MATH 345: LAB 2, SADDLE NODE BIFURCATIONS

Purpose: The purpose of this lab is to use **xppaut** to explore the dynamics and the bifurcation diagrams associated with a logistic growth model subject to predation. The following ODE which models the growth and predation of the budworm population is originally due to Ludwig (1978) and it is given in dimensionless form by

$$\frac{dx}{dt} = rx(1 - x/\kappa) - x^2/(1 + x^2)\,, \qquad x(0) = x_0$$

Here $r > 0$ is the growth rate for small x, x is the budworm density and $\kappa > 0$ is the carrying capacity. The model is described and partially analyzed in section 3.7 of Strogatz's book.

Loading the ODE system: Please copy and paste the file *insect.ode* from my website. It has the form

```
# insect.ode
dx/dt = r*x*(1-x/kap) - x*x/(1+x*x)
param r=.70, kap=12.0
@ total=15, xhi=15, xlo=0.0, yhi=20.0, ylo=0.0
done
```

The parameters are r and κ and their values are as indicated. Either of these values can be changed at the terminal when using **xppaut**.

Calling the Program: To invoke the program follow the steps indicated

1. In your main directory, type *xppaut* and hit *return.* (this calls the executable)
2. You will then get a window that appears where you can click on the file harvest.ode that you wish to load.
3. At this stage XPPAUT opens a big window with MANY options should appear. We will examine only a few of the large number of options in this lab.

In this lab we will use the commands illustrated in Lab 1 and we will focus on some extra features of **xppaut** including:

1. initial conditions with *range* option to get a quick glimpse at the trajectories (to save for plotting one would have to use the freeze coomands after each trajectory as exhibited last week).
2. the window zoom option to focus on a particular limited range for the initial conditions (this is important when there is a wide scale to view, as occurs in this problem).
3. the singularity option which allows one to determine equilibrium points at a fixed value of the parameter provided a good initial guess is available (this initial guess is entered after viewing the trajectories).
4. the file command with *auto* option which allows one to compute branches of equilibrium solutions as a function of a parameter.

1. TUTORIAL OF NEW COMMANDS

Computing Range of Trajectories: Click on the *Initialconds* option and click on the *range* suboption. Choose the values *Start: 0* and *End: 20.* Click on *ok.* This specifies that *20 (=steps)* trajectories will be computed and that the spacing between initial conditions will be uniformly distributed between the starting and ending values. Although you cannot freeze these trajectories for plotting, the resulting plot indicates that there is a stable equilibrium point at $x \approx 10$. These trajectories cannot be frozen for plotting so you will have to freeze individual trajectories as in Lab 1 to obtain a hardcopy (Do this later).

Computing Equilibrium Values: Let's compute the equilibrium value satisfying $x \approx 10$ as can be seen from the plot above. Double click on the *ICs* Icon in the lower part of the screen. Change the value to 10 and replace the Icon using the *minimize* option (recall how this was done last week). This will set an initial guess for the equilibrium value of x. Next, click on the *Sing pts* option and click on the *Go X* suboption. This will invoke Newton's method which will use the initial guess $x = 10$ in trying to compute the equilibrium point. It will ask you to print eigenvalues and invariant sets. Choose *No* for both. At this stage a new Icon called *Equil* should have appeared. Double click on this Icon and record on a piece of paper the equilibrium value of x, which should be $x = 10.361$ (recall $r = .70$). This value gives a point on the bifurcation diagram of x_{eq} versus r, and will be used later. Replace the *Equil* Icon.

Magnifying a Window: Let's check whether there is an equilibrium solution for x small. To see this, we have to magnify the window. To do so, click on the *Window/zoom* option. Click on the *window* suboption. Change *Y Hi* from 20 to 5. This re-scales the vertical axis. Click on *ok.* Click on the *Erase* command to kill all previous trajectories. Now click on *Initialconds* option with the *range* suboption and take the *End:* value to be 5 (and not 20). Click on *ok.* The trajectories that are drawn indicate that there is no equilibrium point for x small and non-zero. *Erase* these trajectories and go back into the *Window/zoom* option to put the vertical scale back at 20.

Freezing Trajectories: With the vertical scale now at 20, take some initial conditions one at a time, freeze each one as shown last week and save the plot with the label $r = .7$, $\kappa = 12$ as a *.ps file* in your directory. Label the horizontal and vertical axes appropriately as indicated last week. A suggested name for the file is *insectr7.ps.* **Note: do not try to put in the file in odes/ subdirectory as this is my directory and there is no write permission**.

Changing the r parameter: Double click on the *par* Icon and change the r parameter to $r = .4$ from $r = .7$. Then you can use *Initialcond* option with the *range* suboption to get a quick glimpse at the trajectories for this case. Now freeze individual trajectories one at a time to get plots of $x(t)$ versus t. Save and print this off as a .ps file. (note: you may need to use the *Windowzoom* option to see what is happening for x small. It might be nice to have a separate .ps file indicating what goes on for x small). Repeat the procedure above for the case $r = .2$ and obtain a hardcopy.

Computing the Bifurcation Diagram: Click on the *file* option and choose the *auto* suboption.

A new window with new options should appear. It is here that we will obtain a plot of the equilibrium solutions versus r (i.e. the bifurcation diagram). Click on the *parameter* option. It will indicate that r is the primary parameter and κ is the secondary parameter. Click on *ok*. Now click on the *axes* option and choose the *Hi* suboption. This is where we can specify the size of the window. The horizonatal (x) axis will be r and should range from 0.0 to .75. The vertical (y) axis denoting the equilibrium solutions should range from 0 to 11. Insert these bounds and click on *ok* when done. Next click on the *Numerics* option. This is where we specify parameters for computing the bifurcation curve.

1. Nmax is the number of steps to be taken along the curve (choose 500).
2. Results will be displayed at the terminal after every NPr steps. (keep it at 50)
3. Ds is the suggested arclength step along the curve. It could be positive or negative depending on which direction the curve is being traversed. (Choose Ds=-.02).
4. Dsmin, Dsmax (both always positive) tell the program the minimum and maximum values of $|Ds|$ that it can compute with. Choose Dsmin=.01 and Dsmax=.030.
5. Now set Parmin=0.0 and Parmax=.75 to indicate the range of r to be computed with.
6. Now set Normmin=0.0 and Normmax=11.0 which indicates the range in the equilibrium values to be computed with. Click on *ok* after these changes have been made.

Now we need to set an initial point on this bifurcation curve. From the calculation above with the *Singpts* option we know that $x = 10.36$ when $r = .7$ and $\kappa = 12$. Use the Icons *ICs* and *Par* to fix these initial values.

Finally we are ready to see something. Choose the *run* option with the *steady state* suboption. An $S - shaped$ curve should appear. This is the bifurcation diagram. Notice the occurrence of two saddle-node bifurcations. The *solid* portions of this curve indicate stable equilibrium solutions, while the lighter middle branch of this curve indicates unstable equilibrium points. The value of r in the Icon should have changed as the curve was traversed. Click on the *file* option with the *postscript* suboption and save this plot in a file. I suggest that you name this file *bifinsect.kap12.ps.*

To quit: Click on the *File* command and then click on the *quit* option in the submenu and choose *yes.*

Printout: To get a printout of the file *yourfile.ps* simply type *lpr yourfile.ps* at the unix command line. You can pop open a unix window at any time to plot any *.ps* file.

2. THE SPECIFIC LAB PROJECT

The Budworm Model:

$$\frac{dx}{dt} = rx(1 - x/\kappa) - x^2/(1 + x^2)\,, \qquad x(0) = x_0$$

1. Compute trajectories of $x(t)$ versus t for a range of initial conditions for the three cases: i) $r = .70$, $\kappa = 12$; ii) $r = .40$, $\kappa = 12$; iii) $r = .20$, $\kappa = 12$. Label the graphs appropriately and indicate the differences in the behavior of the trajectories for each of these cases. (You may need an extra magnified plot for the cases where $r = .40$ and $r = .20$ to illustrate convincingly that there are equilibrium solutions where x is quite small).
2. For $\kappa = 12$, use the *auto* option of **xppaut** to compute the bifurcation diagram exhibiting the equilibrium solutions versus the parameter r for r on the range $0 < r < .75$. Obtain a similar bifurcation diagram for the cases where $\kappa = 20$ and $\kappa = 5$.
3. Write a few paragraphs explaining these pictures in the context of the budworm problem described in Strogatz. In particular, how is the tendency towards budworm infestation affected by changes in r and k? What do the bifurcation diagrams tell us qualitatively?