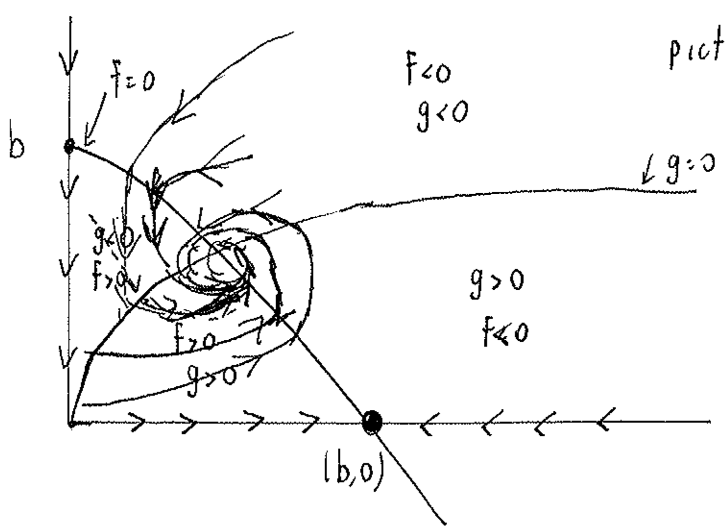
$$a = a_c \equiv \frac{4(b-2)}{b^2(b+2)} \quad \text{WHERE } (X_e, Y_e) \text{ IS A center.}$$

NOW PLOT THE DIRECTION FIELD. ASSUME $\exists$ A UNIQUE EQUILIBRIUM POINT. THEN

$$g = 0 \quad \text{when} \quad y = \frac{X}{a(1+X)} \quad \text{OR} \quad y = 0$$

$$f = 0 \quad \text{when} \quad y = (1+X)(b-X) \quad \text{OR} \quad X = 0$$

picture: let $a > a_c$



$\exists$ A HOPF BIFURCATION WHEN $a = a_c$

<u>REMARK</u> $\exists$ A UNIQUE EQUILIBRIUM POINT X_e IN $0 < X < b$ TO

$$a(1+X)(b-X) = \frac{X}{1+X} \quad \text{NOW} \quad h(X) = a(b-X) - \frac{X}{a(1+X)^2}$$

OR EQUIVALENTLY $h(X) = (b-X)(1+X) - \dfrac{X}{a(1+X)}$. NOTICE $h(0) > 0$, $h(b) < 0$

AND $h'(X) = -(1+X) + (b-X) - \dfrac{1}{a(1+X)^2} < 0$.

PROBLEM 8.3.1

$$x' = 1-(b+1)x + ax^2y$$
$$y' = bx - ax^2y.$$

(i) NOW THE EQUILIBRIUM POINT IS $y = b/ax \rightarrow 1-(b+1)x + \frac{ax^2 b}{ax} = 0 \rightarrow 1-(b+1)x + xb = 0$

THIS IMPLIES THAT $(x_E, y_E) = (1, b/a)$.

NOW $J = \begin{pmatrix} -(b+1)+2axy & ax^2 \\ b-2axy & -ax^2 \end{pmatrix}$

AT THE EQUILIBRIUM POINT

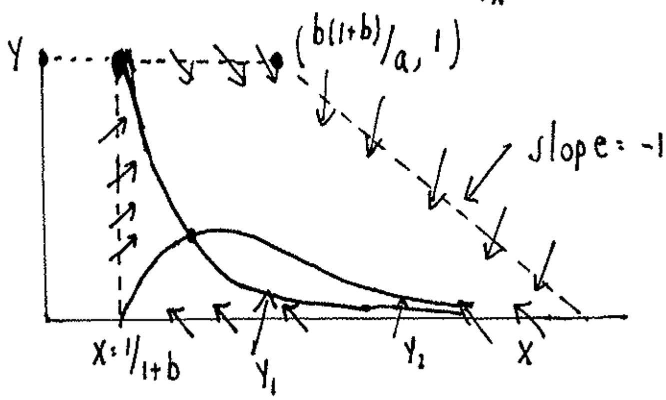
$J = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$ trace $J = b-a-1$
  det $J = a > 0$.

THUS IF $b-1-a > 0 \longrightarrow$ either unstable node OR unstable spiral

$b-1-a < 0 \longrightarrow$ stable node OR stable spiral.

(ii) NOW nullclines are $y_1 = b/ax$ AND $y_2 = \frac{1}{ax^2}[(b+1)x-1]$.

THUS we have



NOW ON $x = \frac{1}{1+b}$ we have $x' = \frac{a}{(1+b)^2}y > 0$ flow is IN

NOW ON $y = 0$ we have $x' < 0$ IF $x > \frac{1}{1+b}$ AND $y' = bx > 0$

THUS AGAIN FLOW IS IN.

NOW TAKE THE LINE $y = \frac{b(1+b)}{a}$. ON THIS LINE WE HAVE

$$y' = bx[1-x(1+b)] < 0 \text{ IF } x > 1/_{1+b}. \text{ Flow is AGAIN IN.}$$

NOW TAKE THE LINE $x + y = K$.

COMPUTE $\underline{x}' \cdot (1,1) = 1-x < 0$ IF $x > 1$. THUS THE DOTTED REGION AS SHOWN ABOVE IS A TRAPPING REGION.

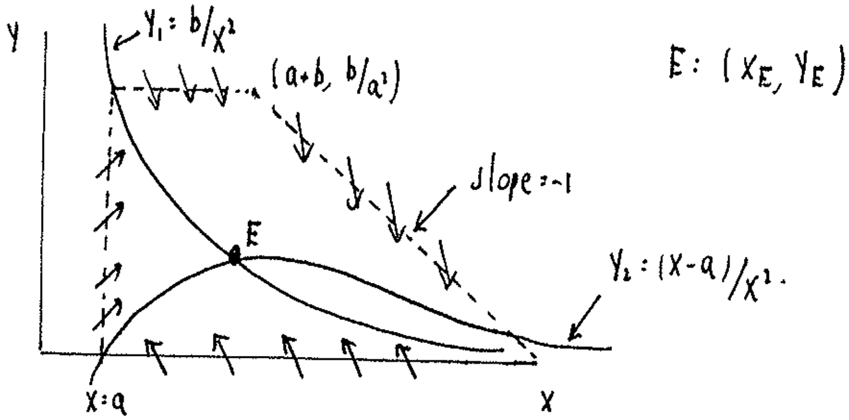
$$x' = a - x + x^2 y$$

PROBLEM 8.3.2 $\quad y' = b - x^2 y$

THE EQ. POINT IS AT $x^2 y = b$ AND $a - x + x^2 y = 0$. THIS GIVES

$$(x_E, y_E) = \left(a+b,\ b/(a+b)^2\right).$$

(i) AFTER DRAWING NULLCLINES we try to construct a trapping region OF THE FORM



NOW ON THE LINE $x = a$ WE HAVE $x' = a^2 y > 0$ (flow points in)

NOW ON LINE $y = 0$ WE HAVE $x' = a - x < 0$ FOR $x > a$ AND $y' = b > 0$

NOW ON LINE $y = b/a^2$ WE HAVE $y' = b - \frac{b}{a^2} x^2 = b\left(1 - x^2/a^2\right) < 0$ FOR $x > a$.

ON THIS LINE $x' = a - x + \frac{x^2 b}{a^2}$.

NOW ON THE LINE $x + y = k$ we have $\underline{x}' \cdot (1,1) = a - x + x^2 y + b - x^2 y = a + b - x$

HENCE $\underline{x}' \cdot (1,1) < 0$ IF $x > a+b$. Flow points in.

Thus the dashed box is a trapping region.

(ii) THE UNIQUE eq. point is $(x_E, y_E) = \left(a+b, \frac{b}{(a+b)^2}\right)$.

THUS $$J = \begin{pmatrix} -1 + 2xy & x^2 \\ -2xy & -x^2 \end{pmatrix} = \begin{pmatrix} -1 + \frac{2b}{a+b} & (a+b)^2 \\ -\frac{2b}{a+b} & -(a+b)^2 \end{pmatrix}$$

NOW $$\det J = -\frac{(b-a)}{(a+b)}(a+b)^2 + 2b(a+b) = -(b-a)(a+b) + 2b(a+b) = (a+b)^2 > 0.$$

$$\operatorname{trace} J = \frac{(b-a)}{(b+a)} - (b+a)^2 = \frac{(b-a) - (b+a)^3}{(b+a)}$$

thus we have an unstable node or spiral if $(b-a) > (b+a)^3$

" " stable node " " if $(b-a) < (b+a)^3$.

(iii) NOW when $(b-a) = (a+b)^3$ we have trace $J = 0$. The eigenvalues are

$$\lambda_\pm = \pm \frac{i\sqrt{+4\det J}}{2} = \pm \frac{2i(a+b)}{2}$$

$$\longrightarrow \quad \lambda_\pm = \pm i(a+b).$$

we have a strict transversality and so the HB theorem applies to give us a periodic solution is born at this point.

(iv) For $(b-a) > (b+a)^3$ we have a periodic solution inside the trapping region by the PB theorem. It is stable and so bifurcation must be supercritical.

(v) NOW we plot the stability region. The boundary is

$$b - a = (b+a)^3.$$

we can write this as $\frac{b-a}{2} + \frac{b+a}{2} = \frac{(b+a)^3 + (b+a)}{2} \longrightarrow b = \frac{(b+a)^3 + (b+a)}{2}$

let $a = \frac{X_E}{2}(1 - X_E^2)$, $b = \frac{X_E}{2}(1 + X_E^2)$ so $b + a = X_E$.

THUS $b = \frac{X_E}{2}(1 + X_E^2)$ $\quad a = X_E - b$.

NOW

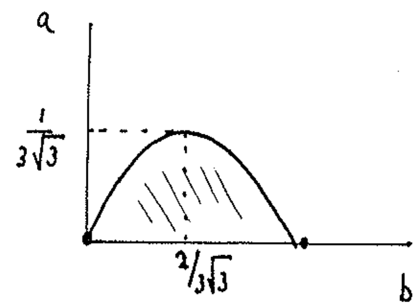
$X_E = 0 \longrightarrow a = 0$ AND $b = 0$

$X_E = 1 \longrightarrow a = 0$ AND $b = 1$.

NOW $\frac{da}{db} = 0$ when $\frac{da}{dX_E} = 0$ OR $\frac{db}{dX_E} = \infty$ which does NOT OCCUR

NOW $\frac{da}{dX_E} = \frac{1}{2} - \frac{3X_E^2}{2} = 0$ WHEN $X_E = \frac{1}{\sqrt{3}} \longrightarrow a_M = \frac{1}{3\sqrt{3}}$, $b_M = \frac{2}{3\sqrt{3}}$

THE CURVE IS SIMPLY



TAKE A POINT INSIDE then trace $J > 0$ (unstable eq. point)

thus in shaded region we get periodic solutions.

(iii) NOW AT THE EQ. POINT $(1, b/a)$ WE HAVE

$$\text{trace } J = 0 \quad \text{when} \quad b = 1 + a.$$

ALSO $\quad \lambda_{\pm} = \pm i\sqrt{|\det J|} = \pm i\sqrt{a}.$

HENCE we have a HOPF-BIFURCATION AT $b_c = 1 + a$. THE PERIODIC SOLUTION EXISTS FOR $b > b_c$.

(iv) THE PERIOD OF OSCILLATION FOR $b \approx b_c$ IS $T \hat{\approx} 2\pi/\sqrt{a}$.