CONSERVATIVE SYSTEMS

CONSIDER SYSTEMS OF THE FORM

$$x'' = F(x)$$

MULTIPLY BY x' TO GET

$$x' x'' = x' F(x)$$

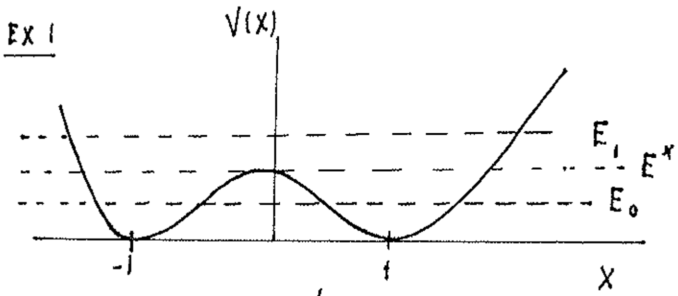
LET $\quad V(x) = -\int^{x} F(\lambda)\, d\lambda$. THEN, $\quad dV/_{dt} = -F'(x)\, dx/_{dt}$. $\qquad V'(x) = -F(x)$

THUS,

$$\frac{d}{dt}\left[\tfrac{1}{2}x'^2 + V(x)\right] = 0$$

HENCE $\quad \frac{1}{2}x'^2 + V(x) = E \qquad$ E total energy.

EX 1



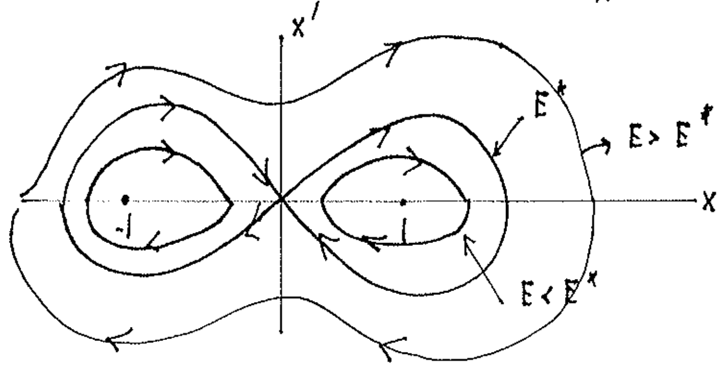
$$V(x) = \frac{1}{4}(1 - x^2)^2$$

$$V'(x) = -F(x)$$

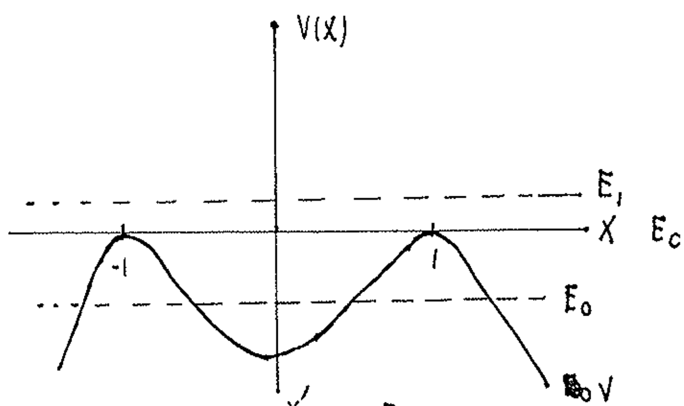
$$\longrightarrow \quad F(x) = x(1 - x^2)$$

$$x'' = x(1 - x^2)$$

$$x'' - x + x^3 = 0$$

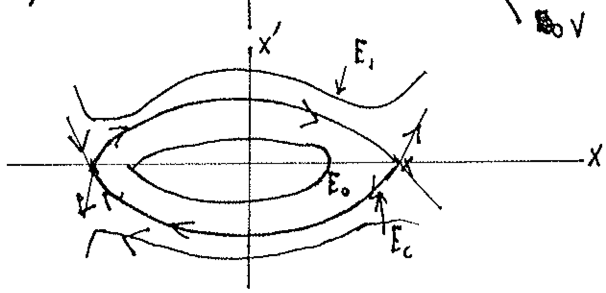


EX 2 $\qquad$ LET $\quad F(x) = -x + x^3 \longrightarrow V(x) = -\frac{1}{4}(1 - x^2)^2$



closed orbits are periodic solutions.

heteroclinic connection $-1 \longrightarrow 1$.



NOTE:

(i) symmetry $\quad \dot{x} \longrightarrow -\dot{x}$

EQUILIBRIA SUPPOSE THAT $V'(X_E) = 0$.

WE LET $X = X_E + \tilde{X}$ WITH $Y \ll 1$ TO GET

$$\tilde{X}'' = -V'[X_E] + -\hat{X}\, V''(X_E) + \dots$$

THUS $$\hat{X}'' + V''(X_E)\, \hat{X} = 0$$

NOW $$\tilde{X} = e^{\lambda t} \longrightarrow \lambda^2 + V''(X_E) = 0$$

(i) IF $V''(X_E) > 0 \rightarrow X_E$ IS LOCAL MINIMA OF V THEN

$\lambda = \pm i\,[V'(X_E)]^{1/2}$.

$\hat{X} \sim A\cos[\omega_E t] + B\sin[\omega_E t]$ $\quad \omega_E = \sqrt{V''(X_E)}$ frequency

local oscillations X_E is called a <u>center</u>

$\hat{X}$ is bounded as $t \to \infty \longrightarrow X_E$ is neutrally stable.

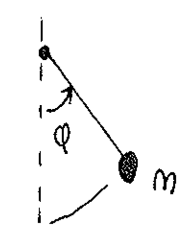
(ii) $V''(X_E) < 0 \longrightarrow \lambda = \pm\sqrt{-V''(X_E)}$

$$\hat{X} \sim A e^{\sqrt{-V''(X_E)}\,t} + B e^{-\sqrt{-V''(X_E)}\,t}$$

exponential growth and decay. X_E is called a saddle point.

EXAMPLE 1 (SIMPLE PENDULUM)

$$\varphi'' + \frac{g}{L}\sin\varphi = 0 \qquad V'(\varphi) = \frac{g}{L}\sin\varphi$$



$V'(\varphi) = \frac{g}{L}\sin\varphi \qquad V(\varphi) = \frac{g}{L}(1 - \cos\varphi)$

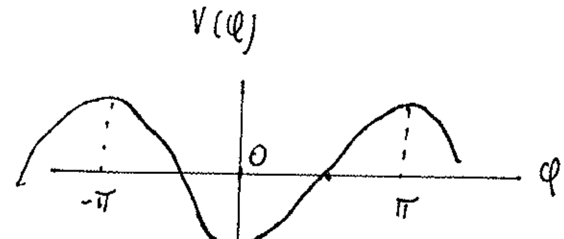
$V(\varphi) = -\frac{g}{L}\cos\varphi$

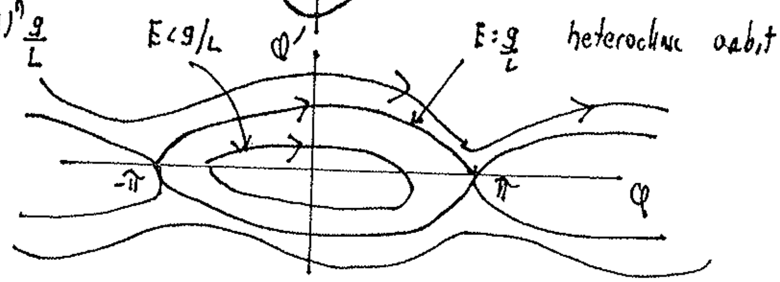
THUS $$\frac{\varphi'^2}{2} + \frac{g}{L}(-\cos\varphi) = E$$

NOW $V'(\varphi) = 0$ AT $\varphi = 0, \pm\pi, \pm 2\pi, \dots$

LET $$\frac{\varphi'^2}{2} + \frac{g}{L}(-\cos\varphi) = E$$

NOW $V''(\varphi) = \frac{g}{L}\cos\varphi = \frac{g}{L}\cos(n\pi) = \pm(-1)^n \frac{g}{L}$





NOW SUPPOSE $\varphi(0) = \varphi_0$, $\varphi'(0) = 0$. THEN,

$$\frac{\varphi'^2}{2} = \frac{g}{L}(\bar{o}\cos\varphi_0 + \cos\varphi)$$

NOW ON $\varphi' \lesssim 0$

$$\varphi' = -\sqrt{\frac{2g}{L}}\left[\cos\varphi - \cos\varphi_0\right]^{+1/2}$$

LET T be the period.

$$\frac{d\varphi}{dt} = -\sqrt{\frac{2g}{L}}\left(\cos\varphi - \cos\varphi_0\right)^{+1/2} \qquad \varphi(0) = \varphi_0 \qquad \varphi\left(\frac{T}{2}\right) = -\varphi_0$$

THUS
$$\int_{\varphi_0}^{-\varphi_0} \frac{d\varphi}{[\cos\varphi - \cos\varphi_0]^{1/2}} = -\sqrt{\frac{2g}{L}}\int_0^{T/2} dt = \frac{-T}{2}\left(2g/L\right)^{1/2}.$$

THUS
$$T = 2\left(\frac{L}{2g}\right)^{1/2}\int_{-\varphi_0}^{\varphi_0} \frac{1}{[\cos\varphi - \cos\varphi_0]^{1/2}}\, d\varphi.$$

NOW LET $\cos\varphi = 1 - 2\sin^2\varphi/2$

$\cos\varphi_0 = 1 - 2\sin^2\varphi_0/2$

$$T = 4\left(L/2g\right)^{1/2}\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2}\,[\sin^2\varphi_0/2 - \sin^2\varphi/2]^{1/2}}.$$

LET $\sin(\varphi/2) = \sin(\varphi_0/2)\sin\phi$

$$\frac{1}{2}\cos\varphi/2\, d\varphi = \sin\varphi_0/2\,\cos\phi\, d\phi \qquad \begin{array}{l}\varphi = 0 \longrightarrow \phi = 0\\ \varphi = \varphi_0 \longrightarrow \phi = \pi/2\end{array}$$

$$T = 4\left(L/2g\right)^{1/2}\int_0^{\pi/2} \frac{2\sin\varphi_0/2\,\cos\phi\, d\phi}{\cos\varphi/2\,\sqrt{2}\,[\sin^2\varphi_0/2 - \sin^2\varphi_0/2\,\sin^2\phi]^{1/2}}\, d\phi$$

$$T = 4\left(L/2g\right)^{1/2}\sqrt{2}\int_0^{\pi/2} \frac{\cos\phi\, d\phi}{\cos\varphi/2\,\cos\phi\, d\phi}$$

BUT
$$\cos\varphi/2 = \sqrt{1 - K^2\sin^2\phi}\, d\phi \qquad K = \sin\varphi_0/2.$$

THUS
$$T = 4\left(L/g\right)^{1/2}\int_0^{\pi/2}\left(\sqrt{1 - K^2\sin^2\phi}\right)^{-1} d\phi$$

$$K = \sin\varphi_0/2$$

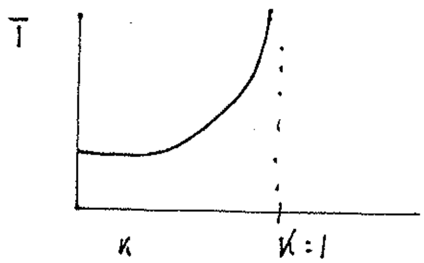
EXPANSION

IN THE SMALL ANGLE APPROXIMATION $K \ll 1$

$$T \approx 4\left(\frac{l}{g}\right)^{1/2} \int_0^{\pi/2} \left(1 + \frac{K^2}{2}\sin^2\phi + \dots\right) d\phi$$

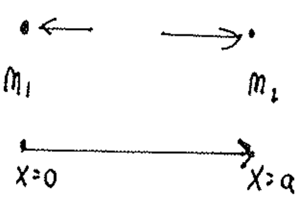
SO $$T \approx 4\left(\frac{l}{g}\right)^{1/2}\left[\frac{\pi}{2} + \frac{K^2}{2}\frac{\pi}{4} + \dots\right]$$

THUS $$T \approx 2\pi\left(\frac{l}{g}\right)^{1/2}\left[1 + \frac{K^2}{4} + \dots\right] \qquad K = \sin\frac{\theta_0}{2}.$$



EXAMPLE 2 PARTICLE BETWEEN TWO MASSES. (attractive)

$$m x'' = \frac{G M_2 m}{(x-a)^2} - \frac{G M_1 m}{x^2}$$



$K_2 = GM_2$

$K_1 = GM_1$

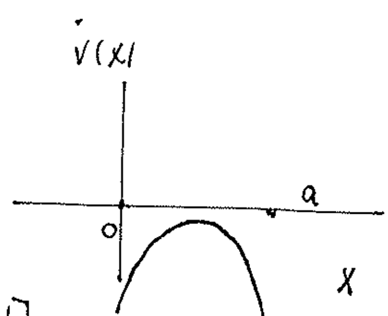
THUS $$x'' = \frac{K_2}{(x-a)^2} - \frac{K_1}{x^2}$$

NOW EQUILIBRIA IS $$\frac{K_2}{K_1} = \left(\frac{x-a}{x}\right)^2 \quad \text{SO} \quad \frac{x-a}{x} = -\left(\frac{K_2}{K_1}\right)^{1/2}$$

$$1 + \left(\frac{K_2}{K_1}\right)^{1/2} = \frac{a}{x}$$

THUS $$x_e = \frac{a}{\left(1 + (K_2/K_1)^{1/2}\right)}$$

NOW $$V'(x) = \frac{K_1}{x^2} - \frac{K_2}{(x-a)^2} \qquad V(x) = \frac{K_2}{x-a} - \frac{K_1}{x}$$

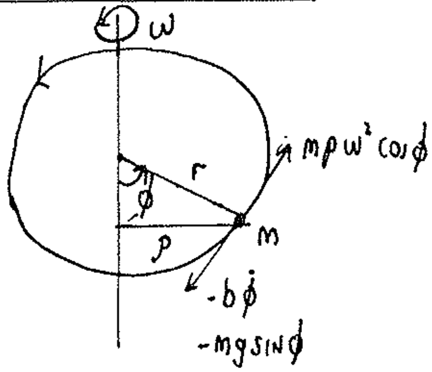


NOW $$V''(x) = -\frac{2K_1}{x^3} + \frac{2K_2}{(x-a)^3} = \frac{2}{x(x-a)}\left[\frac{K_2 x}{(x-a)^2} - \frac{K_1(x-a)}{x^2}\right]$$

BUT $$\frac{K_2}{(x-a)^2} = \frac{K_1}{x^2} \quad \text{SO} \quad V''(x_e) = \frac{2}{x_e(x_e-a)}\left[\frac{K_1}{x_e} - \frac{K_1(x_e-a)}{x_e^2}\right] = \frac{2K_1 a}{x_e^3(x_e-a)} < 0.$$

THUS x_e IS A ~~MINIMUM~~ MAXIMUM $\rightarrow$ INSTABILITY.

EXAMPLE (BEAD ON A WIRE)



$$m r \ddot{\phi} = -b\dot{\phi} - mg\sin\phi + m\rho\omega^2\cos\phi$$

BUT $\rho = r\sin\phi$

SO $$m r \ddot{\phi} = -b\dot{\phi} - mg\sin\phi + m r\omega^2\cos\phi\sin\phi$$

ASSUME NO DAMPING $b = 0$. $\quad t = T\tau$

$$\frac{mr}{T^2}\phi'' = -mg\sin\phi + mr\omega^2\cos\phi\sin\phi$$

$$\frac{1}{T^2\omega^2}\phi'' = -\frac{g}{r\omega^2}\sin\phi + \sin\phi\cos\phi$$

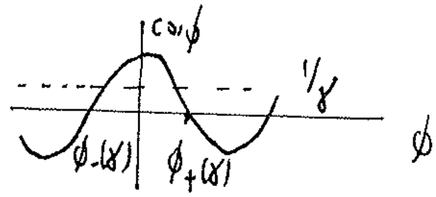
LET $T = 1/\omega$, $\gamma = r\omega^2/g$.

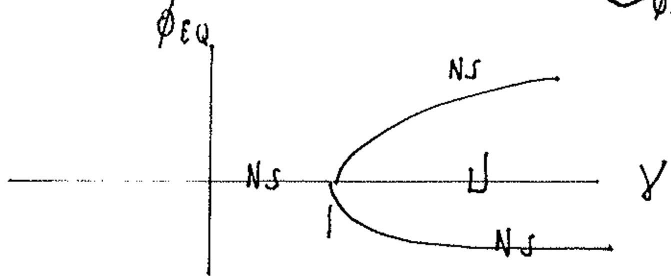
$$\phi'' = -\frac{1}{\gamma}\sin\phi + \sin\phi\cos\phi$$

NOW $$F(\phi) = -\frac{1}{\gamma}\sin\phi + \frac{1}{2}\sin(2\phi)$$

EQUILIBRIA AT $\sin\phi = 0 \longrightarrow \phi = 0, \pi$

$\cos\phi = 1/\gamma \longrightarrow \phi = \phi_c(\gamma), \quad \gamma > 1$





NOW $$V(\phi) = \frac{1}{\gamma}\int^{\phi}\sin u\,du - \frac{1}{2}\int^{\phi}\sin(2u)\,du$$

$$V(\phi) = -\frac{1}{\gamma}\cos(\phi) + \frac{1}{4}\cos 2\phi$$

$$V''(\phi) = \frac{1}{\gamma}\cos\phi - \cos 2\phi$$

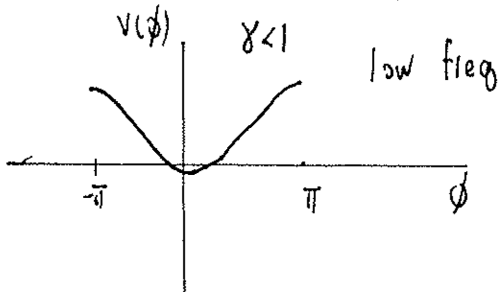
$$V''(0) = \frac{1}{\gamma} - 1 \qquad V''(0) > 0 \longrightarrow \gamma < 1 \quad \text{stable oscillation (low frequency)}$$

$$V''(\pi) = -\frac{1}{\gamma} - 1 < 0 \qquad V''(\pi) < 0 \longrightarrow \gamma > 1 \quad \text{exponential } \cancel{\text{decay}} \text{ growth.}$$

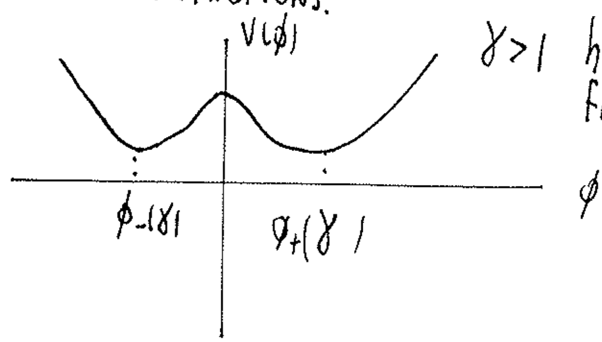
NOW $$V''(\phi_+) = \frac{1}{\gamma}\cos\phi_+ - (2\cos^2\phi_+ - 1) = \frac{1}{\gamma^2} - 2\left(\frac{1}{\gamma^2} - \frac{1}{2}\right) = 1 - 1/\gamma^2 .$$

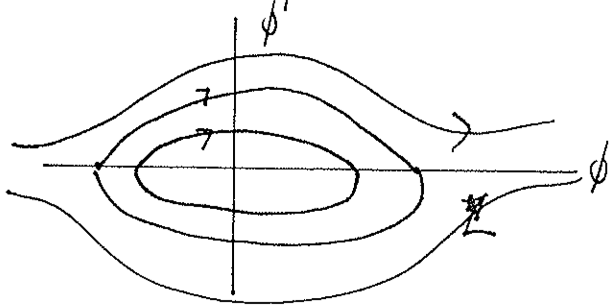
BUT IF $\gamma > 1$ $V''(\phi_+) > 0 \longrightarrow$ stable oscillations.

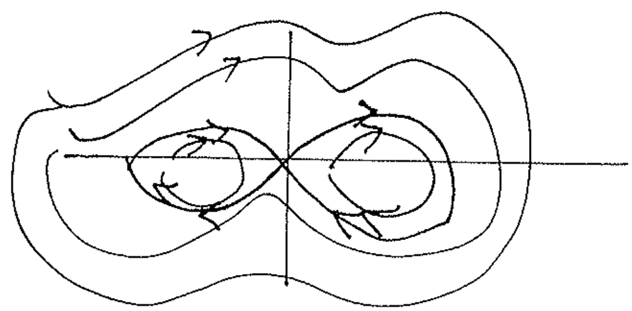
$\gamma<1$ low frequency



$\gamma>1$ high freq.







NOW IF $\gamma > 1$ THEN $\omega = \left(1 - 1/\gamma^2\right)^{1/2}$ is frequency.

This implies that

$$T = \omega\left(1 - 1/\gamma^2\right)^{1/2}, \text{ SINCE } T = 2\pi/\omega .$$

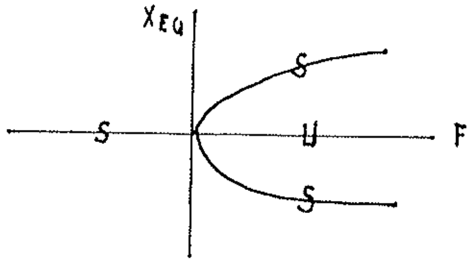
EXAMPLE **(DELAYED BIFURCATION)**

LET $$X'' + KX' = -X(2X^2 - F) \qquad K > 0$$

$$X'(0) = 0, \quad X(0) = X_0 .$$

NOW DIAGRAM IS; EQUILIBRIA AT $X = 0$ AND $X = \pm\sqrt{F/2}$ WHEN $F > 0$.

THE BIFURCATION DIAGRAM IS



$K=0$ BRANCHES are neutrally stable

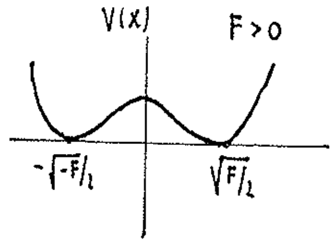
$K>0$ Asymptotically stable

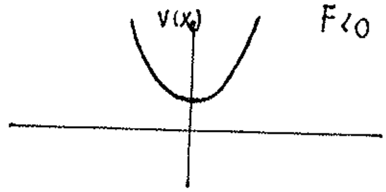
let $F=-1+\varepsilon t$ $\varepsilon=.0025$ to see A delayed BIFURCATION.

$$x'' + Kx' + V'(x) = 0 \qquad V'(x) = 2x^3 - Fx \qquad V = \frac{x^4}{2} - \frac{Fx^2}{2} + C$$

THUS WE CAN WRITE

$$V(x) = \frac{1}{2}\left[x^2 - F/2\right]^2$$





$x'' + V'(x) = 0$

let $x = x_E + y$

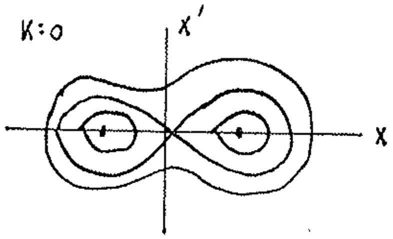
stable if $V''(x_E) > 0$.

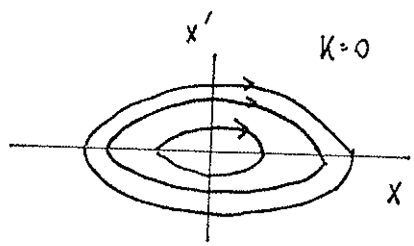
$y = x'$

$y' = -V'(x)$

$$J = \begin{pmatrix} 0 & 1 \\ -V''(x_E) & 0 \end{pmatrix}$$

eigenvalues $\lambda^2 + V''($





NOW IF $K>0$,

$$x'x'' + x'V'(x) = -Kx'^2$$

$$\frac{d}{dt}\left[\frac{1}{2}x'^2 + V(x)\right] = -Kx'^2$$

let $E(x,x') = E(t) = \frac{1}{2}x'^2 + V(x)$. THEN $E \geq 0$ FOR ALL t.

THIS EQUATION GIVES $dE/dt \leq 0$ $E(0) = E_0 \geq 0$, $E(t) \geq 0$.

THUS $E \to 0$ AS $t \to \infty$.

NOW WRITE THIS AS $y = x'$

$$y' + Ky = xF - 2x^3$$

SO

$$\begin{cases} x' = y \\ y' = -Ky + xF - 2x^3 \end{cases}$$

WE NOW PLOT FOR $F > 0$.

LET $\quad x = x_E + \hat{x}, \quad y = y_E + \hat{y}$.

THEN
$$\hat{x}' = \hat{y} \qquad\qquad \tilde{\underline{x}}' = J\,\tilde{\underline{x}}$$
$$\hat{y}' = (F - 6x_E^2)\,\hat{x} - K\hat{y}$$

$$J = \begin{pmatrix} 0 & 1 \\ F - 6x_E^2 & -K \end{pmatrix}$$

NOW $\quad x_E = 0 \longrightarrow$ eigenvalues are $-\lambda(-\lambda - K) - F = 0$

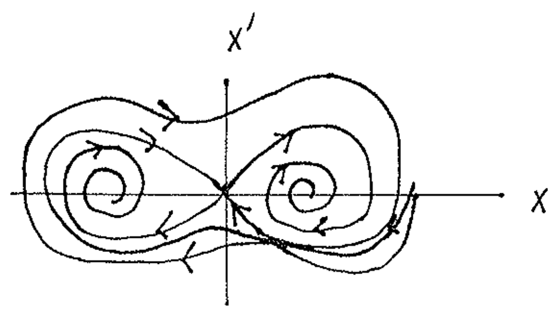
THUS $\quad \lambda^2 + K\lambda - F = 0 \qquad \lambda_\pm = \dfrac{-K \pm \sqrt{K^2 + 4F}}{2}$

IF $\quad F > 0 \;\rightarrow\; \lambda_+ > 0,\ \lambda_- < 0 \;\longrightarrow\; x = 0$ SADDLE POINT.

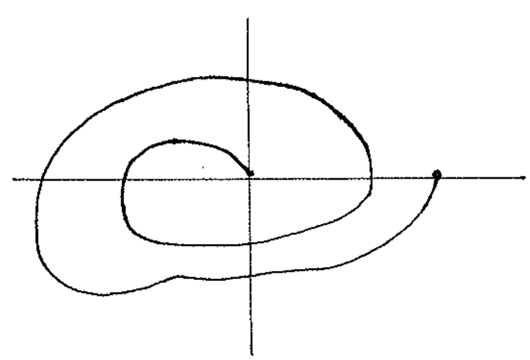
NOW $\quad x_E = \pm\sqrt{F/2} \qquad J = \begin{pmatrix} 0 & 1 \\ -2F & -K \end{pmatrix}$

$$-\lambda(-\lambda - K) + 2F = 0$$
$$\lambda^2 + \lambda K + 2F = 0$$
$$\lambda_\pm = \frac{-K \pm \sqrt{K^2 - 8F}}{2}$$

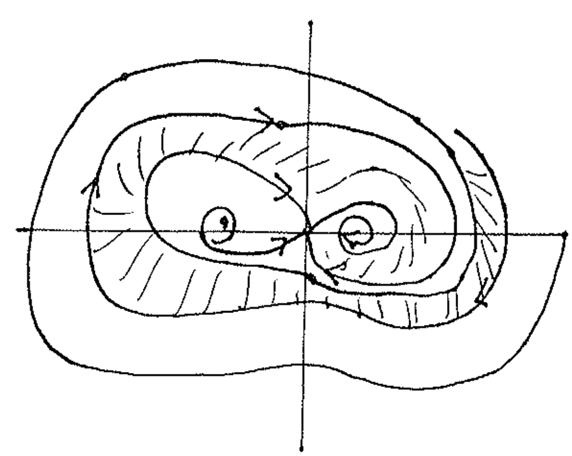
REAL $\lambda_\pm < 0. \longrightarrow$ stable.



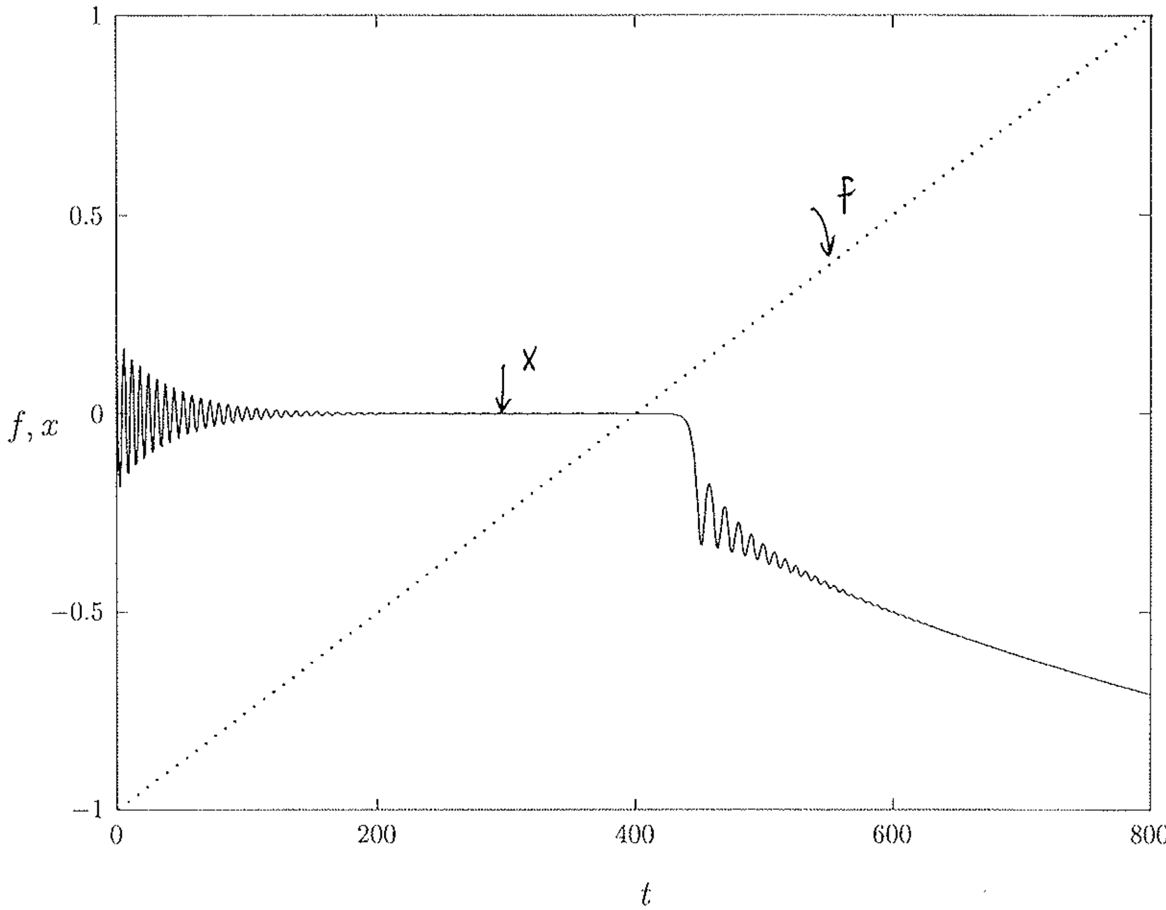
THE BASIN OF $\left(\sqrt{F/2},\ 0\right)$ IS AS SHOWN



BASIN OF ATTRACTION



DELAYED BIFURCATION OF $x'' + Kx' + x(2x^2 - f) = 0 \qquad f = -1 + \varepsilon t$

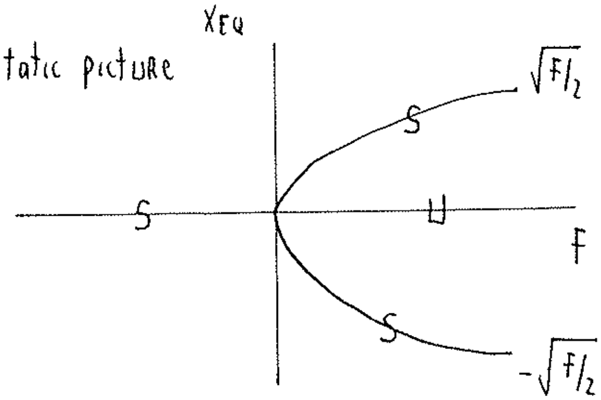


$\varepsilon = .0025 \qquad K = 0.05 \qquad x(0) = 0.20$

$x'(0) = 0$

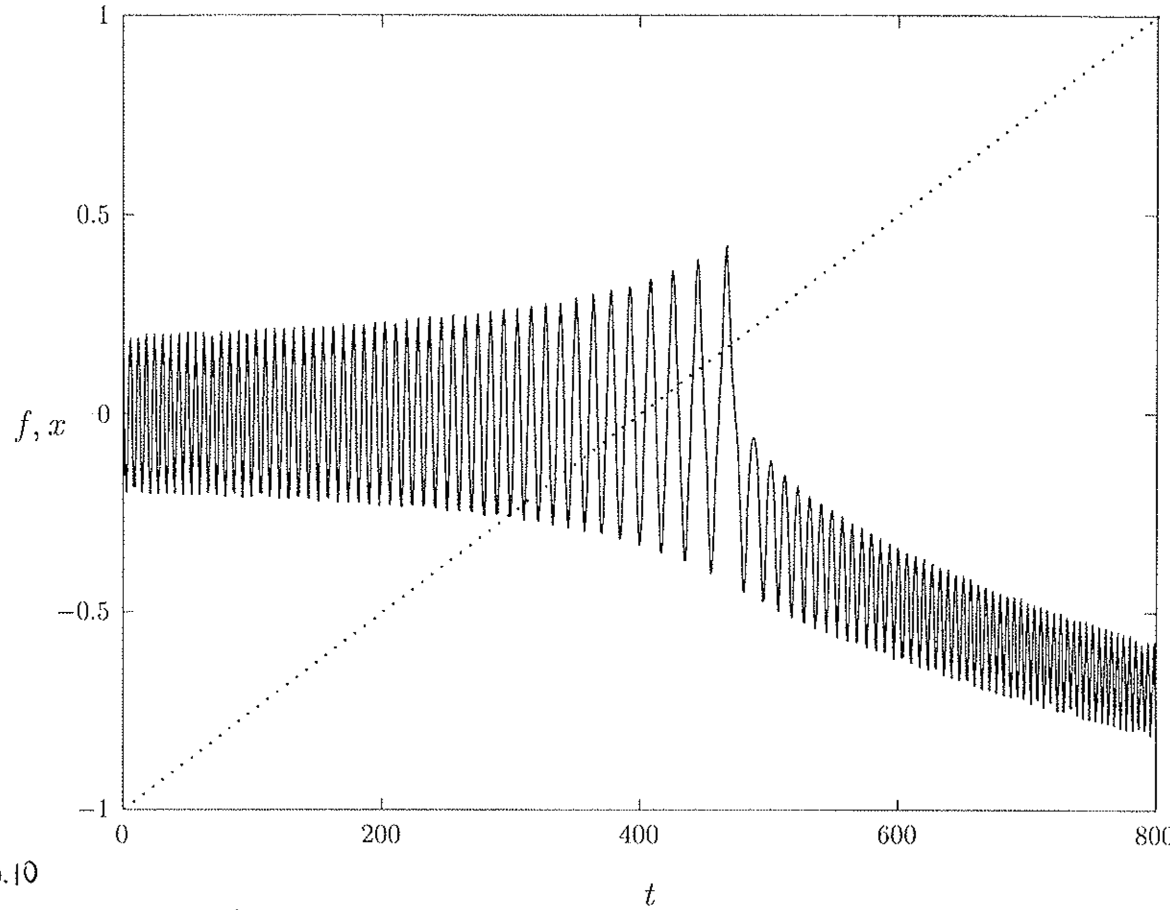
transition based on static theory should occur as f crosses through zero.

static picture



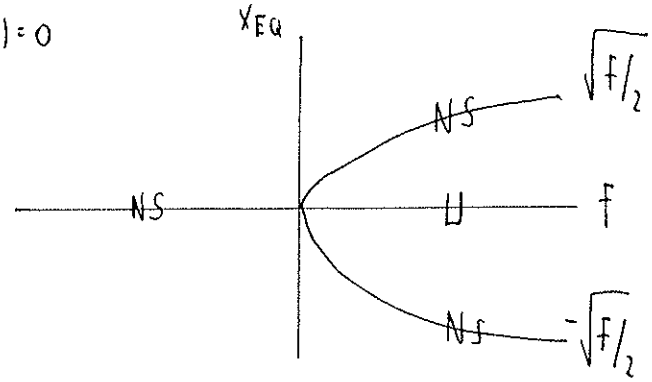
strong stability since $K > 0$.

$$x'' + x(2x^2 - f) = 0 \qquad f = -1 + \varepsilon t$$

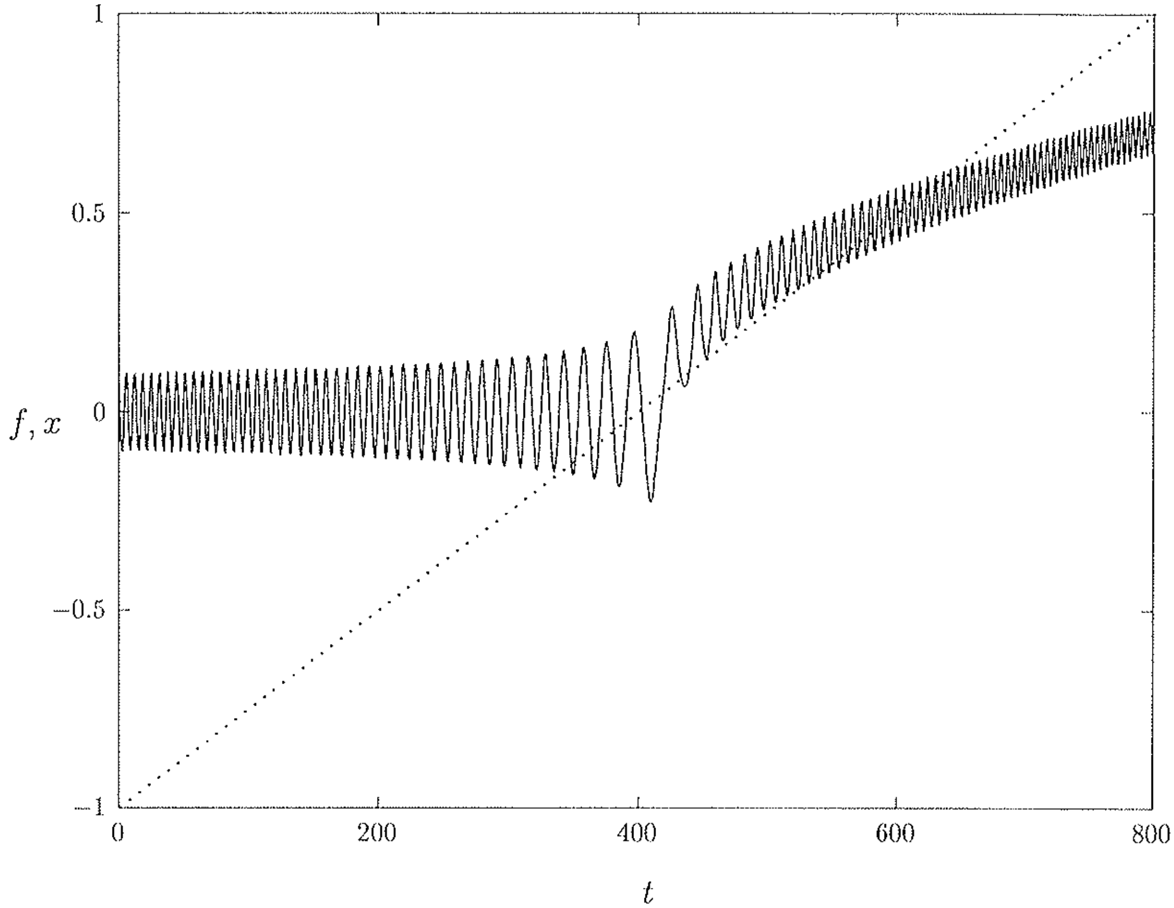


$x(0) = -0.10$

$x'(0) = 0$



Neutrally stable equilibrium branches since no damping.

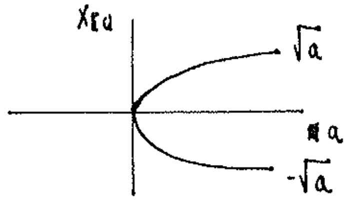


with a differnent initial condition we get other point at $t \to \infty$

$X(0) = 0.10$

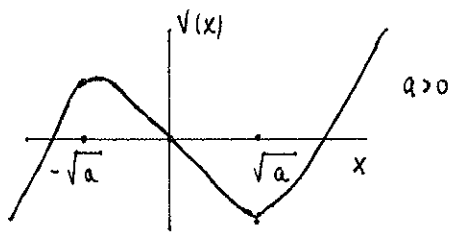
$X'(0) = 0$

EXAMPLE $\quad x'' = a - x^2$



NOW $\quad V'(x) = x^2 - a \qquad V(x) = \dfrac{x^3}{3} - ax + C$

$$V(x) = \frac{x}{3}\left(x^2 - 3a\right) \qquad V(\pm\sqrt{a}) = \pm\frac{\sqrt{a}}{3}\left(a - 3a\right) = \mp\frac{2a}{3}\sqrt{a}.$$



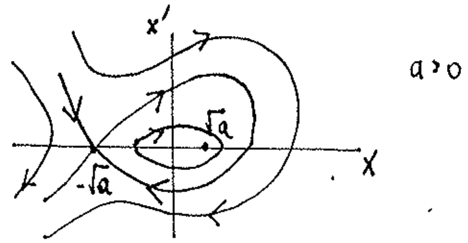
$V''(x) = 2x$

$V''(\sqrt{a}) > 0 \longrightarrow$ center MINIMUM

$V''(-\sqrt{a}) < 0 \longrightarrow$ saddle point MAXIMUM

$V(\sqrt{a}) = -\dfrac{2a}{3}\sqrt{3} \qquad V(-\sqrt{a}) = \dfrac{2a}{3}\sqrt{a}$

THIS suggests a phase portrait as

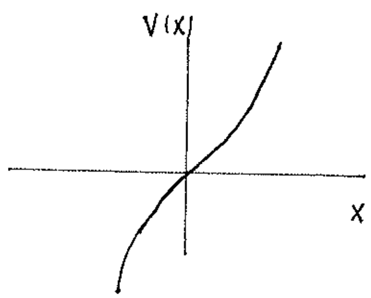


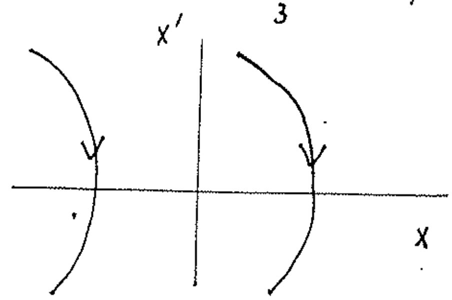
EQUATION OF SEPARATRIX.

$$\frac{x'^2}{2} + V(x) = V(-\sqrt{a})$$

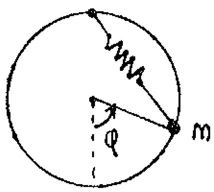
$$\frac{x'^2}{2} + \frac{x}{3}\left(x^2 - 3a\right) = \frac{2a}{3}\sqrt{a}.$$

NOW IF $a < 0$ WE GET $\quad V(x) = \dfrac{x}{3}\left(x^2 - 3a\right)$





EXAMPLE (BEAD ON A WIRE WITH A SPRING)



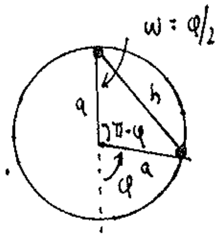
gravity

no damping

spring force : natural length $= L_0$

Now gravity is $mg\sin\varphi$ tangent to the spring.

THUS $$ma\varphi'' = K[h - L]\sin(w) - mg\sin\varphi$$



$$h^2 = a^2 + a^2 - 2a^2\cos(\pi - \varphi) = 2a^2[1 + \cos\varphi]$$

$$\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1$$

$$\cos^2\varphi = \frac{1 + \cos 2\varphi}{2} \longrightarrow h = \sqrt{2}\, a \sqrt{2}\cos(\varphi/2)$$

$$h = 2a\cos(\varphi/2)$$

THUS $$ma\varphi'' = K[2a\cos(\varphi/2) - L]\sin(\varphi/2) - mg\sin\varphi$$

let $t = \tau T$.

$$\frac{ma}{T^2}\ddot{\varphi} = K[2a\cos(\varphi/2) - L]\sin(\varphi/2) - mg\sin\varphi$$

$$\frac{a}{gT^2}\ddot{\varphi} = \frac{2aK}{gm}[\cos(\varphi/2) - \gamma]\sin(\varphi/2) - \sin\varphi \qquad \gamma = \frac{L}{2a}$$

choose $T = \sqrt{a/g}$.

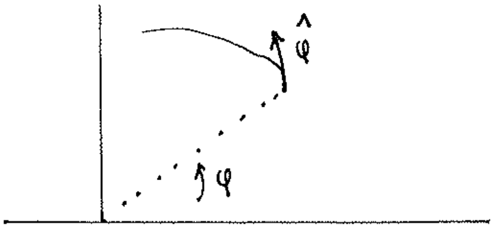
$$\ddot{\varphi} = b[\cos(\varphi/2) - \gamma]\sin(\varphi/2) - \sin\varphi \qquad \gamma = L/2a$$

$$b = 2aK/mg$$

$\gamma > 1$ compression

$\gamma < 1$ tension

EXAMPLE (PLANETARY MOTION) CENTRAL FORCE



$$\hat{r} = (\cos\varphi, \sin\varphi) \qquad \hat{r}\cdot\hat{\varphi} = 0$$

$$\hat{\varphi} = (-\sin\varphi, \cos\varphi)$$

$$\dot{\hat{r}} = \dot{\varphi}\,\hat{\varphi}$$

$$\dot{\hat{\varphi}} = -\dot{\varphi}\,\hat{r}$$

NOW $\quad X(t) = R(t)\cos(\varphi(t))$

$\quad Y(t) = R(t)\sin(\varphi(t))$

$$\dot{x} = \dot{R}\cos\varphi - R\sin\varphi\,\dot{\varphi}$$

$$\dot{y} = \dot{R}\sin\varphi + R\cos\varphi\,\dot{\varphi}$$

SO $\quad \underline{v} = \dot{R}\,\hat{r} + R\dot{\varphi}\,\hat{\varphi}$

$$\dot{\underline{V}} = \ddot{R}\,\hat{r} + \dot{R}\,\dot{\hat{r}} + (R\dot{\varphi})'\,\hat{\varphi} + R\dot{\varphi}\,\dot{\hat{\varphi}} = \ddot{R}\,\hat{r} + (\dot{R}\dot{\varphi})\hat{\varphi} + (R\dot{\varphi})(-\dot{\varphi}\,\hat{r}) + (R\dot{\varphi})^{\cdot}\,\hat{\varphi}$$

$$\underline{a} = (\ddot{R} - R\dot{\varphi}^2)\,\hat{r} + (R\ddot{\varphi} + 2\dot{R}\dot{\varphi})\,\hat{\varphi}$$

NOW SUPPOSE WE HAVE A CENTRAL FORCE $\quad \underline{F} = (F_0, 0) = F_0\,\hat{r}$

THEN $\quad m\,\underline{a} = \underline{F}$

YIELDS,

$$m(\ddot{R} - R\dot{\varphi}^2) = F_0$$

$$R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = 0 \longrightarrow R^2\ddot{\varphi} + 2R\dot{R}\dot{\varphi} = 0$$

$$\longrightarrow (R^2\dot{\varphi})^{\cdot} = 0$$

THUS $\quad R^2\dot{\varphi} = L \qquad L = \text{angular momentum}.$

$$\ddot{R} - \frac{R L^2}{R^4} = \frac{F_0(R)}{m}$$

DEFINE $\quad V'(R) = -\,L^2/_{R^3} - {}^1/_m\,F_0(R)$

NOW IF $\quad F_0 = -KR^p \qquad K > 0 \quad p < -1 \quad$ (attractive force)

$$V(R) = \frac{L^2}{2R^2} + \frac{K}{m(1+p)}\,R^{1+p}$$

NOW $$V'(R) = -\frac{L^2}{R^3} + \frac{K}{m} R^P$$

$V' > 0$ FOR $R > R_0$ $p > -3$

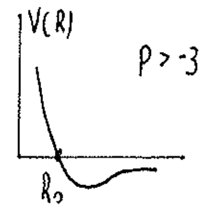
$V' < 0$ FOR $R < R_0$ $p \not> -3$

NOW $V' = 0$ WHEN $R_0 = \left(\frac{mL^2}{K}\right)^{1/p+3}$.

THUS $$V''(R_0) = \frac{3L^2}{R_0^4} + \frac{Kp}{m} R_0^{p-1} = \frac{1}{R_0^4}\left(3L^2 + \frac{Kp}{m} R_0^{p+3}\right) = \frac{1}{R_0^4}\left(3L^2 + \frac{Kp}{m}\frac{mL^2}{K}\right)$$

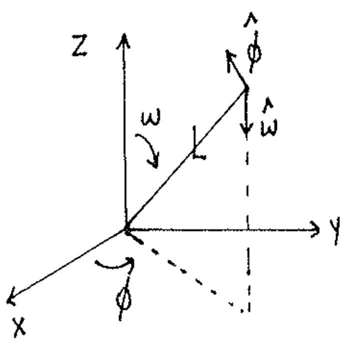
THIS GIVES $V''(R_0) = \frac{L^2}{R_0^4}(3+p)$ IF $p > -3$ $V''(R_0) > 0$ stability.

Frequency $\omega = \frac{L}{R_0^2}(3+p)^{1/2}$



SPHERICAL PENDULUM

RIGID ROD HAS LENGTH L



$X = L \sin w \cos\phi$

$Y = L \sin w \sin\phi$

$Z = L \cos w$

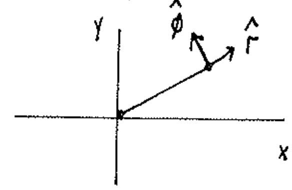
$\dot{\hat{r}} = \dot{\phi}\,\hat{\phi}$

$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$

$\hat{r} = (\cos\phi, \sin\phi, 0)$

$\hat{w} = (0, 0, 1)$

IN $w = \pi/2$ plane



NOW $$\underline{X} = L \sin w\, \hat{r} + L\cos w\, \hat{w}$$

$$\underline{V} = L\cos w\, \dot{w}\, \hat{r} + L \sin w\, \dot{\hat{r}} - L \sin w\, \dot{w}\, \hat{w}$$

$$\underline{V} = L\cos w\, \dot{w}\, \hat{r} + L \sin w\, \dot{\phi}\, \hat{\phi} - L \sin w\, \dot{w}\, \hat{w}.$$

NOW Kinetic energy $$K.E. = \frac{1}{2} m |\underline{V}|^2 = \frac{1}{2} m L^2 \dot{w}^2 + \frac{1}{2} m L^2 \sin^2 w\, \dot{\phi}^2$$

$$P.E. = mgL\cos w.$$

HENCE $$L = K.E. - P.E. = \frac{1}{2} m L^2 \dot{w}^2 + \frac{1}{2} m L^2 \sin^2 w\, \dot{\phi}^2 - mgL\cos w.$$

THE EULER-LAGRANGE EQUATIONS ARE

$$\frac{d}{dt}\left(\partial L/\partial\dot\phi\right) - \partial L/\partial\phi = 0 \longrightarrow \frac{d}{dt}\left(mL^2 \sin^2 w\,\dot\phi\right) = 0$$

$$\frac{d}{dt}\left(\partial L/\partial\dot w\right) - \partial L/\partial w = 0 \longrightarrow \frac{d}{dt}\left(mL^2\dot w\right) - mL^2\dot\phi^2 \sin w\cos w - mgL\sin w = 0$$

THUS we get 2 equations:

$$\begin{cases} \ddot w - \dot\phi^2 \sin w\cos w - \dfrac{g}{L}\sin w = 0 \\ \dot\phi = h/\sin^2 w \qquad h: \text{CONSTANT} \end{cases}$$

RE-WRITING:

$$\ddot w - h^2 \cot w\csc^2 w - \frac{g}{L}\sin w = 0.$$

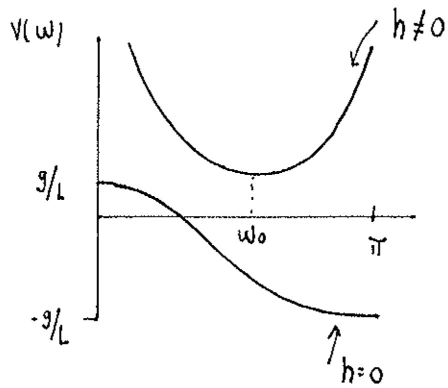
$$\dot\phi = h/\sin^2 w$$

NOW DEFINE THE POTENTIAL BY

$$V'(w) = -h^2\cot w\csc^2 w - \frac{g}{L}\sin w$$

$$V(w) = \frac{h^2}{2}\csc^2 w + \frac{g}{L}\cos(w) \qquad 0 \le w \le \pi$$

HENCE $\ddot w + V'(w) = 0.$



HENCE IF $h=0$ THEN $w=0$ unstable (top of motion)
$w=\pi$ stable (bottom of sphere)

NOTICE h is "angular momentum" in plane of constant w. $\longrightarrow$ how fast it is whirling around. thus if $h=0$ (bottom of sphere and top of sphere are stable and unstable respectively).

NOW IF $h>0$ we only have one equilibrium point at w_0. IT will be stable and corresponds to a whirling motion in the fixed latitude plane

NOTICE $V'(w_0) = 0 \longrightarrow \dfrac{Lh^2}{g} = \dfrac{\sin^4(w_0)}{\cos(w_0)}$.

NOTICE THAT AS $h \to \infty$ THEN $\omega_0 \simeq \pi/2$. WE expect that THE inequality $\omega_0 > \pi/2$ (IN southern hemisphere) should hold for any $h>0$.

NOW SUPPOSE h IS LARGE. Let

$$t = T\tau.$$

$$\frac{1}{T^2}\,\omega'' - h^2 \cot\omega \csc^2\omega - \frac{g}{l}\sin\omega = 0$$

$$\frac{1}{T}\,\phi' = h/\sin^2\omega \qquad\qquad \omega' = d\omega/d\tau$$

CHOOSE $T = 1/h$. THEN we get the approximate system:

$$\omega'' - \cot\omega \csc^2\omega = 0$$

$$\phi' = 1/\sin^2\omega$$

$$V(\omega) = \frac{1}{2}\csc^2\omega.$$