MATH 345: LAB 6, CHAOTIC DYNAMICS

Purpose: The purpose of this lab is to use **xppaut** to explore and visualize very complicated behavior for solutions to certain special nonlinear ODE systems in 3 dimensions. Some of the ODE's below can exhibit chaotic dynamics. Although the existence of chaos is difficult to quantify analytically, it is associated with an extreme sensitive dependence on initial conditions.

The three problems that we will look at are: The Lorenz equations; a resonant double-well potential; a model problem where the route to chaos is more easily understood. **Note: Please copy all 3 of the .ode files below to your home directory before using xppaut**.

Double-Well Potential at Resonance

The motion of a long thin elastic stell beam with its top end embedded in a pulsating frame and the bottom end hanging just above two magnets can be modeled by the forced Duffing equation

$$x^{''} + cx^{'} - x + x^3 = a\cos t$$

Note that this problem can be written as a 3-D autonomous system by introducing the variables $y = x^{'}$ and $\theta = t$, so that $\theta^{'} = 1$. The file is contained in *xpp/doubwell.ode*.

1. Use **xppaut** to load in this system.
2. Choose the damping coefficient as $c = 0.25$ and $a = 0$. Take the initial condition $x = -1.5$ at $t = 0$ with either $y = 1.6$, $y = 1.55$, $y = 1.5$, $y = 1.45$, $y = 1.4$, $y = 1.35$, and $y = 1.3$. Plot these trajectories in the (x, y) phase-plane. Notice that some trajectories are captured by the right well and some by the left well. **Turn in a plot of this picture**. The x-axis should range from $-2 < x < 2$ and the y-axis from $-2 < y < 2$. (Recall that the command ./ will halt the integration along a particular trajectory). See if you can follow the two manifolds that arise out of the origin.
3. Change the parameters to $c = .25$ and $a = .18$. In the *numerics* (*total*) option change the total time for integration to $t = 50$. Plot several solutions in the phase plane and **turn in this picture. Also turn in plots of x versus t and y versus t for a particular initial condition**. You should observe a limit cycle behavior. You may need to adjust the stepsize dt in the *numerics* option together with the *total* time.
4. Now change the parameters to $c = .25$ and $a = .25$. Change the total time for integration to *total=200* in the *numerics* option, and change the stepsize *dt* to .05. Click on *method* in

the numerics option and use *Runge-Kutta*. Use the *Viewaxes (2-D)* option to plot the $y-x$ plane on the range $|x| < 1.5$, $|y| < 1.5$ for the initial condition $x_0 = .20$ and $y_0 = .10$. **Turn in plots of the trajectory in the phase plane together with plots of the individual components x and y versus t.** You should observe a transient version of chaos for this initial condition.

5. Now change the parameters to $c = .25$ and $a = .40$. Change the total time for integration to *total=400* in the *numerics* option, and change the stepsize *dt* to .05. Click on *method* in the numerics option and use *Runge-Kutta*. Use the *Viewaxes (2-D)* option to plot the $y-x$ plane on the range $|x| < 2.0$, $|y| < 2.0$ for the initial condition $x_0 = 0.0$ and $y_0 = 0.0$. **Turn in plots of the trajectory in the phase plane together with plots of the individual components x and y versus t.** You should observe a 'chaotic' response of the system.
6. Now change the parameters to $c = .25$ and $a = .60$. Change the total time for integration to *total=400* in the *numerics* option, and change the stepsize *dt* to .05. Click on *method* in the numerics option and use *Runge-Kutta*. Use the *Viewaxes (2-D)* option to plot the $y-x$ plane on the range $|x| < 2.0$, $|y| < 2.0$ for the initial condition $x_0 = 0.0$ and $y_0 = 0.0$. **Turn in plots of the trajectory in the phase plane together with plots of the individual components x and y versus t.** You should observe a large amplitude limit cycle of the system.
7. Now we plot some Poincare sections. Use the *Viewaxes (2-D)* option to plot the $y-x$ plane on the range $|x| < 2.0$, $|y| < 2.0$. Take the total time to be 10000.0 and click on *graphics edit curve* and select linetype 0, which will plot points instead of curves. Now click on *numerics Poincare* and click on *section*. Choose the variable to be T and section to be $2\pi = 6.283185$ and direction to be 1. Keep the stop on section command the same. Then, take the parameter values to be either $c = .25$, $a = .25$ where we observed transient chaos or $c = .25$, $a = .40$, where chaos was evident. Choose the initial condition $x = .2$, $y = .1$ in each case and **turn in plots of the Poincare map**.
8. **In a few paragraphs discuss summarize your experimental observations on this system**

Lorenz Equations

The Lorenz equations, which were formulated in the 1960's in connection with a toy mathematical model of the atmosphere, is a famous example where chaotic dynamics can occur. The model is

$$\begin{aligned} x' &= \sigma(-x+y) \\ y' &= rx - y - xz \\ z' &= -bz + xy\,. \end{aligned}$$

Here the parameter values are chosen to be $r = 28$, $\sigma = 10$, and $b = 8/3$. The file is contained in *odes/lorenz.ode*. **You need to turn in a few pictures for this example. In addition, you need to describe the observations seen below in item 6**. We will discuss some of this in class.

1. Use **xppaut** to load in this system.
2. Use *initialconds* (*go*) option to integrate the ODE from the initial conditions given in the .ode file. The system will integrate up to time $t = 500$ (overwrite the storage as needed). Notice that the trajectory never settles down to a particular limit but that it begins to fill a dense but thin set in 3-D space. This is the attractor. You can type ./ at any time to halt the integration. **Turn in this picture.**
3. To get a cross-section of the attractor in the *x,z* plane click on *Viewaxes* and enter x for the horizontal axis, z for the vertical axis and let the horizontal axis range from -20 to 20 while the vertical axis ranges from 0 to 50. Erase the current picture so that we can re-do it below.
4. Now use the *Initialconds* (*mouse*) option and click on an initial data point near $x = -7.5$ and $z = 30$ (note $y = 0$). Notice that the trajectory alternates between two different lobes of the attractor and that a dense set is filled up as t increases.
5. Now use the *Xi vs t* option to plot z versus t. Erase the current picture. Go into the *window/zoom* (*window x* option and make the horizontal time axis go between 0 and 35 and the vertical axis between 0 and 50. Now use the *initialconds (new)* option with the initial conditions $x = -7.5$, $y = 0$, $z = 30$ and integrate in time. Kill the integration with a ./ when the integration goes past $t = 35$ (or alternatively do not overwrite the storage). Now choose the slightly different initial conditions $x = -7.5$, $y = 0$, $z = 30.001$ and integrate in time. Notice that the two curves are very different after about $t = 12$. This indicates an extreme sensitivity to initial conditions. **Turn in this picture.**
6. **Now change the parameter value of r to $r = 313$ and then to $r = 166.3$. Write a short paragraph on the type of motion observed in each case**. Note: that some of the variables x, y, z will need to be re-scaled rather drastically in this case since r is large. You need to figure out the re-scaling so that your plot is in the window.