MATH 345 HW #7

PROBLEM 7.5.3

PROBLEM 8.1.11, 8.2.14, 8.7.3

PROBLEM 9.2.1, 9.2.2 $\Longleftarrow$ SEE ONLINE NOTES.

MATH 345 HW #7

PROBLEM 7.5.3 CONSIDER $\ddot{x} + K(x^2-4)\dot{x} + x = 1$ FOR $K \gg 1$.

WE LET $F(x) = \int_0^x (s^2-4)\,ds = \left(\frac{x^3}{3} - 4x\right)$.

THEN $$\frac{d}{dt}\left[\dot{x} + K F(x)\right] = 1 - x.$$

LET $\tilde{y} = \dot{x} + K F(x)$. THEN

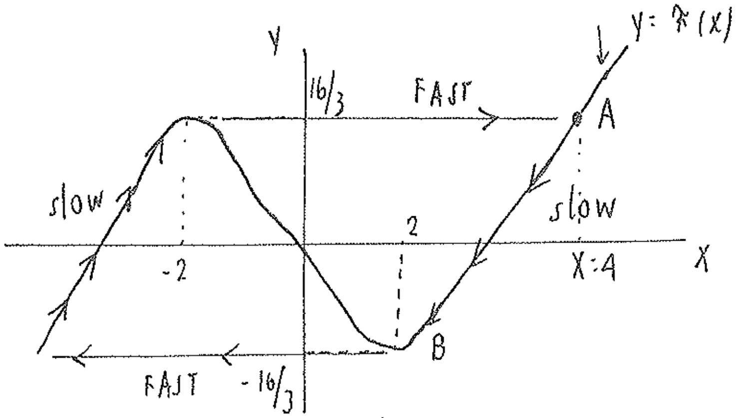
$$\dot{x} = -K F(x) + y, \qquad \dot{y} = 1 - x.$$

NOW LET $\tilde{y} = Ky$ AND $t = K\tau$. THEN $\frac{dx}{dt} = \frac{dx}{d\tau}\frac{1}{K}$. $\frac{d\tilde{y}}{dt} = K\frac{dy}{dt} = \frac{dy}{d\tau}$.

THEREFORE, WE OBTAIN

$$\begin{cases} K^{-2}x' = -F(x) + y & \text{WITH } K \gg 1. \\ y' = 1 - x \end{cases}$$

NOW WE DRAW THE CURVE $y = F(x) = \frac{x^3}{3} - 4x$.



NOTICE $F(-2) = \frac{x^3}{3} - 4x \Longrightarrow x = 4$ ALSO

- NOW FOR $K \gg 1$ WE HAVE $y \approx F(x)$ AND $y' = 1 - x \begin{cases} < 0 & \text{IF } x > 1 \\ > 0 & \text{IF } x < 1 \end{cases}$

- NOTICE THAT IF $y - F(x) > 0$ THEN $x' \approx K^2(y - F(x)) \gg 0$ SO x INCREASES RAPIDLY.

THE PERIODIC SOLUTION OR LIMIT CYCLE IS SKETCHED ABOVE WITH THE ARROWS.

THE PERIOD IS ESTIMATE AS TWICE THE TIME IT TAKES TO GO FROM A TO B $\longrightarrow$ $\tau_{AB} = \int_4^2 \frac{F'(x)}{1-x}\,dx \Longrightarrow \tau_{period} \simeq 2\tau_{AB}$

TO OBTAIN THIS INTEGRAL NOTICE $y \cong F(x)$ AND $\frac{dy}{dt} = 1 - x$ YIELDS THAT $F'(x)\,dx/dt = 1 - x$. SEPARATING VARIABLES GIVES τ_{AB}.

NOW $\hat{f}'(x) = x^2 - 4.$

SO $\displaystyle\int_4^2 \frac{\hat{f}'(x)}{-(x-1)}\,dx = \int_2^4 \frac{\hat{f}'(x)}{x-1}\,dx = \int_2^4 \frac{x^2-4}{x-1}\,dx$

SO $\displaystyle\tau_{period} \simeq 2\int_2^4 \left(\frac{x^2-4}{x-1}\right)dx = 2\int_2^4 \left[\left(\frac{x^2-1}{x-1}\right) - \frac{3}{(x-1)}\right]dx$

$$\tau_{period} \simeq 2\left[\int_2^4 (x+1)\,dx - 3\int_2^4 \frac{1}{x-1}\,dx\right]$$

$$\cong 2\left[\frac{(x+1)^2}{2}\Big|_2^4 - 3\log(x-1)\Big|_2^4\right]$$

$$\tau_{period} \cong 2\left[\frac{25}{2} - \frac{9}{2} - 3\log 3\right]$$

$$\tau_{period} \simeq 16 - 6\log 3$$

SO IN ORIGINAL TIME VARIABLE THE PERIOD IS ASYMPTOTICALLY

$$T \approx (16 - 6\log 3)\,K$$

PROBLEM 8.2.14

$$x' = \mu x + y - x^3$$
$$y' = -x + \mu y + 2y^3$$

WE WRITE THIS AS $\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}}_{J} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}$

WITH $f(x,y) = -x^3$, $g(x,y) = 2y^3$. NOW trace $J = 0$ when $\mu = 0$ AND det $J = 1$ when $\mu = 0$.

HENCE, $\mu = 0$ IS A HOPF BIFURCATION POINT. WE NOW CALCULATE THE QUANTITY a DEFINED FOR $w = -1$ BY

$$16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{w}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy} \right]$$

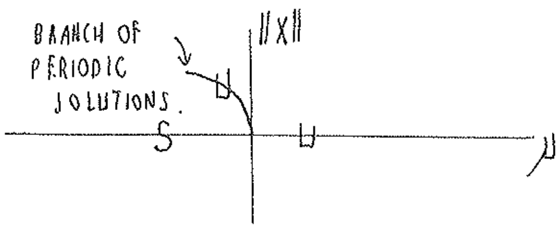
NOW AT THE HOPF BIFURCATION POINT $(x,y) = (0,0)$ WHEN $\mu = 0$ AND $w = -1$ WE CALCULATE

$$f_{xxx} = -6, \quad g_{yyy} = 12$$

NOW ALL OTHER DERIVATIVES ARE ZERO AT $(0,0)$ (i.e. $f_{xy}, f_{xx}, f_{yy}, g_{xy}, g_{yy}, g_{xx}, g_{xxy}, f_{xyy} = 0$).

HENCE $16a = -6 + 12 = 6 \longrightarrow a = 3/8 > 0$

THUS, THE BIFURCATION IS SUBCRITICAL



PROBLEM 8.1.11

$$u' = a(1-u) - uv^2$$
$$v' = uv^2 - (a+K)v$$

WE CONSIDER ONLY $u \geq 0, v \geq 0$ SINCE THEY ARE CHEMICAL CONCENTRATIONS

NOW WE FIND THE EQUILIBRIA.

NOTICE THAT $(u,v) = (1,0)$ IS AN EQUILIBRIUM.

THEN, WE SET $uv = (a+K)$ $\quad (v' = 0)$

THIS YIELDS $a(1-u) - \frac{u}{u^2}(a+K)^2 = 0, \quad (u' = 0).$

WHICH CAN BE SIMPLIFIED TO

$$u(1-u) - \frac{(a+K)^2}{a} = 0, \implies u^2 - u + \frac{1}{a}(a+K)^2 = 0.$$

THEREFORE,

$$u_\pm = \frac{1 \pm \sqrt{1 - \frac{4}{a}(a+K)^2}}{2}, \quad v_\pm = \frac{(a+K)}{u_\pm}$$

GIVE TWO ADDITIONAL EQUILIBRIA (IN ADDITION TO $(u,v) = (1,0)$).
THIS SHOWS THAT THERE ARE THREE EQUILIBRIA WHEN

$1 - \frac{4}{a}(a+K)^2 > 0$. THE NEW EQUILIBRIA EMERGE FROM SN BIFURCATION

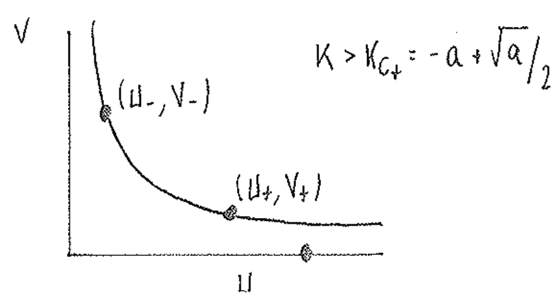
AND ONLY ONE EQUILIBRIUM AT $(1,0)$ WHEN $1 - \frac{4}{a}(a+K)^2 < 0$.

SOLVING FOR K_c WE GET $(a+K_c)^2 = a/4$ OR $K_c = -a \pm \sqrt{a}/2$

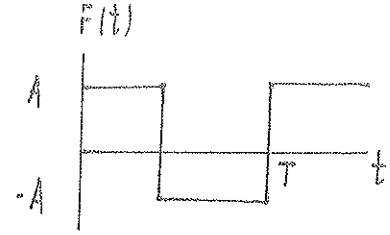
IS THE SADDLE-NODE BIFURCATION.

NOW WHEN $(a+K_c) = \sqrt{a}/2$ WE HAVE $u_\pm = \frac{1}{2}$, $v_\pm = \frac{\sqrt{a}}{2(1/2)} = \sqrt{a}$.

NOTICE THAT THE EQUILIBRIA LIE ON THE HYPERBOLA $uv = (a+K)$



PROBLEM 8.7.3



We consider $\dot{x} + x = F(t) = \begin{cases} A, & 0 < t < T/2 \\ -A, & T/2 < t < T \end{cases}$

Assume that $f(t+T) = f(t)$ for all t.

(i) Let $x(0) = x_0$. We calculate now $x(T)$.

We solve the first order ODE:

$$x = \begin{cases} A + (x_0 - A)e^{-t}, & 0 \le t \le T/2 \\ -A + \alpha e^{-(t - T/2)}, & \frac{T}{2} \le t \le T. \end{cases}$$

To find α we impose continuity:

$$A + (x_0 - A)e^{-T/2} = -A + \alpha.$$

Thus $\alpha = 2A + (x_0 - A)e^{-T/2}$.

This gives

$$x = \begin{cases} A + (x_0 - A)e^{-t}, & 0 \le t \le T/2 \\ -A + \left[2A + (x_0 - A)e^{-T/2}\right]e^{-(t - T/2)}, & T/2 \le t \le T. \end{cases}$$

We calculate

$$x(T) = -A + \left[2A + (x_0 - A)e^{-T/2}\right]e^{-T/2}$$

$$x(T) = -A + 2Ae^{-T/2} + (x_0 - A)e^{-T}$$

$$x(T) = x_0 e^{-T} - A\left[1 - 2e^{-T/2} + e^{-T}\right].$$

So $\left\{ x(T) = x_0 e^{-T} - A(1 - e^{-T/2})^2. \right.$

(ii) NOW SET $X(T) = X_0$.

THEN, $$X_0 = X_0 e^{-T} - A(1-e^{-T/2})^2$$

SO $$(e^{-T}-1)X_0 = A(e^{-T/2}-1)^2$$

SO $$(e^{-T/2}-1)(e^{-T/2}+1)X_0 = A(e^{-T/2}-1)^2.$$

THUS $$X_0 = A\,\frac{e^{-T/2}-1}{e^{-T/2}+1} = A\,\frac{e^{T/4}(e^{-T/2}-1)}{e^{T/4}(e^{-T/2}+1)} = -A\,\frac{(e^{T/4}-e^{-T/4})}{(e^{T/4}+e^{-T/4})}$$

THUS $$X_0 = -A\,\frac{2\sinh(T/4)}{2\cosh(T/4)} = -A\tanh(T/4).$$

(iii) NOW $T \to 0$ MEANS SHORT PERIOD WHILE $T \gg 1$ MEANS LONG PERIOD. WE USE $\tanh(z) \sim \begin{cases} z & \text{AS } z \to 0 \\ \infty & \text{AS } z \to +\infty \end{cases}$

TO GET $$X_0 \sim \begin{cases} -AT/4 & \text{FOR } T \text{ small also } X(T) \approx X_0 \\ -A & \text{FOR } T \text{ large. also } X(T) \approx -A. \end{cases}$$

(iv) LET $X_1 = X(T)$, $X_2 = X(2T)$ ETC..

THUS $$X(T) = X(0)\,e^{-T} - A(1-e^{-T/2})^2.$$

CLEARLY IF WE DEFINE $X_n \equiv X[nT]$, WE OBTAIN

THAT $$X_{n+1} = X_n e^{-T} - A(1-e^{-T/2})^2.$$

THUS $X_{n+1} = P(X_n)$ WHERE THE POINCARÉ MAP IS

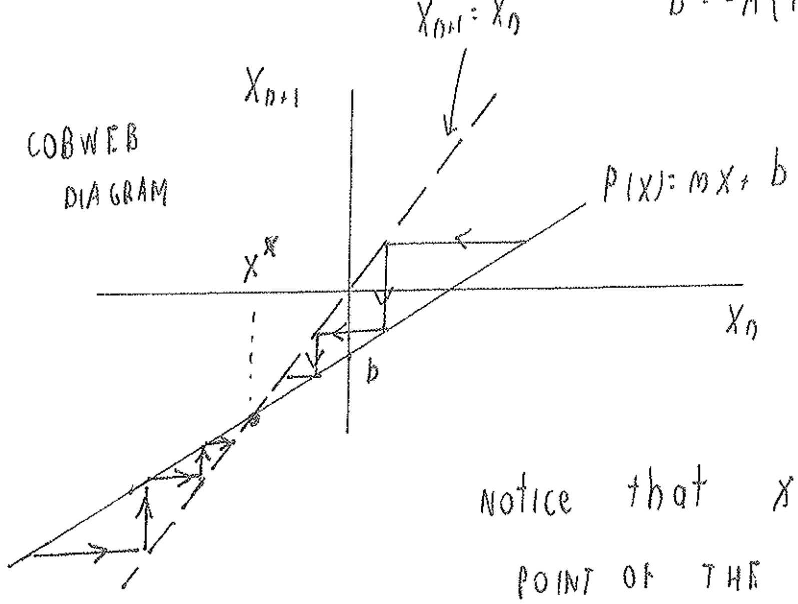
$$P(X_n) \equiv X_n e^{-T} - A(1-e^{-T/2})^2.$$

NOW THE POINCARÉ MAP HAS THE FORM

$$P(X) = mX + b \qquad m = e^{-T} \text{ WITH } 0 < m < 1$$

$$b = -A\left(1 - e^{-T/2}\right)^2 \text{ WITH } -A < b < 0.$$

COBWEB DIAGRAM



Notice that X^* is an attracting fixed point of the map. Follows since $|P'(X^*)| < 1$.

THUS THE ODE PROBLEM

$$X' + X = F(t) = \begin{cases} A & 0 \le t \le T/2 \\ -A & T/2 \le t \le T \end{cases}$$

$$X(0) = X_0$$

HAS A UNIQUE STABLE PERIODIC ORBIT

WHEN $\quad X_0 = -A \tanh(T/4).$

PROBLEM 9.2.1

$$x' = \sigma(y-x) = f(x,y,z)$$

$$y' = rx - xz - y = g(x,y,z)$$

$$z' = xy - bz = h(x,y,z)$$

(i) THE NON-TRIVIAL EQUILIBRIUM STATES ARE CALCULATED AS

$$\underline{X}_{\pm} \equiv (\pm\alpha, \pm\alpha, r-1) \quad \text{WITH} \quad \alpha = \sqrt{b(r-1)} \quad \text{FOR} \quad r>1.$$

THE JACOBIAN IS $J = \left.\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}\right|_{\underline{x}=\underline{X}_{\pm}} = \begin{pmatrix} -\sigma & 0 & 0 \\ 1 & -1 & \mp\alpha \\ \pm\alpha & \pm\alpha & -b \end{pmatrix}$

NOW SET $\det(J-\lambda I) = 0$. EXPANDING IN CO-FACTORS GIVES

$$0 = \det\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\alpha \\ \pm\alpha & \pm\alpha & -b-\lambda \end{pmatrix} = (-\sigma-\lambda)\det\begin{pmatrix} -1-\lambda & \mp\alpha \\ \pm\alpha & -b-\lambda \end{pmatrix} - \sigma\det\begin{pmatrix} 1 & \mp\alpha \\ \pm\alpha & -b-\lambda \end{pmatrix}$$

$$0 = -(\lambda+\sigma)\left((\lambda+1)(\lambda+b)+\alpha^2\right) - \sigma\left(-(\lambda+b)+\alpha^2\right).$$

EXPANDING THIS OUT GIVES THAT λ SATISFIES $f(\lambda)=0$ WHERE

$$f(\lambda) \equiv \lambda^3 + \lambda^2(b+\sigma+1) + \lambda(b+\sigma b+\alpha^2) + 2\sigma\alpha^2 = 0.$$

BUT $\alpha^2 = b(r-1)$ SO THAT λ IS A ROOT OF $f(\lambda)=0$ WHERE

$$f(\lambda) \equiv \lambda^3 + \lambda^2(b+\sigma+1) + \lambda b(r+\sigma) + 2\sigma b(r-1)$$

(ii) NOW $f(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - [\lambda_1+\lambda_2+\lambda_3]\lambda^2 + \dots - \lambda_1\lambda_2\lambda_3.$

THUS

$$\left.\begin{aligned} \lambda_1+\lambda_2+\lambda_3 &= -(b+\sigma+1) \\ \lambda_1\lambda_2\lambda_3 &= 2\sigma b(1-r). \end{aligned}\right\} \quad (*)$$

NOW LET $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$ AND λ_3 REAL. THEN $(*)$ BECOMES

$\lambda_3 = -(b+\sigma+1)$

AND $\quad -\omega^2 [b+\sigma+1] = 2\sigma b (1-r)$

SO $\quad \omega^2 = \dfrac{2\sigma b (r-1)}{b+\sigma+1} \quad (+).$

NOW WE NEED 1 FURTHER EQUATION. SET $\operatorname{IM}[f(i\omega)] = -i\omega^3 + i\omega b (r+\sigma) = 0.$

THUS $\quad \omega^2 = b(r+\sigma).$

COMBINE WITH $(+)$ TO GET $\quad b(r+\sigma) = \dfrac{2\sigma b (r-1)}{b+\sigma+1}.$

NOW SOLVE FOR r: $\quad r\left[b - \dfrac{2\sigma b}{b+\sigma+1}\right] = -\dfrac{2\sigma b}{b+\sigma+1} - b\sigma$

OR $\quad r\left[1 - \dfrac{2\sigma}{b+\sigma+1}\right] = \dfrac{-2\sigma}{b+\sigma+1} - \sigma = \dfrac{-2\sigma - b\sigma - \sigma^2 - \sigma}{b+\sigma+1}$

SO $\quad r[b+1-\sigma] = -3\sigma^2 - b\sigma - \sigma^2.$

THUS $\quad r = r_{HOPF} = \dfrac{\sigma(3\sigma + b + \sigma)}{\sigma - (b+1)} \cong 24.74$ IF $\sigma = 10$, $b = 8/3$

$\omega = \sqrt{b(\sigma + r_{HOPF})}$. WE NEED $\sigma > b+1$ SO THAT $r > 0$.

(iii) EARLIER IN (ii) WE GOT $\lambda_3 = -(b+\sigma+1).$

PROBLEM 9.2.2

SHOW THAT THERE IS AN ELLIPSOIDAL REGION E OF THE FORM $rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leq C$ SUCH THAT ALL TRAJECTORIES OF THE LORENZ EQUATIONS REMAIN IN E FOR ALL TIME.

$$\begin{cases} x' = \sigma(y-x) \\ y' = rx - y - xz \\ z' = xy - bz \end{cases}$$

SOLUTION WE DEFINE

$$E \equiv rx^2 + \sigma y^2 + \sigma(z-2r)^2$$

WE WANT TO SHOW THAT $dE/dt = \nabla E \cdot \underline{x}' < 0$, ON E = CONSTANT PROVIDED THAT THERE IS SOME CONSTRAINT THAT (x,y,z) IS A POINT SUFFICIENTLY FAR FROM THE ORIGIN.

WE CALCULATE

$$\nabla E \cdot \underline{x}' = 2rxx' + 2\sigma yy' + 2\sigma(z-2r)z'$$

$$= 2rx\left[\sigma(y-x)\right] + 2\sigma y\left[rx - y - xz\right] + 2\sigma(z-2r)(xy-bz)$$

$$= \underline{2r\sigma xy} - 2r\sigma x^2 + \underline{2r\sigma xy} - 2\sigma y^2 - 2\sigma xyz + 2\sigma xyz - 2\sigma bz^2 \underline{- 4\sigma rxy} + 4\sigma rbz$$

$$= -2r\sigma x^2 - 2\sigma y^2 - 2\sigma bz^2 + 4\sigma rbz$$ (underlined terms cancel).

THUS $$\frac{dE}{dt} = \nabla E \cdot \underline{x}' = -2\sigma\left[rx^2 + y^2 + b(z^2 - 2rz)\right]$$

NOW COMPLETE THE SQUARE

$$\frac{dE}{dt} = \nabla E \cdot \underline{x}' = -2\sigma\left[rx^2 + y^2 + b(z-r)^2 - r^2 b\right]$$

THUS ON E = CONSTANT, WE HAVE $dE/dt < 0$ PROVIDED THAT

$$(*) \quad rx^2 + y^2 + b(z-r)^2 > r^2 b.$$

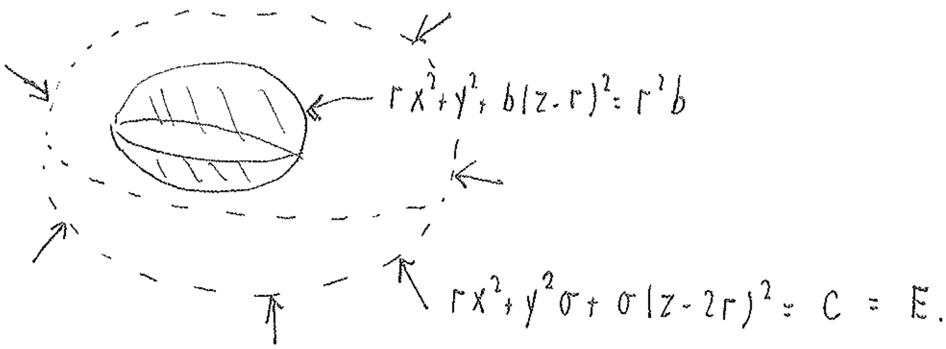
THUS ON ELLIPSE E = CONSTANT THE FLOW IS DIRECTED INWARDS PROVIDED THAT WE SATISFY $(*)$.

WE DEFINE THE ELLIPSE F BY

$$F = \{ (x,y,z) \mid rx^2 + y^2 + b(z-r)^2 - r^2 b = 0 \}.$$

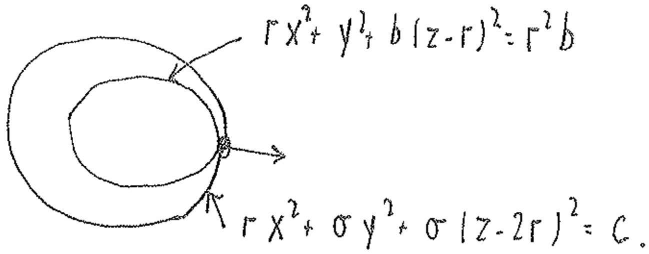
NOTICE THAT F GOES THROUGH THE ORIGIN $(0,0,0)$.

IN ORDER TO HAVE ON $E = C$ (CONSTANT) THAT $dE/dt < 0$, WE MUST CHOOSE C SUFFICIENTLY LARGE SO THAT THE ELLIPSOID $E = C$ LIES COMPLETELY OUTSIDE THE ELLIPSOID F; i.e. WE MUST HAVE A QUALITATIVE PICTURE AS SHOWN



CLEARLY, WE CAN CHOOSE C SUFFICIENTLY BIG SO THAT THE FLOW IS POINTING INWARDS ON $rx^2+\sigma y^2+\sigma(z-2r)^2 = C$. THE ELLIPSOID $E = C$ IS THEN A TRAPPING REGION FOR THE LORENZ SYSTEM.

TO DETERMINE THE MINIMUM VALUE OF C FOR WHICH $E = C$ IS IN REGION $rx^2+y^2+b(z-r)^2 \geq r^2 b$ WE MUST HAVE A PICTURE AS SHOWN, WHERE THE ELLIPSOIDS INTERSECT AT SOME POINT.



THE CONDITION FOR CONTACT IS CLEARLY THAT THEY INTERSECT

$$rx^2+y^2+b(z-r)^2-r^2 b = rx^2+\sigma y^2+\sigma(z-2r)^2 - C$$

AND OUTWARD NORMALS ARE PARALLEL.

$$\nabla\left(rx^2+y^2+b(z-r)^2-r^2 b\right) = K\,\nabla\left(rx^2+\sigma y^2+\sigma(z-2r)^2 - C\right)$$

SO $$r x^2 + y^2 + b(z^2 - 2zr + r^2) - r^2 b = r x^2 + \sigma y^2 + \sigma(z^2 - 4zr + 4r^2) - C \qquad (1)$$

$$2 r x = 2 K r x \qquad (2)$$

$$2 y = 2 K \sigma y \qquad (3)$$

$$2 b (z - r) = 2 \sigma K (z - 2r) \qquad (4)$$

THE SOLUTION TO (2) WITH $X \neq 0$ IS $K = 1$ (NOTE $X = 0$ WILL NOT WORK).

THEN (3) YIELDS $y = 0$.

THE SOLUTION TO (4) IS

$$b(z - r) = \sigma(z - 2r) \longrightarrow (b - \sigma) z = br - 2r\sigma$$

SO $$z = \frac{r(b - 2\sigma)}{b - \sigma}. \qquad (5)$$

FINALLY EQUATION (1) YIELDS

$$y^2 + bz^2 - 2zbr = \sigma y^2 + \sigma(z^2 - 4zr + 4r^2) - C$$

WHICH GIVES $$C = (\sigma - 1) y^2 + (\sigma - b) z^2 + 4\sigma r^2 + 2zbr - 4z\sigma r. \qquad (6)$$

FINALLY PUTTING (5) INTO (6) WE OBTAIN $C = C_{MIN}$, WHERE

$$C_{MIN} = \frac{r^2 (b - 2\sigma)(b + 2\sigma - 2)}{\sigma - b} + 4r^2\sigma.$$

THUS IF $C > C_{MIN}$, WE CONCLUDE THAT THE LORENZ FLOW IS INWARD ON THE ELLIPSE $E = C$, WHERE $E = rX^2 + \sigma y^2 + \sigma(z - 2r)^2$.