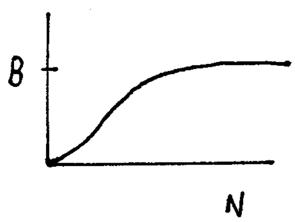
EXAMPLE (INSECT OUTBREAK)

$$\frac{dN}{dt} = RN\left(1 - N/K\right) - P(N)$$



$$P(N) = \frac{BN^2}{A^2 + N^2}.$$

LAB 2 SOLUTIONS

r grow rate for small x

K foliage left on trees (carrying capacity)

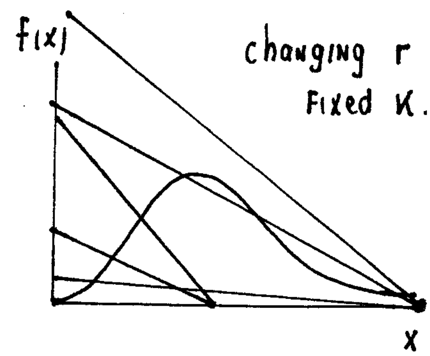
NOW LET $N = Ax$, $t = T\tau$

$$\frac{A}{T}x' = RAx\left(1 - \frac{A}{K}x\right) - B\frac{x^2}{1+x^2}$$

$$\frac{A}{BT}x' = x\frac{RA}{B}\left(1 - x/K\right) - \frac{x^2}{1+x^2}$$

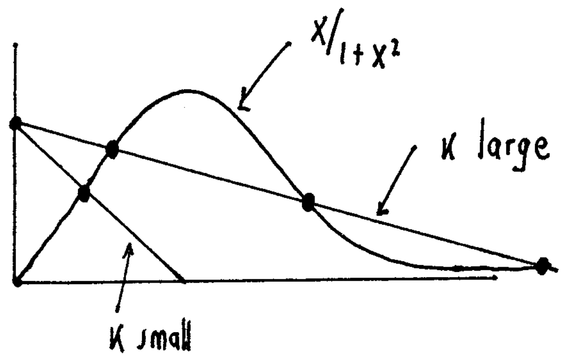
LET $T = A/B$, $r = RA/B$ AND $K = K/A$.

THEN $$\frac{dx}{d\tau} = rx\left(1 - x/K\right) - x^2/_{1+x^2} = f(x)$$



THUS $\left.\begin{array}{l} r(1 - x/K) = x/_{1+x^2} \\ x = 0 \end{array}\right\}$ EQUILIBRIA

changing K



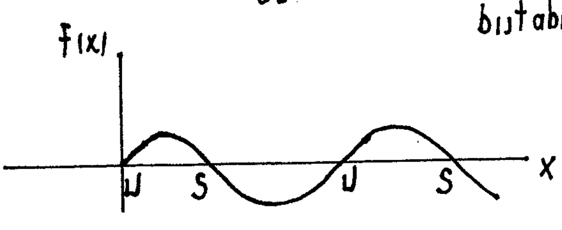
plot $r(1 - x/K)$ AND $x/_{1+x^2}$

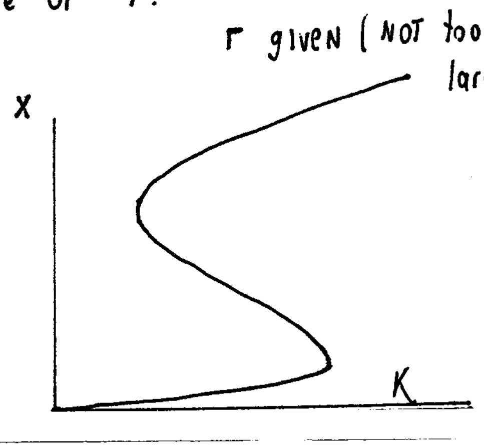
- K SMALL → 1 EQ. POINT
- K large → three equilibrium points

} certain range of r.

NEAR $x = 0$ $\frac{dx}{dt} \approx rx$ $x = 0$ is UNSTABLE.

bistability range





UPPER BRANCH : OUTBREAK OF INSECTS

LOWER BRANCH : "SAFE" LEVEL OF INSECTS.

THUS IF r IS FIXED AND K IS TOO LARGE (carrying capacity), THEN THERE CAN BE AN OUTBREAK OF INSECTS.

SOLVE FOR K TO GET $$K = \frac{rX(1+X^2)}{r(1+X^2) - X}$$

THE PLOTS ARE SHOWN.

NOW WHERE IN (r, K) PARAMETER space is there bistability?

THE CONDITIONS FOR A FOLD POINT ARE $f(X)=0$, $f'(X)=0$

$$r(1 - X/K) = X/_{1+X^2}$$

$$\frac{d}{dx}\left[r(1-X/K)\right] = \frac{d}{dx}\left[\frac{X}{(1+X^2)}\right]$$

THIS YIELDS, $$-\frac{r}{K} = \frac{1-X^2}{(1+X^2)^2} \quad \text{so} \quad r = -\frac{K(1-X^2)}{(1+X^2)^2}$$

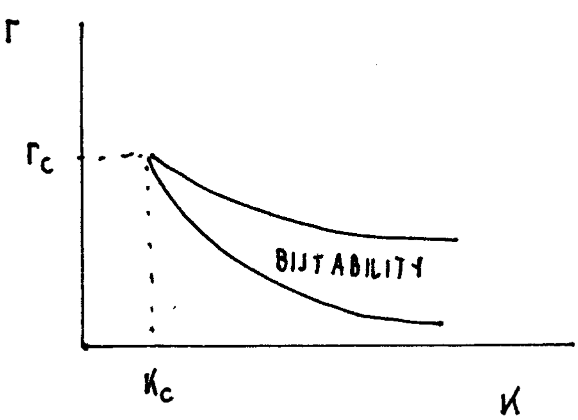
$$r(1 - X/K) = X/_{1+X^2}$$

NOW $$r - \frac{r}{K}X = \frac{X}{1+X^2} \longrightarrow r + \frac{X(1-X^2)}{(1+X^2)^2} = \frac{X}{1+X^2} \longrightarrow r = \frac{X}{1+X^2} - \frac{X(1-X^2)}{(1+X^2)^2}$$

THIS GIVES $$\left.\begin{aligned} r &= \frac{2X^3}{(1+X^2)^2} \\ K &= \frac{2X^3}{X^2-1} \end{aligned}\right\} \quad X \geq 1 \qquad K = r\frac{(1+X^2)^2}{X^2-1} \longrightarrow \text{gives } K$$

THIS GIVES A PARAMETRIC CURVE as shown

$$K \to \begin{cases} \infty & \text{AS } X \to 1^+ \\ \infty & \text{AS } X \to \infty \end{cases}$$

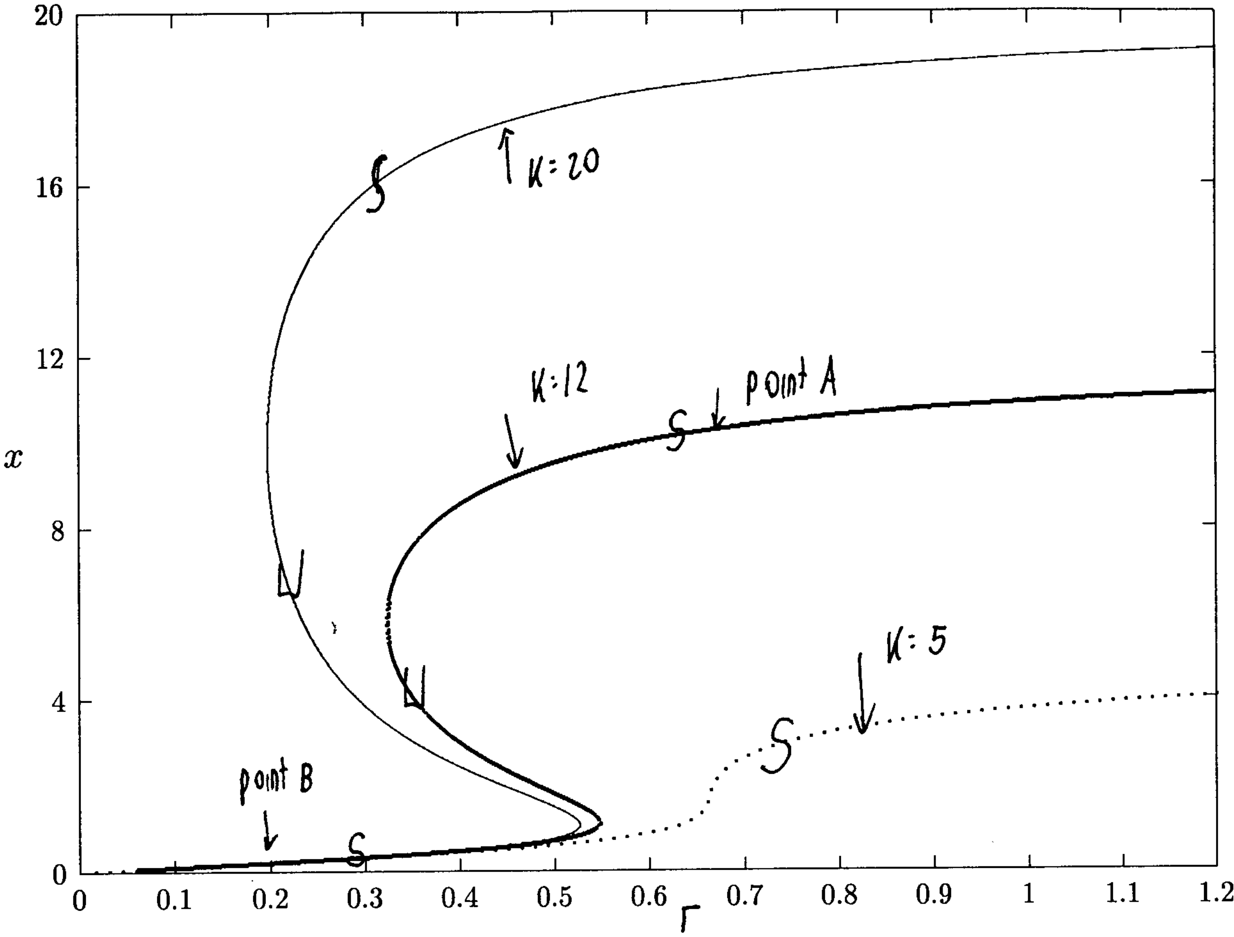


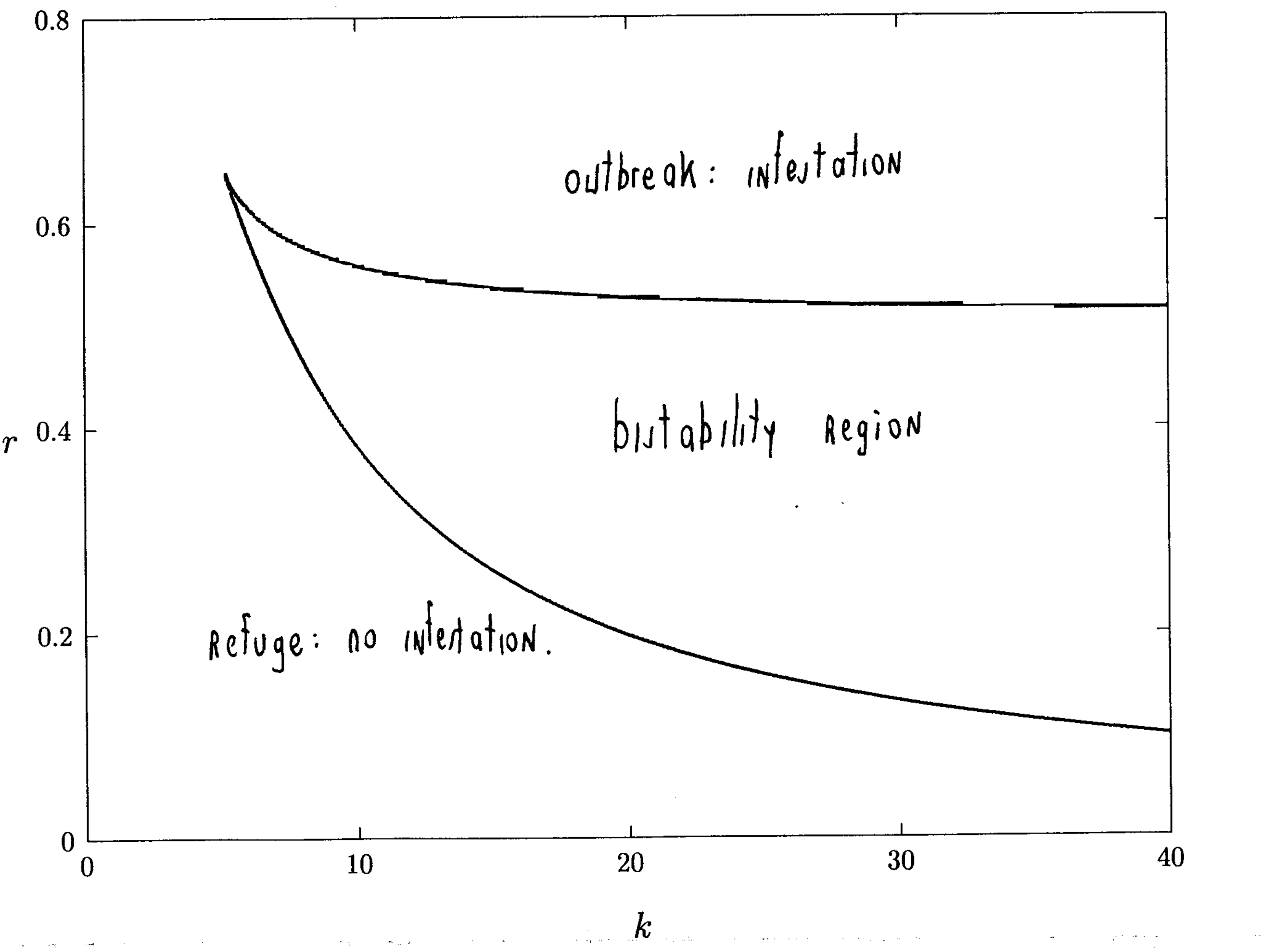
NOW $$K = 2X \mp \frac{2X}{1 \mp X^2}$$

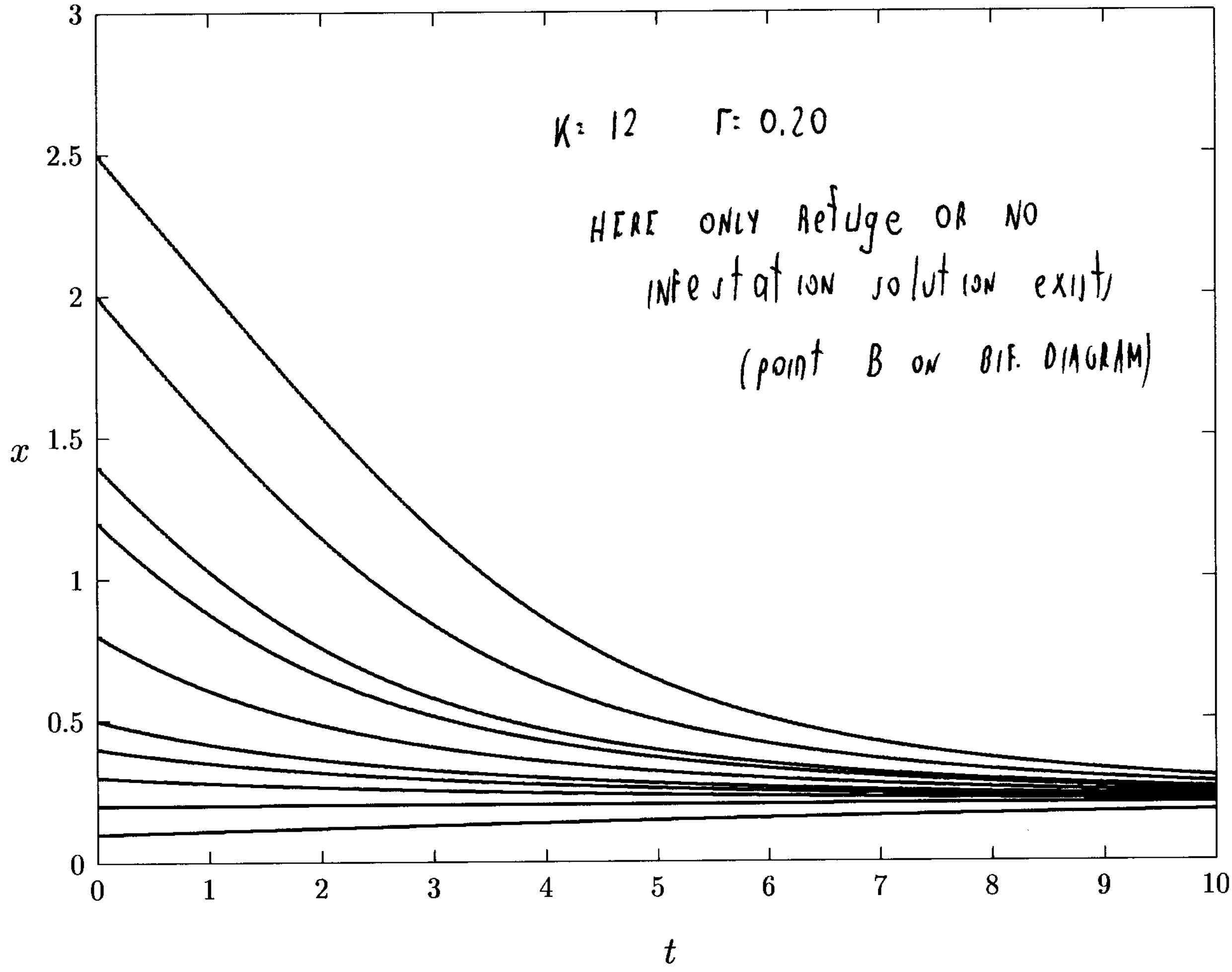
$$K_X = 2 + \cancel{} - \frac{2(1+X^2)}{(1-X^2)^2}$$

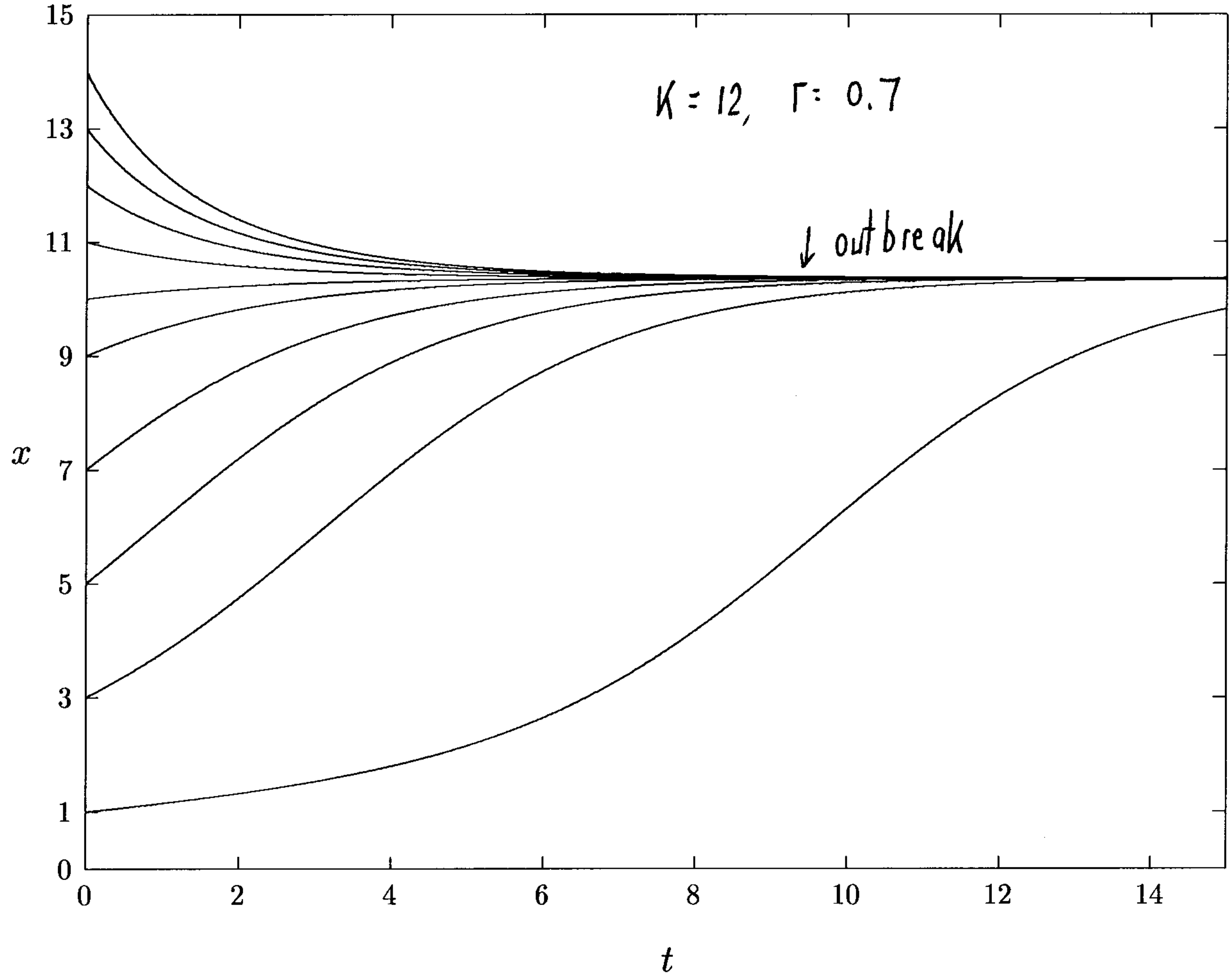
$K_X = 0$ when $X = \sqrt{3}$

THUS $r_c = 3\sqrt{3}/8 \quad K_c = 3\sqrt{3}$.

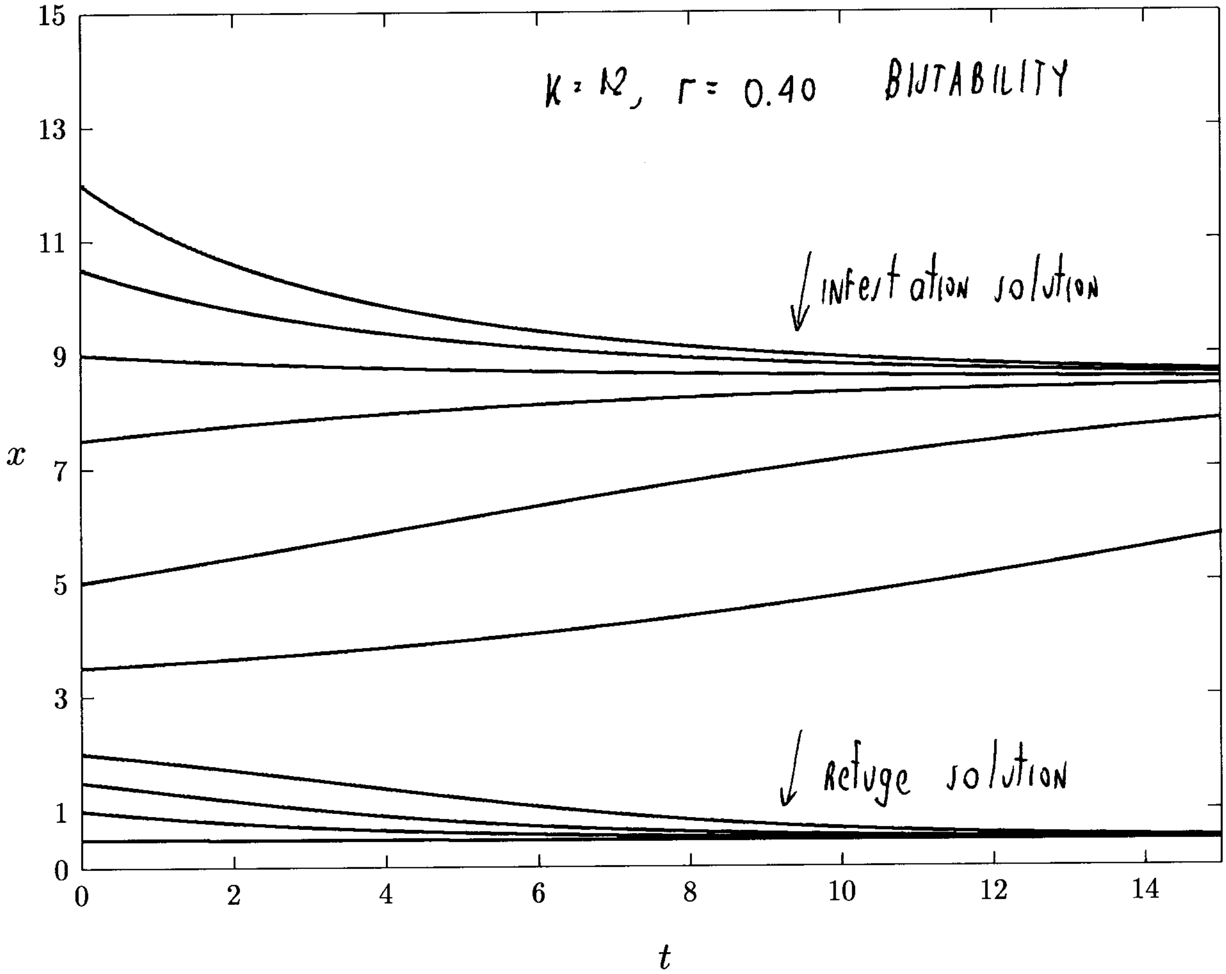

20
16
12
8
4
0
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
Γ
K=20
K=12
point A
K=5
point B
S
U


0.8
0.6
0.4
0.2
0
r
0
10
20
30
40
k
outbreak: infestation
bistability region
Refuge: no infestation.


K= 12 r= 0.20
HERE ONLY Refuge OR NO infestation solution exists
(point B on BIF. DIAGRAM)
x
t
0
0.5
1
1.5
2
2.5
3
0
1
2
3
4
5
6
7
8
9
10



HERE ONLY THE "INFESTATION" SOLUTION EXISTS
(POINT A ON BIF. DIAGRAM)



K = 12, r = 0.40 BISTABILITY
infestation solution
refuge solution
x
t