LAB 3 SOLUTIONS

PROBLEM 1 THE SIMPLE PENDULUM IS

$$\varphi'' + \mu \varphi' + \sin\varphi = 0$$

(i) FOR $\mu = 0$ (NO damping) we have

$$\varphi'' + \sin\varphi = 0 \quad \text{OR} \quad \frac{\varphi'^2}{2} + 1 - \cos\varphi = E. \qquad E = \text{CONSTANT.}$$

WE PLOT THE PHASE-PLANE BELOW USING XPPAUT.

THE GRAPH OF φ VS t WITH $\varphi(0) = \varphi_0$ AND $\varphi'(0) = 0$ SHOWS THAT THE PERIOD T DEPENDS ON φ_0 AND HAS

$$dT/d\varphi_0 > 0 \qquad T \to +\infty \quad \text{AS} \quad \varphi_0 \to \pi^-.$$

NOW AS DERIVED IN CLASS THE PERIOD IS

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \qquad k = \sin(\varphi_0/2)$$

AS $\varphi_0 \to \pi^-$, $k \to 1^-$ AND THE INTEGRAL DIVERGES.

TO DETERMINE HOW IT DIVERGES LET $\varphi = \pi + V$ AND LINEARIZE

$$V'' + \sin(\pi + V) = V'' - V + \ldots = 0$$

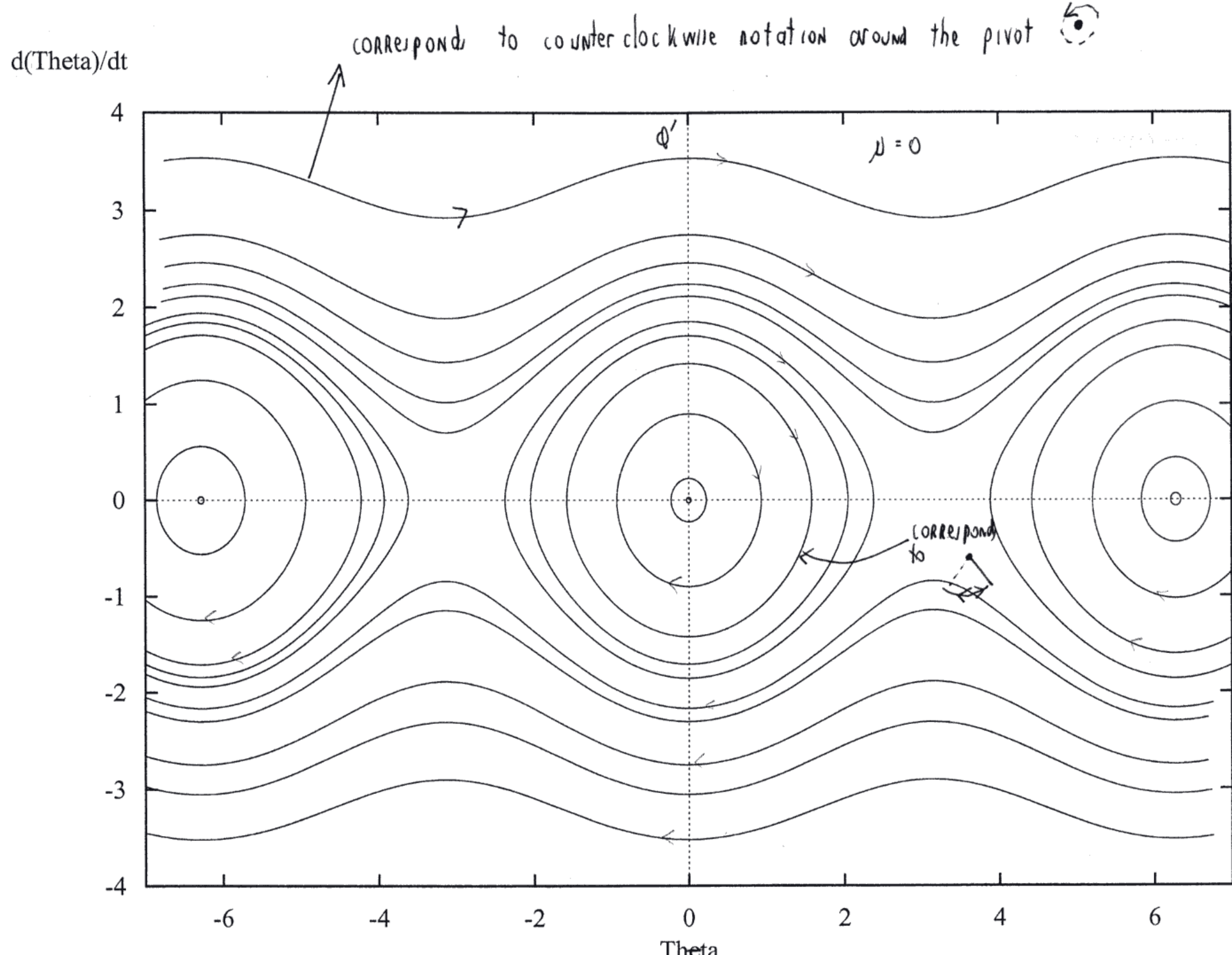
SO $\quad V = Ae^{-t} + Be^{t}.$

NOW SUPPOSE WE ESTIMATE T BY HOW LONG IT STAYS NEAR $\varphi = \pi$. SO USING DECAYING EXPONENTIAL,

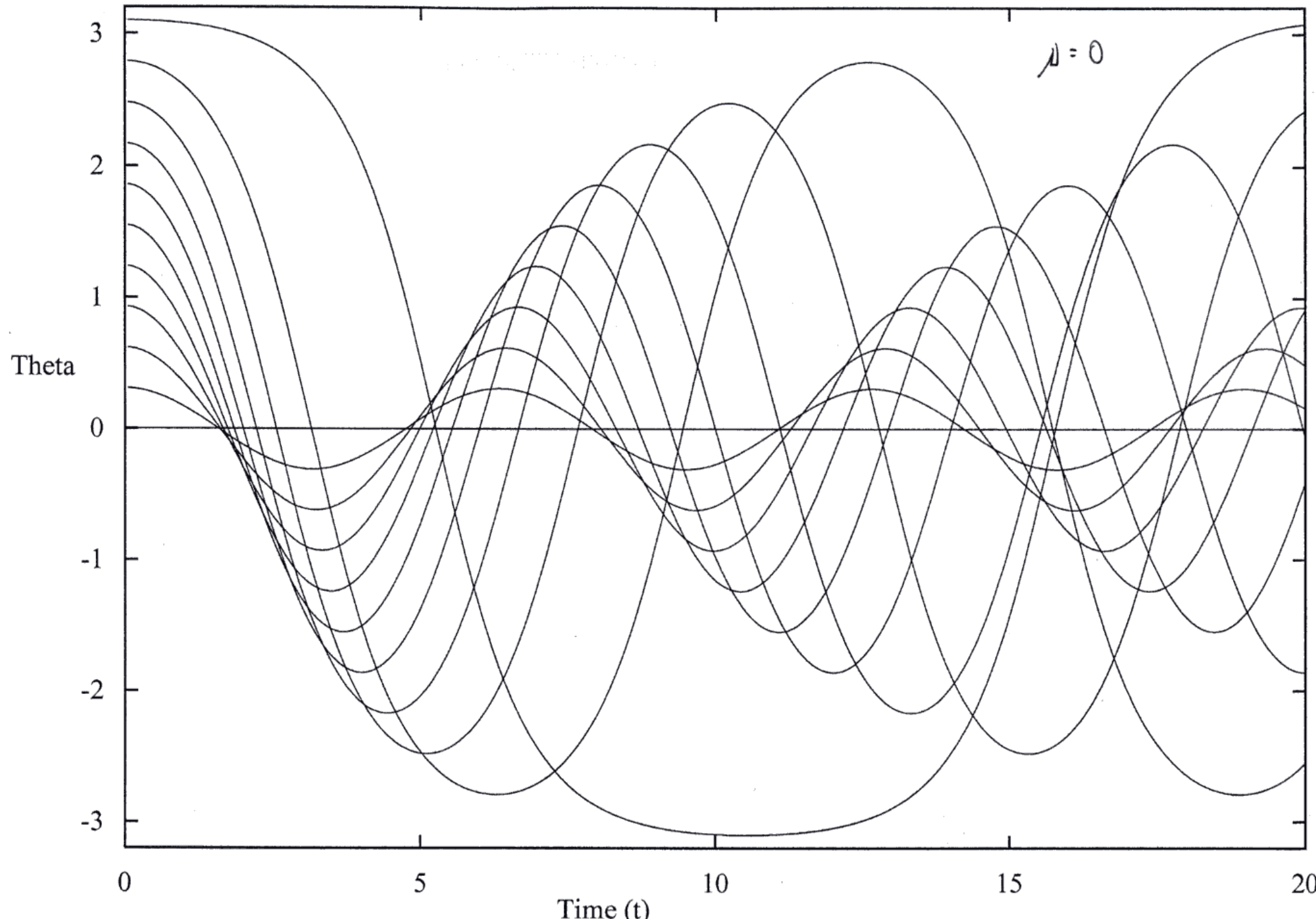
$$\varphi \sim \pi + Ae^{-T} \quad \text{OR} \quad T = -\log\left|\frac{\pi - \varphi}{A}\right| \sim -\log(\pi - \varphi) + O(1)$$

THUS THE PERIOD HAS $T(\varphi_0) \sim -\log(\pi - \varphi_0)$ AS $\varphi_0 \to \pi^-$.

LOGARITHMIC BEHAVIOR.



correspond to counterclockwise rotation around the pivot
d(Theta)/dt
Theta
ϕ'
$\mu = 0$
correspond to
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6



μ = 0
Theta
Time (t)
0
5
10
15
20
3
2
1
0
-1
-2
-3

(ii) NOW CONSIDER THE DAMPED PENDULUM WITH

$$\varphi'' + \mu \varphi' + \sin\varphi = 0. \qquad \mu = 0.4.$$

MULTIPLYING BY φ' we get

$$\frac{d}{dt} E = -\mu \varphi'^2 \qquad E = \frac{1}{2}\varphi'^2 + 1 - \cos\varphi \ \text{(total energy)}$$

WE SHOW THE FOLLOWING PLOTS AND IN EACH PLOT INDICATE THE PHYSICAL MOTION OF PENDULUM

- φ vs t FOR $\mu = 0.4$ $\quad \varphi \to 0$ AS $t \to \infty$ FOR CERTAIN INITIAL CONDITIONS. NOT ENOUGH ENERGY TO GO OVER THE TOP.
- A PLOT OF dE/dt vs t showing $E \to 0$ AS $t \to \infty$
- THE PHASE PLANE
- PHASE PLANE φ' vs φ, SHOWING TRAJECTORIES THAT GO OVER THE TOP ONCE EITHER IN CLOCKWISE OR COUNTERCLOCKWISE ORIENTATION

NEAR $(0,0)$ WE HAVE (OR $(0, \pm 2n\pi)$)

$$\varphi'' + \mu\varphi' + \varphi \hat{=} 0$$

so $\varphi = e^{\lambda t} \to \lambda^2 + \mu\lambda + 1 = 0$

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2} \qquad \text{SPIRAL (STABLE) since } \mu = 0.4$$

eigenvalues are $.780, -1.180$

NEAR $(0, n\pi)$ $\quad n = \pm 1, \pm 3 \ldots$

THEN $\sin(n\pi + \varphi) \sim \cos(n\pi)\varphi + \ldots$

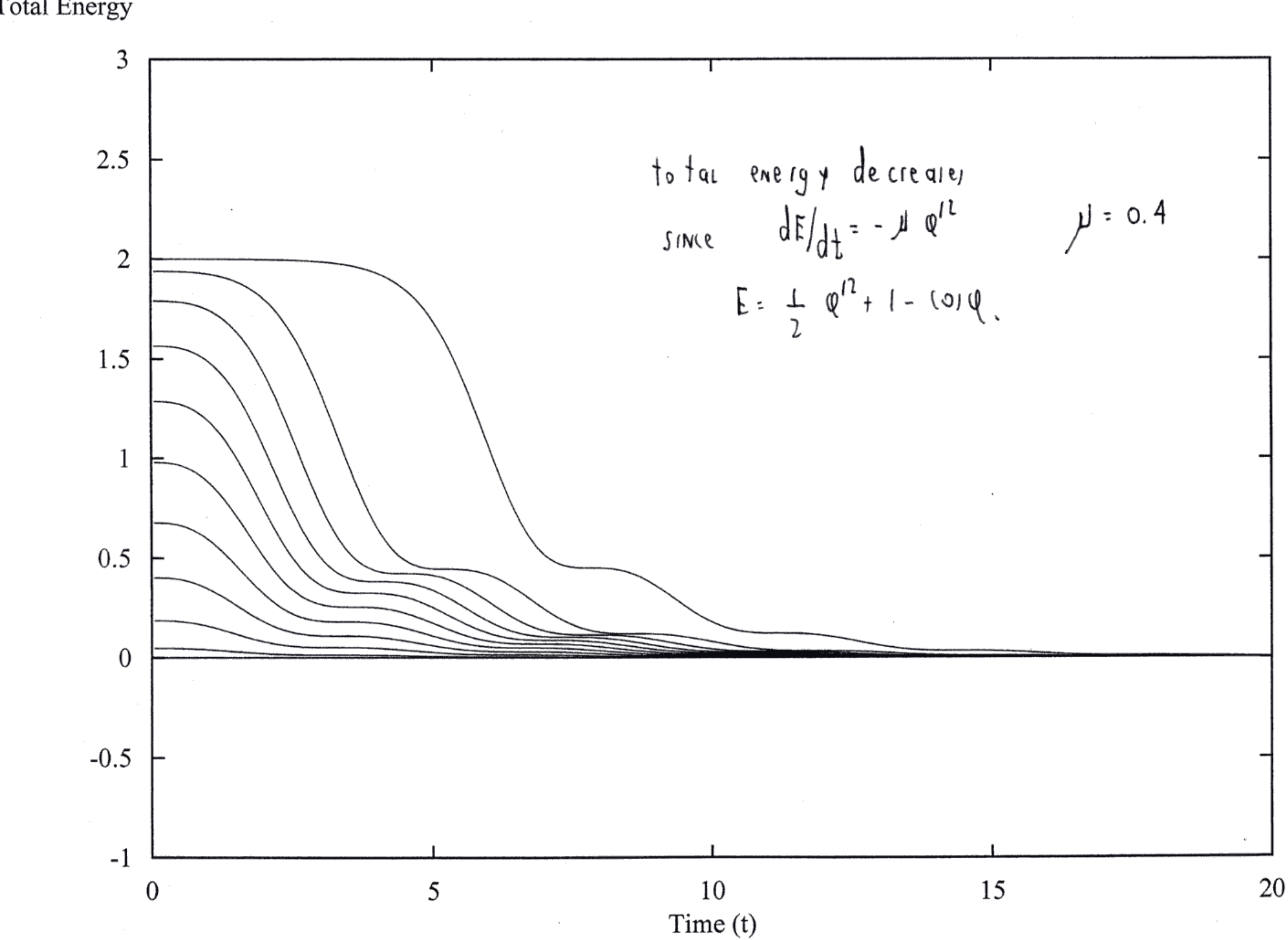
THUS $\varphi'' + \mu\varphi' - \varphi = 0$ linearized equation.

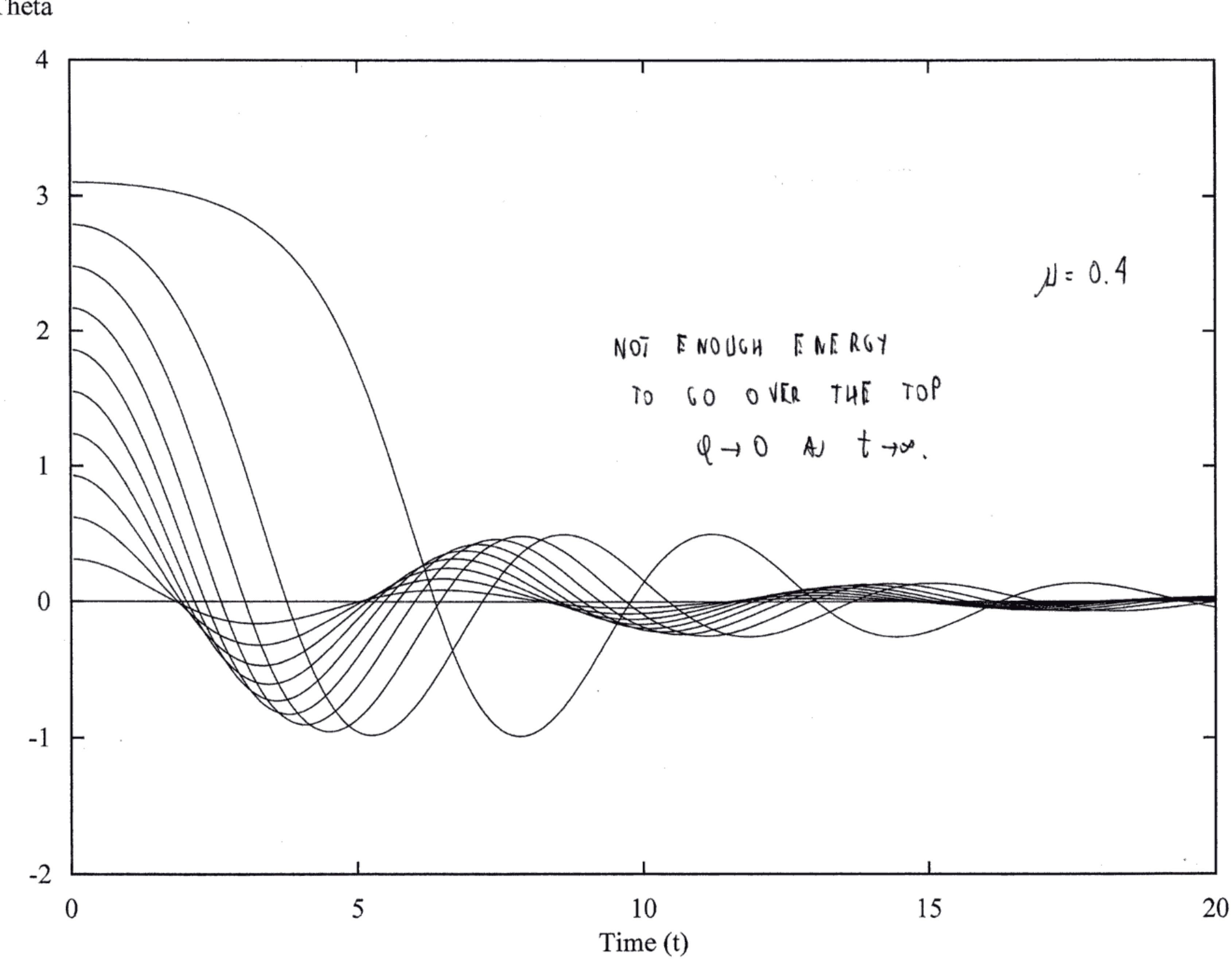
Let $\varphi = e^{\lambda t}$

$$\lambda^2 + \mu\lambda - 1 = 0 \to \lambda = \frac{-\mu \pm \sqrt{\mu^2 + 4}}{2}$$

$\to$ saddle point.

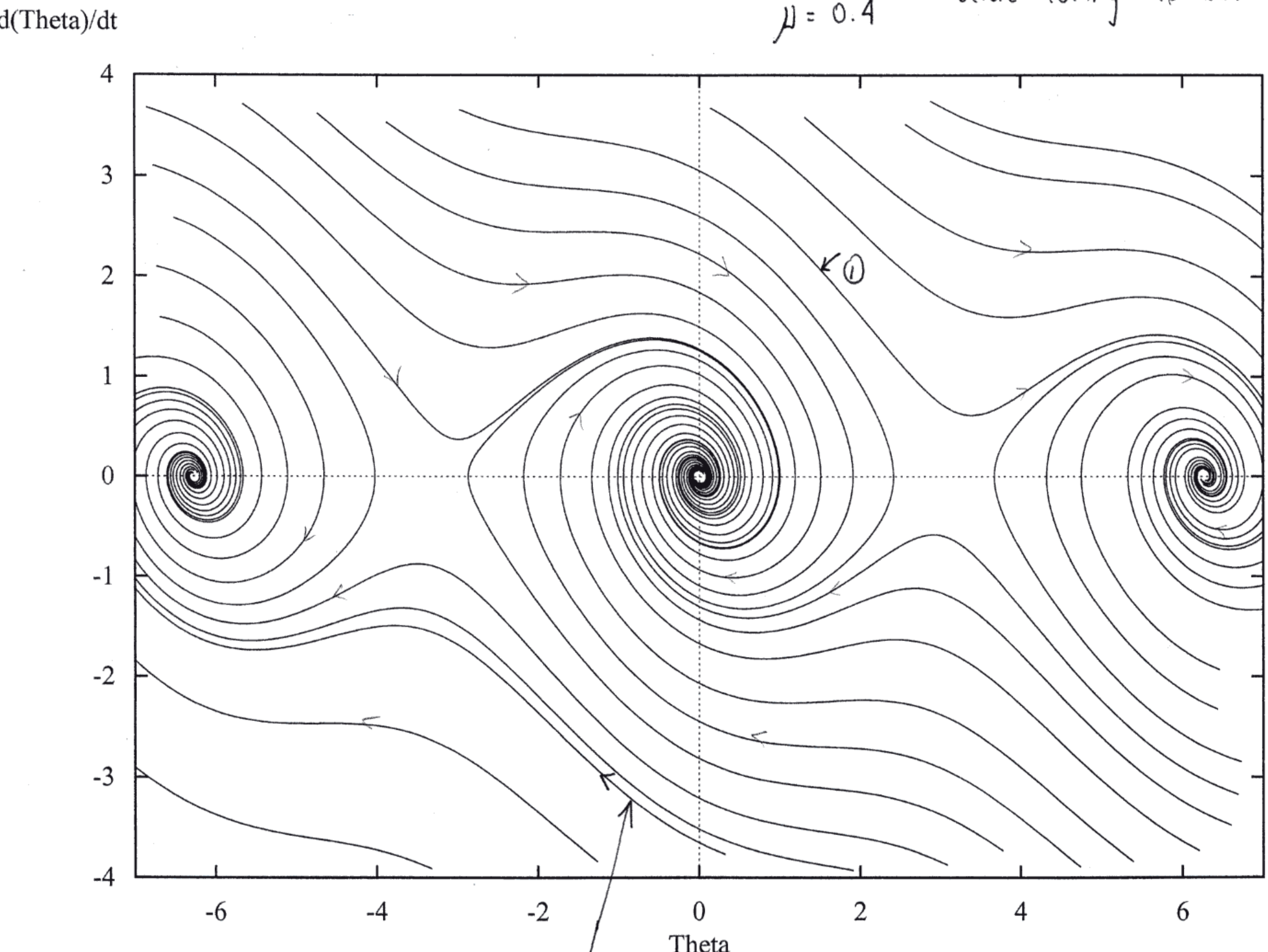
when $\mu = 0.4$ $\quad \lambda = .820, -1.220.$


Total Energy
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
Time (t)
total energy decreases
since $dE/dt = -\mu \varphi'^2$
$E = \frac{1}{2} \varphi'^2 + 1 - \cos\varphi$.
$\mu = 0.4$


heta
4
3
2
1
0
-1
-2
0
5
10
15
20
Time (t)
$\mu = 0.4$
NOT ENOUGH ENERGY
TO GO OVER THE TOP
$\varphi \to 0$ AS $t \to \infty$.

① trajectory make one (counterclockwise) revolution over the top before coming to rest

$\mu = 0.4$



② trajectory make one clockwise revolution over the top before coming to rest.

PROBLEM 2

$$v' = -\sin\varphi - Dv^2 = f$$
$$v\varphi' = -\cos\varphi + v^2 = g$$

$v \geq 0$. v = speed.

$0 \leq \varphi < 2\pi$

(i) suppose $D = 0$. Then,

$$v' = -\sin\varphi \qquad \frac{dv}{d\varphi} = \frac{-v\sin\varphi}{v^2 - \cos\varphi}$$
$$v\varphi' = v^2 - \cos\varphi$$

Now $\quad v^2\,dv = \cos\varphi\,dv - v\sin\varphi\,d\varphi = d(v\cos\varphi)$

Hence $\quad \frac{1}{3}v^3 - v\cos\varphi = C.$

Thus conserved quantity is $\quad E = v^3 - 3v\cos\varphi.$

Equilibria are at $\quad v' = 0 \longrightarrow \varphi = 0$ or $\pm\pi$

$\varphi' = 0 \longrightarrow \cos\varphi = v^2$

Thus only equilibrium is when $\varphi_E = 0$ and $v_E = 1$.

Now compute Jacobian:

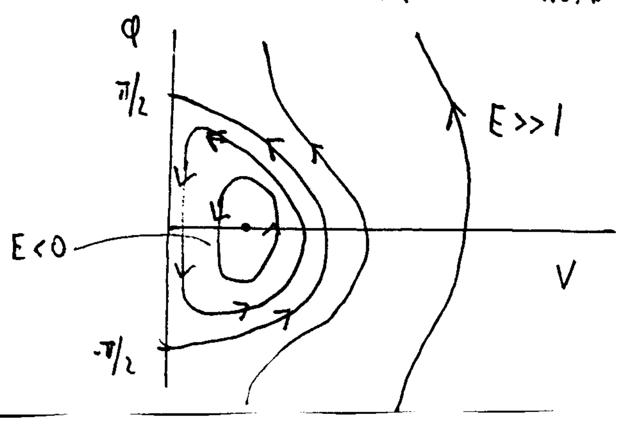
$$J = \begin{pmatrix} f_v & f_\varphi \\ g_v & g_\varphi \end{pmatrix} = \begin{pmatrix} 0 & -\cos\varphi \\ 1 + \frac{\cos\varphi}{v^2} & \frac{\sin\varphi}{v} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \quad \text{AT } (\varphi_E, v_E) = (0$$

eigenvalues are $\lambda = \pm\sqrt{2}\,i \longrightarrow (0,1)$ is a linear center.

However since the system has a conserved quantity, there are indeed periodic orbits around $\varphi_E = 0$, $v_E = 1$.

To plot phase plane we plot level lines of E. If $E = 0$, then $v^2 = 3\cos\varphi$ or $v = 0$. Now $v = 0$ implies $\varphi = \pm\pi/2$ (otherwise vector field is singular).

NOTE: $v' < 0$ in $(0, \pi/2)$



PHYSICAL INTERPRETATION

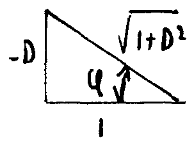
- EQUILIBRIUM $\varphi = 0,\ V = 1$ STRAIGHT LEVEL FLIGHT
- SMALL CLOSED ORBITS 
- $E > 1$  φ INCREASES STEADILY (this is what a closed orbit in V, φ plane corresponds to)

THE OBJECT DOES LOOPS IF YOU THROW TOO HARD
THIS IS THE TRAJECTORY THAT IS NOT IN THE "WELL".

NOW FOR $D > 0$ WE EXPECT A STABLE SPIRAL TO REPLACE THE CENTER (WHEN D IS SMALL) OR A STABLE NODE (WHEN D IS LARGE).

NOW SET $\dot V = 0$ TO GET $DV^2 = -\sin\varphi$. SET $\dot\varphi = 0$ SO THAT $V^2 = \cos\varphi$.

HENCE, $\tan\varphi = -D$ AT EQUILIBRIUM



$V^2 = \cos\varphi = [1+D^2]^{-1/2}$

THUS THE EQUILIBRIUM IS $\tan\varphi_e = -D$ AND $V_E = (\sqrt{1+D^2})^{-1/2} = (1+D^2)^{-1/4}$.

NOW THE JACOBIAN IS

$$J = \begin{pmatrix} -2DV & -\cos\varphi \\ 1+\cos\varphi/V^2 & \sin\varphi/V \end{pmatrix} = \begin{pmatrix} -2DV_e & -V_e^2 \\ 2 & -DV_e \end{pmatrix} \quad \text{AT EQUILIBRIUM}$$

NOW $\det J = 2D^2 V_e^2 + 2V_e^2 = 2V_e^2(1+D^2) = 2\sqrt{1+D^2} \equiv \Delta$

$\operatorname{trace} J = -3DV_e = \dfrac{-3D}{\sqrt[4]{1+D^2}} = \tau$.

$$\lambda^2 - \lambda \operatorname{Trace} J + \det J = 0$$

NOW $\tau < 0$ stable fixed point.
$\det J > 0$

we now compute $\lambda = \left(\operatorname{trace} J \pm \sqrt{(\operatorname{trace} J)^2 - 4\det J}\right)/2$.

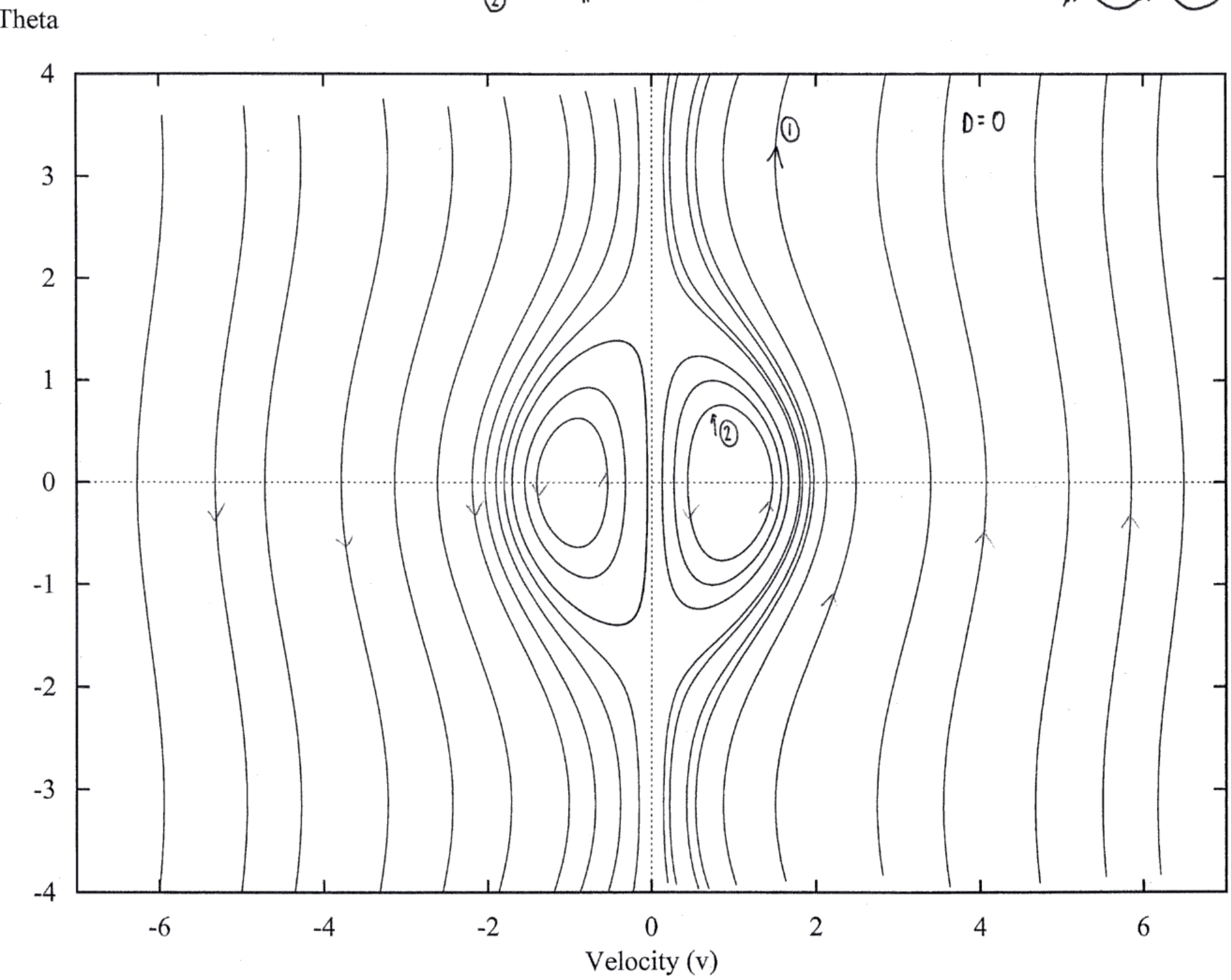
so $(\operatorname{trace} J)^2 - 4\det J = \tau^2 - 4\Delta = \dfrac{9D^2}{\sqrt{1+D^2}} - 8\sqrt{1+D^2} = \dfrac{D^2-8}{\sqrt{D^2+1}}$

so λ complex $\longrightarrow$ stable spiral when $0 < D < 2\sqrt2$
$\lambda < 0$ Real $\longrightarrow$ stable node when $D > 2\sqrt2$

$\Longrightarrow$ IN either case ($D < 2\sqrt2$ OR $D > 2\sqrt2$) the glider settles into steady flight for most initial conditions with its nose pointing down since $\varphi_e < 0$.

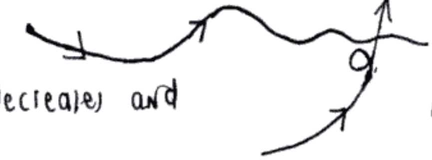
1 CORRESPONDS TO TRAJECTORY WITH FLIGHT PATH

2 " " " " " "



① actual "physical" trajectory

② corresponds to a stall. v decreases and the angle increases



but it slows down.