DISCRETE MAPS

DISCRETE DYNAMICAL SYSTEMS OCCUR WHEN TIME IS CONSIDERED TO BE DISCRETE RATHER THAN CONTINUOUS.

A 1-D DISCRETE-TIME MAP HAS THE FORM

$$X_{n+1} = g(X_n) \quad \text{FOR } n = 0, 1, 2, \ldots \text{ WITH } X_0 \text{ given}$$

AND $g(X)$ A SMOOTH FUNCTION OF X; OR POSSIBLY PIECEWISE SMOOTH.

<u>REMARK</u> (i) APPLICATIONS TO ECOLOGY: DISCRETE LOGISTIC GROWTH

(ii) NEWTON'S METHOD $X_{n+1} = X_n - f(X_n)/f'(X_n)$ FOR FINDING THE ZEROES OF $f(x) = 0$.

(iii) POINCARÉ MAPS FROM NONLINEAR DYNAMICAL SYSTEMS

(iv) FEED-BACK SYSTEMS

FIXED POINTS AND LINEAR STABILITY

CONSIDER $X_{n+1} = f(X_n)$

THEN X^* IS A FIXED POINT OF THE MAP IF

$$X^* = f(X^*).$$

WE LINEARIZE $X_n = X^* + y_n$ WITH $y_n \ll 1$. THEN,

$$X^* + \underline{y_{n+1}} = f(X^* + y_n) = f(X^*) + \underline{f'(X^*)\, y_n} + \text{quadratic terms}$$

SINCE $X^* = f(X^*)$, THEN

$$y_{n+1} = f'(X^*)\, y_n, \quad \text{SO} \quad y_n = [f'(X^*)]^n\, y_0 \text{ FOR } n \geq 0.$$

DEFINE $\lambda = f'(X^*)$ AS THE MULTIPLIER

(i) IF $|\lambda|<1$ THEN X^* IS AN <u>ATTRACTOR</u> (STABLE)

(ii) IF $|\lambda|>1$ THEN X^* IS A <u>REPELLER</u> (UNSTABLE)

(iii) IF $\lambda=0$ THEN X^* IS CALLED <u>SUPERSTABLE</u>

(iv) IF $|\lambda|=1$ THEN STABILITY CANNOT BE DETERMINED BY LINEARIZATION; WE NEED TO CONSIDER HIGHER ORDER TERMS.

<u>EXAMPLE 1</u> LET $X_{n+1}=f(X_n)$ WITH $f(X_n)=1-X_n^2$.

THE FIXED POINTS SATISFY $X=f(X)$, SO $X=1-X^2$.

THUS $X^2+X-1=0 \rightarrow X_\pm = \frac{-1}{2} \pm \sqrt{5}/2$

NOW $f'(X)=-2X$

$$f'(X_+) = -2\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right) = 1-\sqrt{5} = -1.236 \rightarrow |f'(X_+)|>1$$

UNSTABLE

$$f'(X_-) = -2\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right) = 1+\sqrt{5} = 3.236 \rightarrow |f'(X_-)|>1$$

UNSTABLE

THUS BOTH X_+ AND X_- ARE UNSTABLE FIXED POINTS.

NOTICE THAT IF $0\le X_n\le 1$ THEN $0\le X_{n+1}\le 1$. THUS ON RANGE $0\le X\le 1$, THE ONLY FIXED POINT IS $X^*=(-1+\sqrt{5})/2$.

WE NOW LOOK FOR 2-CYCLES; OR PERIOD-TWO POINTS. WE WANT TO FIND FIXED POINTS OF $X=f(f(x))\equiv f^2(x)$. WE CALCULATE

$$f(f(x)) = 1-[f(x)]^2 = 1-(1-x^2)^2 = 2x^2-x^4.$$

WE WANT TO FIND ROOTS X TO $x=f[f(x)]$, i.e.

$$x=2x^2-x^4.$$

NOTICE THAT $X=0,1$ ARE SOLUTIONS SO THAT AFTER FACTORING

$$X(X-1)(X^2+X-1)=0 \longrightarrow X=\frac{-1+\sqrt{5}}{2},\quad X=0,\quad X=1.$$

SINCE $X^* = \frac{-1+\sqrt{5}}{2}$ SATISFIES $X^* = F(X^*)$ THEN $F(X^*) = F[F(X^*)] \equiv F^2(X^*)$,
SO THAT $X^* = F^2(X^*)$

NOW DEFINE $p = 0$, $q = 1$.

THEN $F(p) = q$, $F(q) = p$.

HENCE, WE COULD HAVE A 2-CYCLE IF WE CHOOSE $X_0 = 0$, FOR THEN

$$\{X_0, X_1, X_2, X_3, X_4, X_5, \ldots\} = \{0, 1, 0, 1, 0, 1, \ldots\}.$$

NOW TO CALCULATE THE STABILITY OF THE 2-CYCLE.

WE HAVE $X_{n+2} = F[F(X_n)]$.

IF $X_n = 0 \;\forall n$, WE LINEARIZE TO GET $X_n = 0 + Y_n$ $Y_n \ll 1$

$$Y_{n+2} = \frac{d}{dx} F[F(X)]\Big|_{X=0} Y_n.$$

WE HAVE STABILITY IF

$$\lambda = \frac{d}{dx} F[F(X)] = F'(F(X))\, F'(X)\Big|_{X=0} \quad \text{SATISFIES } |\lambda| < 1$$

WE CALCULATE $\lambda = F'(F(0))\, F'(0) = F'(1)\, F'(0)$

NOW $F'(X) = -2X$ SO $F'(0) = 0$, $F'(1) = -2$.

$\longrightarrow \lambda = 0$ so 2-cycle is superstable.

<u>EXAMPLE 2</u> NEWTON'S METHOD TO COMPUTE THE ZEROES OF $g(X) = 0$ IS

$$X_{n+1} = X_n - \frac{g(X_n)}{g'(X_n)}. \quad \text{DEFINE } F(X) \equiv X - g(X)/g'(X).$$

FIXED POINTS ARE SOLUTIONS TO $X^* = X^* - g(X^*)/g'(X^*) \longrightarrow g(X^*) = 0$

NOW LET $g(X^*) = 0$. WE CALCULATE $F'(X^*) = 1 - \frac{g'(X^*)}{g'(X^*)} + \frac{g(X^*)\, g''(X^*)}{[g'(X^*)]^2}$.

THUS $\lambda = F'(X^*) = 0 \longrightarrow$ NEWTON'S METHOD IS SUPERSTABLE!

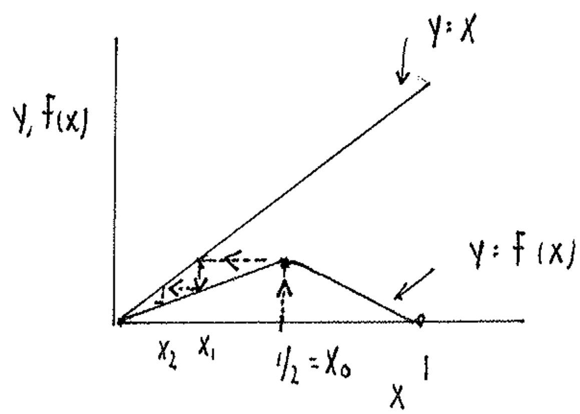
i.e. VERY FAST CONVERGENCE NEAR THE ROOT

COBWEB DIAGRAMS FOR $X_{n+1} = f(X_n)$

FROM AN INITIAL POINT X_0 DRAW A VERTICAL LINE UP TO $f(X_0)$. FROM THIS POINT DRAW A HORIZONTAL LINE EITHER LEFT OR RIGHT TO JOIN THE DIAGONAL $y = X$. THE X-ORDINATE IS THE ITERATE $X_1 = f(X_0)$. FROM THE POINT $(X_1, f(X_0))$ DRAW A VERTICAL LINE UP OR DOWN TO JOIN $f(X)$. DRAW A HORIZONTAL LINE FROM THIS POINT TO THE DIAGONAL AT THE POINT $(X_2, f(X_1))$. ETC..

EXAMPLE 3 DRAW A COBWEB DIAGRAM FOR $X_{n+1} = f(X_n)$ WHERE

$$f(X) = \begin{cases} X/2 , & 0 \le X \le 1/2 \\ 1/2\,(1-X) , & 1/2 \le X \le 1 \end{cases} \qquad \text{WITH} \quad X_0 = 1/2 .$$



THE ITERATES X_n TEND TO $X = 0$ AS $n \to \infty$ AS SHOWN.

NOTICE THE FIXED POINTS SATISFY $X = f(X)$.

THUS ON $0 \le X \le 1/2 \longrightarrow X = X/2 \longrightarrow X = 0$

ON $1/2 \le X \le 1 \longrightarrow X = \frac{1}{2}(1-X) \longrightarrow X = 1/3$ NOT IN RANGE.

SO $X = 0$ IS ONLY FIXED POINT AND $\lambda = |f'(0)| = 1/2 < 1$.

HENCE $X = 0$ IS AN ATTRACTOR (STABLE).

EXAMPLE 4: FIND THE FIXED POINTS AND CLASSIFY THEIR STABILITY FOR

(i) $X_{n+1} = \sqrt{X_n}$ ON $X_0 \geq 0$

(ii) $X_{n+1} = 1 + \frac{1}{2} \sin(X_n)$ ON $0 \leq X_n < \infty$

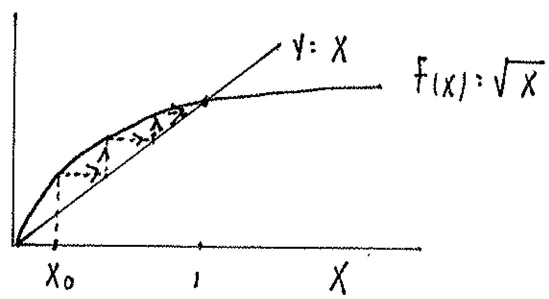
SOLUTION

(i) THE FIXED POINTS SATISFY $X = \sqrt{X}$ SO $X^* = 0$ OR $X^* = 1$.

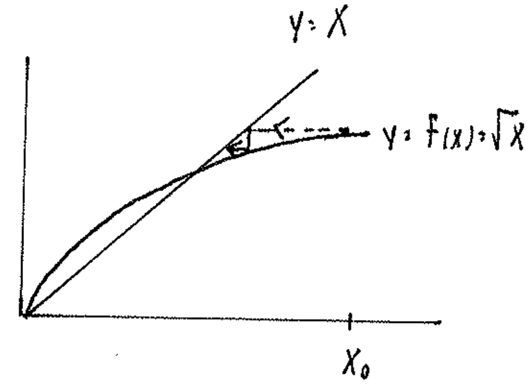
NOW $F(X) = \sqrt{X}$ SO $F'(X) = \dfrac{1}{2\sqrt{X}}$ SO $F'(0) = \infty$, $F'(1) = 1/2$.

THUS $X^* = 0$ IS UNSTABLE, WHILE $X^* = 1$ IS STABLE.

THE COBWEB DIAGRAM IS



X_n CONVERGES TO $X^* = 1$ AS $n \to \infty$ WHEN $0 < X_0 < 1$



$X_n \to 1$ AS $n \to \infty$ WHEN $X_0 > 1$.

(ii) HERE $F(X_n) = 1 + \frac{1}{2} \sin(X_n)$.

THE FIXED POINTS X^* ARE ROOTS OF $g(X) = 0$ WHERE

$$g(X) = X - 1 - \frac{1}{2} \sin X. \qquad g'(X) = 1 - \frac{1}{2} \cos X > 0 \text{ ON } 0 \leq X < \infty$$

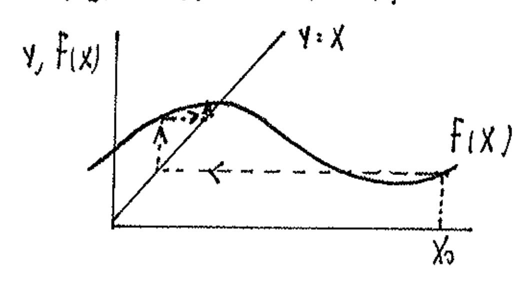
NOW $g(0) = -1$, $g(\frac{\pi}{2}) = \frac{\pi}{2} - 1 - \frac{1}{2} = .57 > 0$.

SINCE $g(0) < 0$, $g(\pi/2) > 0$ AND $g'(X) > 0$ ON $0 < X < \pi/2$ THEN THERE IS A UNIQUE ROOT TO $g(X) = 0$ ON $0 < X < \pi/2$. THIS IS ONLY ROOT ON $0 < X < \infty$. USING NEWTON'S METHOD WE COMPUTE $X^* = 1.4987$.

NOW $F'(X^*) = \frac{1}{2} \cos(X^*)$

SO $|\lambda| = |F'(X^*)| < 1$

$\longrightarrow X^*$ IS STABLE



LOGISTIC MAP

WE BEGIN WITH

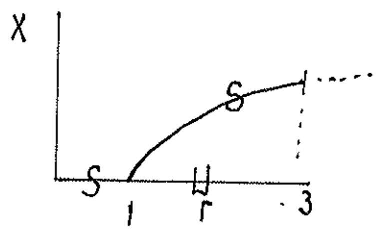
$$X_{n+1} = r X_n (1 - X_n), \quad X_n \geq 0.$$

WE CONSIDER ONLY $0 \leq X_n \leq 1$ AND $0 \leq r \leq 4$. THUS $0 \leq X_{n+1} \leq 1$, SO THAT THE ITERATES REMAIN IN $[0,1]$.

THE FIXED POINTS ARE

$$X = r X (1 - X) \quad \text{SO} \quad X = 0 \quad \text{AND} \quad X = 1 - 1/r.$$

A PLOT OF BIFURCATION DIAGRAM IS



TRANSCRITICAL BIFURCATION AT $r=1$.

NOW THE LINEARIZED STABILITY WITH $f(X,r) = rX(1-X)$ IS DETERMINED BY $f'(X,r) = r[1-2X]$.

THE MULTIPLIER IS $\Lambda = f'(X, r)$

- FOR $X = 0$, THEN

  $f'(0,r) = r \rightarrow$ IF $0 < r < 1$, THEN $X=0$ IS AN ATTRACTOR (STABLE)

  IF $r=1$, THEN $|f'(0,1)| = 1 \rightarrow$ INCONCLUSIVE

  IF $r > 1$, $X=0$ IS UNSTABLE

- FOR $X = 1 - 1/r$ THEN $f'(X,r) = r[-1 + 2/r] = 2 - r$.

  IF $0 < r < 1 \rightarrow |f'| > 1 \Rightarrow$ UNSTABLE (BUT $X<0$ SO WE DON'T CARE)

  IF $1 < r < 3 \rightarrow |f'| < 1 \Longrightarrow$ STABLE ATTRACTOR

  IF $r = 3$, THEN $\Lambda = -1$ SO INCONCLUSIVE. LINEARIZATION IS $X_n = 2/3 + Y_n \longrightarrow Y_{n+1} = -Y_n$.

  THIS IS A FLIP BIFURCATION.

  IF $3 < r \leq 4 \rightarrow X = 1 - 1/r$ IS UNSTABLE REPELLOR.

NOW WE CONSIDER THE BIFURCATION POINT

$$r_c = 3 \quad \text{WITH} \quad X_c = 1 - 1/r_c = 2/3.$$

THE LINEARIZATION IS SIMPLY

$$y_{n+1} = -y_n.$$

THE ORBIT OF THE LINEARIZATION IS $y_0, -y_0, y_0, -y_0, y_0, \ldots$
SO EVERY ORBIT OF LINEARIZATION IS A PERIOD 2-CYCLE.

WE NOW LOOK FOR PERIOD-2 CYCLES IN THE ORIGINAL SYSTEM.
LOOK FOR X_0 SUCH THAT $X_2 = X_0$.

$$X_0, \; X_1 = F(X_0), \; X_2 = F(F(X_0)) \equiv F^2(X_0).$$

$$\text{WANT} \quad X_2 = X_0 \rightarrow X_0 = F^2(X_0)$$

<u>NOTE</u> WE DEFINE $F^2(X) \equiv F[F(X)]$

IT DOES NOT MEAN $[F(X)]^2$.

<u>PERIOD 2 CYCLES</u> $F(X) = Xr(1-X)$

$$F^2(X) = F(F(X)) = r[rX(1-X)][1 - rX(1-X)]$$

$$F^2(X) = r^2 X(1-X)(1 - rX + rX^2).$$

WE SET $F^2(X) = X \rightarrow \left. r^2 X(1-X)(1 - rX + rX^2) = X. \right\} (*)$

NOW SET $X_* = 0$ AND $X_* = 1 - 1/r$ BOTH SATISFY $F(X_*) = X_*$, THEN THEY ALSO SATISFY $F[F(X^*)] = F(X^*) = X^*$, i.e. $F^2(X^*) = X^*$.

SO DIVIDE $r^2 X(1-X)(1 - rX + rX^2) = X$ BY $X = 0, 1 - 1/r$. (BOTH MUST BE FACTORS)

SO $X = 0$ AND $X = 1 - 1/r$ BOTH DIVIDE $(*)$.

NOW WRITE

$$r^2(1-X)(1 - rX + rX^2) - 1 = 0$$

$$r^2[-rX^3 + rX^2 - X + rX^2 - rX + 1] - 1 = 0.$$

WE WRITE THIS AS

$$r^3 x^3 - 2r^3 x^2 + r^2(r+1)x + 1 - r^2 = 0.$$

NOW $x=0$ AND $x = 1 - 1/r$ ARE ROOTS. THUS $rx - r + 1$ DIVIDES INTO THIS POLYNOMIAL

$$\begin{array}{r} r^2x^2 - (r^2+r)x + (r+1) \\ rx + (1-r) \,\big)\, r^3x^3 - 2r^3x^2 + (r^3+r^2)x + 1 - r^2 \\ -\left[r^3x^3 + (r^2-r^3)x^2\right] \\ \hline -(r^3+r^2)x^2 \\ -\left[-(r^3+r^2)x^2 - (r^2+r)(1-r)x\right] \\ \hline x\left[r^3+r^2+(r^2+r)(1-r)\right] \\ x\left[r^3+r^2-r^3+r-r^2+r^2\right] \\ x(r^2+r) + (1-r^2) \\ -\left[r(r+1)x + (1-r^2)\right] \\ \hline 0 \end{array}$$

THUS WE OBTAIN THAT THE PERIOD 2 POINTS ARE ROOTS OF

set this $=0$.

$$x\left(rx + 1 - r\right)\left(r^2x^2 - (r^2+r)x + (r+1)\right) = 0.$$

WE OBTAIN

$$x = \frac{(r^2+r) \pm \sqrt{(r^2+r)^2 - 4r^2(r+1)}}{2r^2}$$

THIS BECOMES

$$x = \frac{r+1}{2r} \pm \frac{\sqrt{r^2(r+1)(r-3)}}{2r^2}$$

WE DEFINE

$$p = \frac{r+1 - \sqrt{(r-3)(r+1)}}{2r} \qquad q = \frac{r+1+\sqrt{(r-3)(r+1)}}{2r}$$

AS $r \to 3^+$ WE GET $p, q \to x^* = 2/3$ FIXED POINT.

THUS AT $r=3$, WE HAVE EMERGENCE OF A PERIODIC 2 SOLUTION.

WE REFER TO SUCH A SOLUTION AS A 2-CYCLE.

WE NOW SHOW THAT

$$f(p) = q.$$

<u>PROOF</u> $f(p) = rp(1-p) = \frac{1}{r}\left[-r^2p^2 + r^2 p\right]$

BUT $r^2p^2 - (r^2+r)p + r + 1 = 0$ (since it satisfies quadratic).

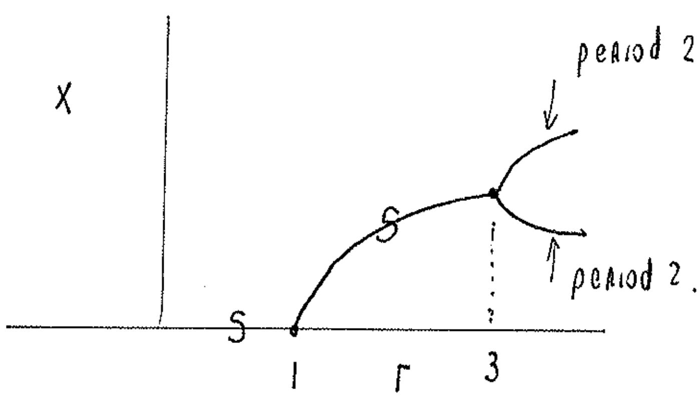
THUS $f(p) = \frac{1}{r}\left[-rp + r + 1\right]$ USING $-r^2p^2 + r^2 p = -rp + r + 1$

$$= -p + \frac{1}{r}(r+1) = \frac{r+1}{2r} + \frac{\sqrt{(r-3)(r+1)}}{2r}$$

$$f(p) = q$$

SINCE $f(f(p)) = f^2(p) = p$ IT FOLLOWS THAT $f(q) = p$.

HENCE OUR BIFURCATION DIAGRAM IS AS SHOWN



NOW LINEARIZED STABILITY OF p, q REQUIRES CALCULATING THE MULTIPLIER

$$\lambda = \frac{d}{dx} f(f(x))\Big|_{x=p} = f'(f(x))\, f'(x)\Big|_{x=p}$$

THUS $$\lambda = f'(q)\, f'(p)$$

NOW RECALL THAT $X = p, q$ ARE THE ROOTS OF

$$r^2 X^2 - (r^2 + r) X + r + 1 = 0$$

THUS $\quad (X - p)(X - q) = X^2 - (p + q) X + pq = 0.$

WE CONCLUDE THAT

$$p + q = \frac{r^2 + r}{r^2} \qquad pq = \frac{r+1}{r^2}.$$

NOW

$$\begin{aligned} \lambda = f'(q) f'(p) &= r^2 (1 - 2q)(1 - 2p) \\ &= r^2 \left(1 - 2(p+q) + 4pq\right) \\ &= r^2 \left[1 - 2\left(\frac{r^2 + r}{r^2}\right) + \frac{4(r+1)}{r^2} \right] \\ &= r^2 - 2(r^2 + r) + 4(r+1) = -r^2 + 2r + 4. \end{aligned}$$

WE CONCLUDE THAT FOR $r \geq 3$, THE MULTIPLIER DETERMINING THE STABILITY OF A 2-CYCLE IS

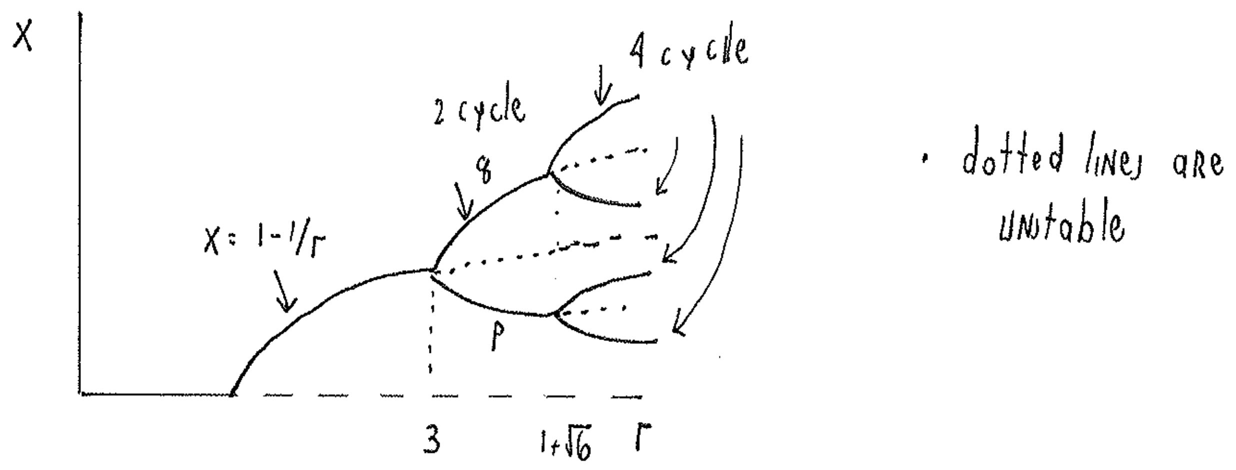
$$\lambda = -r^2 + 2r + 4. \qquad d\lambda/dr = -2r + 2 < 0 \quad \text{ON } r \geq 3.$$

WHEN $r = 3 \longrightarrow \lambda = 1$ AT BIFURCATION POINT. NOTICE THAT $\lambda \downarrow$ AS $r \uparrow$, AND WE GET A FLIP BIFURCATION WHEN $\lambda = -1$.

THIS OCCURS WHEN $\quad -r^2 + 2r + 4 = -1 \longrightarrow r^2 - 2r - 5 = 0.$

SOLVING WE OBTAIN $\quad r = 1 + \sqrt{6}$

THUS WHEN $r = 1 + \sqrt{6} \cong 3.449$ WE OBTAIN THAT THE 2-CYCLE LOSES ITS STABILITY TO THE CREATION OF A 4-CYCLE.



FOR $r > \sqrt{6}+1$ F^2 HAS A 2-CYCLE $\Longrightarrow$ F HAS A FOUR cycle.

NOW IF WE USE NUMERICS VIA XPPAUT WE OBTAIN THE FOLLOWING NUMERICAL RESULTS FOR VALUES OF r CORRESPONDING TO THE <u>PERIOD-DOUBLING CASCADE</u>

$r_1 = 3$ } stable 2-cycle $r_1 < r < r_2$

$r_2 = 1+\sqrt{6} \doteq 3.449$ } stable 4 cycle $r_2 < r < r_3$

$r_3 \doteq 3.54404$ } stable 8 cycle $r_3 < r < r_4$

$r_4 = 3.56440$ } stable 16 cycle $r_4 < r < r_5$.

$r_5 \doteq 3.568759$

WE HAVE STABLE 2^m CYCLE FOR $r_m < r < r_{m+1}$. THIS is A PERIOD-DOUBLING CASCADE

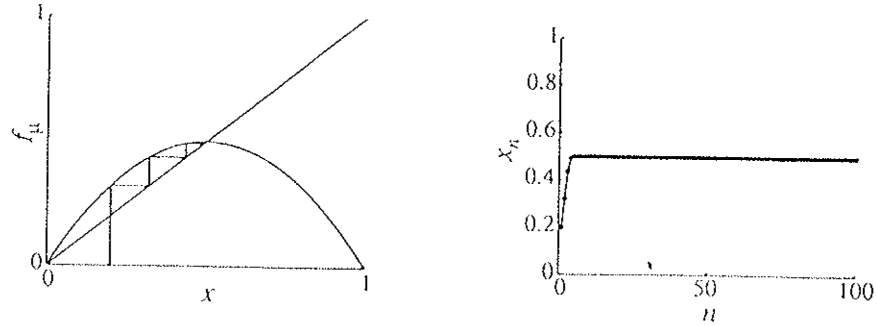
δ FEIGENBAUM CONSTANT $\approx$ 1975

$$\lim_{m\to\infty} r_m = r_\infty = 3.569446.$$

$$\lim_{m\to\infty} \frac{r_m - r_{m-1}}{r_{m+1} - r_m} = \delta = 4.669 \quad \text{so} \quad r_m = 3.569446 + O\left(1/\delta^m\right)$$

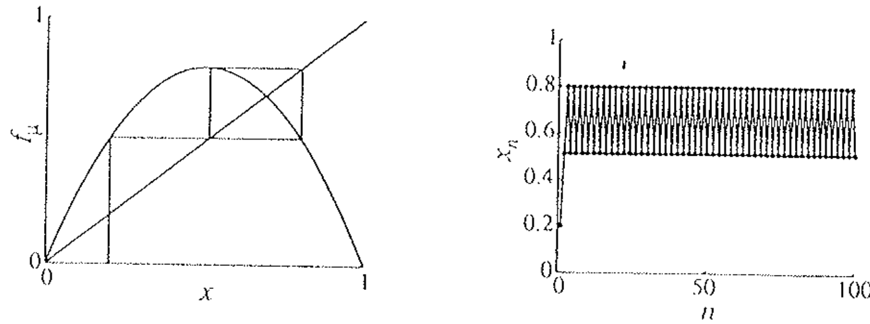
FOR SOME BUT NOT ALL VALUES OF r IN $r_\infty < r \le 4$, THERE SEEMS TO BE A STRANGE OR CHAOTIC ATTRACTOR, ITERATES APPEAR RANDOM.

ITERATES OF LOGISTIC MAP



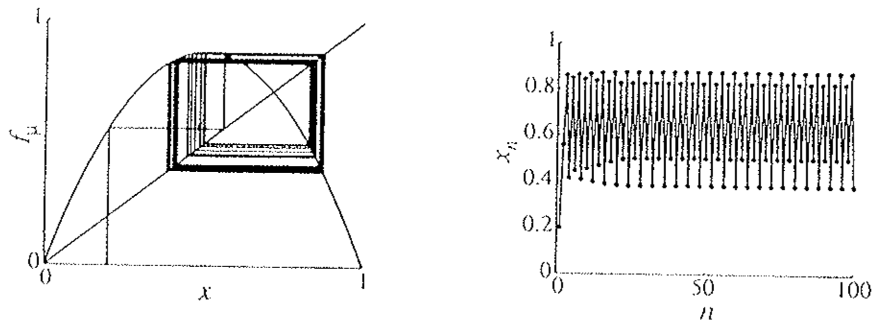
(i)

$X_0 = 0.2$
$\Gamma = 2$
stable FIXED pt



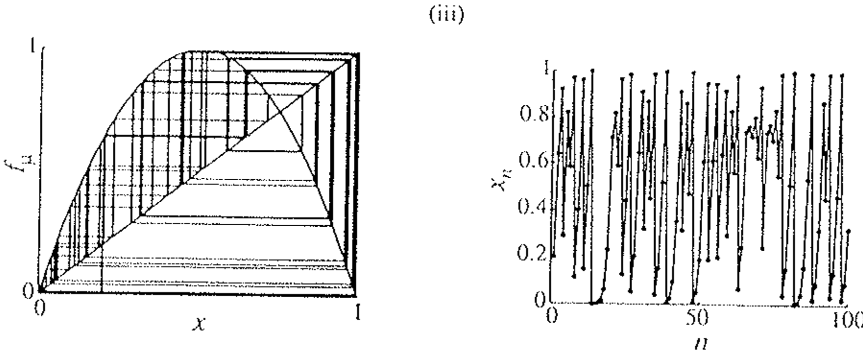
(ii)

$X_0 = 0.2$
$\Gamma = 3.2$
period 2-cycle



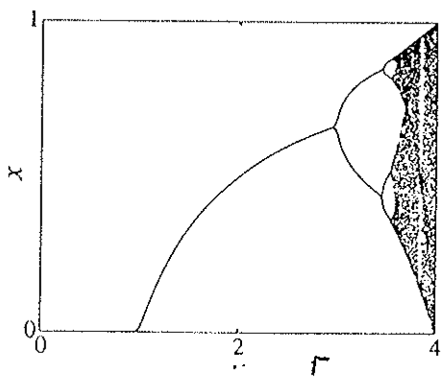
(iii)

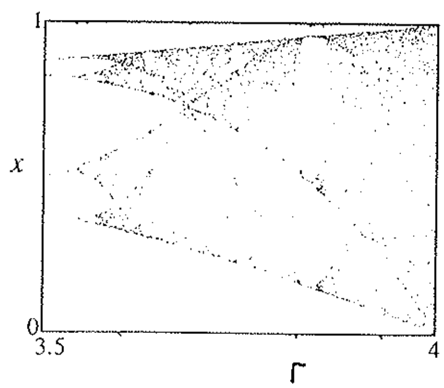
$X_0 = 0.2$
$\Gamma = 3.5$
4-cycle.



$X_0 = 0.2$
$\Gamma = 4$
" chaotic "

BIFURCATION DIAGRAM OF LOGISTIC MAP



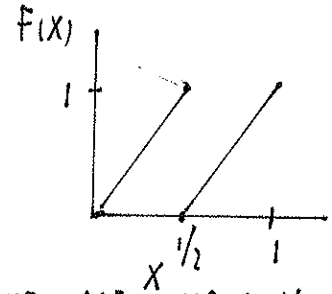


ZOOM OF LEFT FIGURE

EXAMPLE (PIECEWISE-LINEAR MAP)

CONSIDER A PIECEWISE LINEAR MAP

$$X_{n+1} = F(X_n) = \begin{cases} 2X_n & \text{IF } 0 \leq X_n < 1/2 \\ 2X_n - 1 & \text{IF } 1/2 < X_n \leq 1 . \end{cases}$$



NOTICE THAT IF $0 \leq X_n \leq 1$, THEN $0 \leq X_{n+1} \leq 1$. FIXED POINTS ARE SIMPLY $X=0$, $X=1$.

NOW CALCULATE

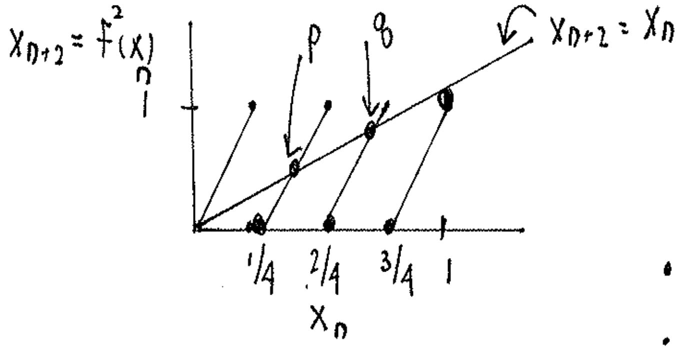
$$F(F(X_n)) = \begin{cases} 2F(X_n) & \text{IF } 0 \leq F(X_n) < 1/2 \\ 2F(X_n) - 1 & \text{IF } 1/2 < F(X_n) \leq 1 \end{cases}$$

$$= \begin{cases} 2(2X_n) & \text{IF } 0 \leq 2X_n < 1/2 \rightarrow 0 \leq X_n < 1/4 \\ 2[2X_n - 1] & \text{IF } 0 \leq 2X_n - 1 < 1/2 \rightarrow 1/2 \leq X_n < 3/4 \\ 2(2X_n) - 1 & \text{IF } 1/2 < 2X_n \leq 1 \rightarrow 1/4 < X_n \leq 1/2 \\ 2[2X_n - 1] - 1 & \text{IF } 1/2 < 2X_n - 1 \leq 1 \rightarrow 3/4 < X_n \leq 1 \end{cases}$$

THUS WE HAVE, REORDERING THE ABOVE.

$$F^2(X_n) = F[F(X_n)] = \begin{cases} 4X_n , & 0 \leq X_n < 1/4 \\ 4X_n - 1 , & 1/4 < X_n \leq 1/2 \\ 4X_n - 2 , & 1/2 \leq X_n < 3/4 \\ 4X_n - 3 , & 3/4 < X_n \leq 1 \end{cases}$$

WE PLOT $F^2(X)$ TO OBTAIN



WE LOOK FOR FIXED POINTS OF

$$X = F^2(X) = F[F(X)].$$

THEY ARE SHOWN TO THE LEFT AT

$X = P$ AND $X = Q$.

- IF $1/4 < X_n \leq 1/2 \longrightarrow X_n = 4X_n - 1 \Rightarrow X_n = 1/3$
- IF $1/2 < X_n < 3/4 \longrightarrow X_n = 4X_n - 2 \Rightarrow X_n = 2/3$.

THUS $P = 1/3$ AND $Q = 2/3$. WE CALCULATE $F(P) = Q$, $F(Q) = P$.

THUS $\{p, q\}$ FOR THE TWO-CYCLE. THE ITERATES ALTERNATE AS

$$f(p),\quad f(f(p)),\quad f(f(f(p))),\ \ldots$$

$$\downarrow$$

$$f(p),\quad f(q),\quad f(p),\ \ldots$$

$$2/3,\ 1/3,\ 2/3,\ 1/3,\ \ldots$$

NOW OBSERVE THAT $|f'(x_n)| > 1$ FOR ALL x_n.

<u>LEMMA</u> SUPPOSE $\{p_1, \ldots, p_N\}$ IS A CYCLE OF PERIOD N; i.e.

$$f(p_1) = p_2,\quad f(p_2) = p_3, \ldots,\quad f(p_N) = p_1.$$

SO EACH p_i IS A FIXED POINT OF THE NTH ITERATE, AND SATISFIES $f^N(p_i) = p_i$ FOR $i = 1, \ldots, N$.

THEN THE N CYCLE IS UNSTABLE.

<u>PROOF</u> DEFINE THE MULTIPLIER λ BY

$$\lambda = \frac{d}{dx} f^N(x)\Big|_{x=p_i}$$

IF $|\lambda| < 1 \longrightarrow$ N-cycle is stable

IF $|\lambda| > 1 \longrightarrow$ N-cycle IS UNSTABLE

WE CALCULATE

$$\frac{d}{dx} f^N(x) = \frac{d}{dx} f[f^{N-1}(x)]$$

$$= f'[f^{N-1}(x)] \frac{d}{dx} f^{N-1}(x)\Big|_{x=p_i}$$

$$= f'[f^{N-1}(x)] \frac{d}{dx} f^{N-2}[f(x)]\Big|_{x=p_i}$$

$$= f'[f^{N-1}(x)]\, f'[f^{(N-2)}(x)] \frac{d}{dx} f^{N-2}(x)\Big|_{x=p_i}$$

$$= f'(p_N)\, f'(p_{N-1}) \ldots f'(p_1).$$

THUS

$$\frac{d}{dx} f^N(x)\Big|_{x=p_i} = f'(p_1) \ldots f'(p_N) = \prod_{i=1}^{N} f'(p_i).$$

THUS,

$$|\lambda| = \prod_{i=1}^{N} |f'(p_i)| > 1 \quad \text{SINCE} \quad |f'(x)| > 1 \quad \forall x.$$

WE CONCLUDE THAT THE N-CYCLE MUST BE UNSTABLE.

IN FACT IT CAN BE PROVED THAT ALL MAPS WITH THE "SAME GENERAL SHAPE" AS $x_{n+1} = f(x_n)$ HAVE INFINITELY MANY CYCLES, OF ARBITRARILY LONG PERIOD.

<u>ATTRACTOR</u> FOR $x_{n+1} = f(x_n)$ IS A CLOSED BOUNDED SET A SUCH THAT

(i) A IS INVARIANT; i.e. $A = f(A)$.

(ii) A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS; i.e. THERE IS AN OPEN SET U CONTAINING A SUCH THAT IF $x_0 \in U$, THEN $\operatorname{dist}(x_n, A) \to 0$ AS $n \to \infty$, WHERE

$$\operatorname{dist}(x_n, A) = \min_{x \in A} |x_n - x|.$$

THE LARGEST SUCH OPEN SET U IS CALLED THE <u>BASIN OF ATTRACTION</u> OF A.

(iii) A IS MINIMAL: NO PROPER CLOSED SUBSET OF A SATISFIES (i) AND (ii).

FOR THE LOGISTIC MAP

- IF $1 < r \leq 3$ THEN $A = \{x^*\}$ WITH $x^* = 1 - 1/r$ IS AN ATTRACTOR. THE BASIN OF ATTRACTION IS THE OPEN INTERVAL $(0,1)$.
- IF $3 < r \leq 1 + \sqrt{6}$ THEN $A = \{p, q\}$ IS AN ATTRACTOR (WHAT IS BASIN OF ATTRACTION?). NOTICE THAT $\{p, x^*, q\}$ SATISFIES (i) AND (ii) ABOVE, BUT NOT (iii).

AN ATTRACTOR IS STRANGE, OR CHAOTIC, IF IT EXHIBITS SENSITIVE DEPENDENCE ON INITIAL CONDITIONS (SDIC). WE HAVE SDIC IF TWO NEARBY INITIAL POINTS x_0 AND $\hat{x}_0$ WITH

$|x_0 - \hat{x}_0|$ SUFFICIENTLY SMALL, GENERATE ORBITS $\{x_n\}, \{\hat{x}_n\}$ THAT SEPARATE EXPONENTIALLY FAST; i.e. THERE IS A $\lambda_{LYAP} > 0$ SUCH THAT IF $|x_0 - \hat{x}_0|$ IS SUFFICIENTLY SMALL THEN

$$|x_n - \hat{x}_n| \approx |x_0 - \hat{x}_0| e^{\lambda_{LYAP}\, n}$$

λ_{LYAP} IS CALLED THE LYAPUNOV EXPONENT.

LYAPUNOV EXPONENT

CONSIDER $x_{n+1} = f(x_n)$.

CONSIDER TWO NEARBY INITIAL POINTS x_0 AND $\hat{x}_0$ WITH $\delta_0 = \hat{x}_0 - x_0$ AND $|\delta_0|$ SMALL. THE CORRESPONDING ORBITS ARE $\{x_n\}$ AND $\{\hat{x}_n\}$. WE DEFINE $\delta_n = \hat{x}_n - x_n$. THEN λ_{LYAP} IS DEFINED BY

$$|\delta_n| \cong |\delta_0| \exp[\lambda_{LYAP}\, n] \qquad \lambda_{LYAP} \text{ IS LYAPUNOV EXPONENT.}$$

THUS $$\lambda_{LYAP} \cong \frac{1}{n} \ln\left(\left| \frac{\delta_n}{\delta_0} \right| \right).$$

IF $\lambda_{LYAP} > 0$ THEN THERE IS A SENSITIVE DEPENDENCE ON INITIAL CONDITIONS (SDIC).

A MORE PRECISE DEFINITION OF λ_{LYAP} IS OBTAINED AS FOLLOWS:

$$\delta_n = \hat{x}_n - x_n = f^n(\hat{x}_0) - f^n(x_0) = f^n(x_0 + \delta_0) - f^n(x_0)$$

SINCE $|\delta_0| << 1$, THEN $$\delta_n \cong \frac{d}{dx} f^n(x)\Big|_{x=x_0} \delta_0 .$$

NOW BY CALCULATING USING THE CHAIN RULE

$$\frac{d}{dx} f^n(x)\Big|_{x=x_0} = f'(x_{n-1})\, f'(x_{n-2}) \ldots f'(x_0)$$

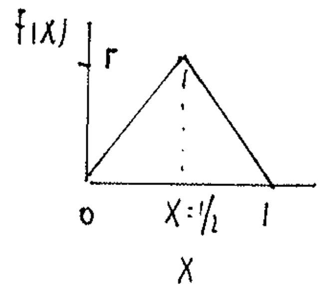
THUS $$\left| \frac{\delta_n}{\delta_0} \right| = \prod_{i=0}^{n-1} |f'(x_i)| \longrightarrow \lambda_{LYAP} \equiv \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{|\delta_n|}{|\delta_0|} \right) \text{ (DEFINITION)}$$

WE THEN GET THE KEY FORMULA $$\left\{ \lambda_{LYAP} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \qquad (*) \right.$$

NOTE $$\ln\left(\prod_{i=0}^{n-1} |f'(x_i)| \right) = \sum_{i=0}^{n-1} \ln |f'(x_i)| .$$

EXAMPLE FIND THE LYAPUNOV EXPONENT FOR THE TENT MAP

$$X_{n+1} = f(X_n)$$



WITH
$$f(X) = \begin{cases} 2rX & 0 \le X \le 1/2 \\ 2r(1-X) & 1/2 \le X \le 1, \end{cases}$$

AND $r > 0$.

NOW WE CALCULATE

$$f'(X) = \begin{cases} 2r & \text{FOR } 0 \le X \le 1/2 \\ -2r & \text{FOR } 1/2 \le X \le 1 \end{cases}$$

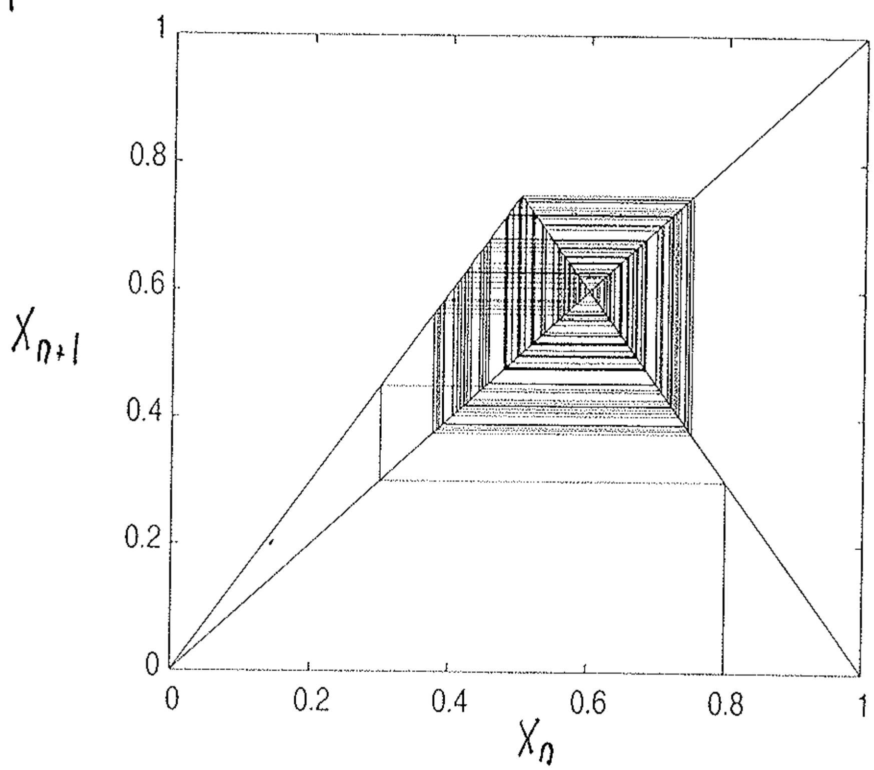
NOW FOR A TYPICAL ORBIT $\{X_0, X_1, X_2, X_3, \dots\}$ WE CALCULATE

$$\lambda_{LYAP} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(X_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|2r| = \ln(2r)$$

NOW IF $2r < 1 \rightarrow r < 1/2$ THEN $\lambda_{LYAP} < 0 \rightarrow$ NO SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

IF $2r > 1 \rightarrow r > 1/2$, THEN $\lambda_{LYAP} > 0 \rightarrow$ SENSITIVE DEPENDENCE ON INITIAL CONDITIONS.

AN ORBIT $r = 3/4$ IS SHOWN BELOW IN A COBWEB PLOT



EXAMPLE CONSIDER $X_{n+1} = f(X_n)$ AND ASSUME THAT X^* IS AN ATTRACTING FIXED POINT SO THAT $|f'(X^*)| < 1$. SUPPOSE THAT X_0 IS IN THE BASIN OF ATTRACTION OF X^*, AND THAT THE SET OF ITERATES IS $\{X_0, X_1, \ldots\}$, WITH $X_n \to X^*$ AS $n \to \infty$.

WE CALCULATE THE LYAPUNOV EXPONENT AS

$$\lambda_{LYAP} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(X_i)| = \lim_{n\to\infty} \frac{1}{n}\Big(\ln|f'(X_0)| + \cdots + \ln|f'(X_n)| \Big).$$

BUT $f'(X_n) \to f'(X^*)$ AS $n \to \infty$. HENCE

$$\lambda_{LYAP} = \lim_{n\to\infty} \frac{1}{n}\Big[n \ln|f'(X^*)| \Big] = \ln|f'(X^*)|$$

HOWEVER, SINCE $|f'(X^*)| < 1 \longrightarrow \ln|f'(X^*)| < 0$ AND $\lambda_{LYAP} < 0$.

CONSEQUENTLY, THERE IS NO SENSITIVE DEPENDENCE ON INITIAL CONDITIONS.

EXAMPLE CONSIDER THE QUADRATIC MAP

$$X_{n+1} = F(X_n) \quad \text{WITH} \quad F(X_n) = X_n^2 + r.$$

THE FIXED POINTS SATISFY $X = F(X)$ SO THAT

$$X^2 - X + r = 0 \quad \longrightarrow \quad X_\pm^* = \frac{1 \pm \sqrt{1-4r}}{2}$$

- IF $r > 1/4 \longrightarrow$ NO FIXED POINTS
- IF $r = 1/4 \longrightarrow$ ONE FIXED POINT
- IF $0 < r < 1/4 \longrightarrow$ TWO FIXED POINTS $X_\pm^*$.

THUS $r = 1/4$ IS A SADDLE-NODE BIFURCATION POINT.

NOW $F'(X) = 2X$ SO

- $\lambda = |F'(X_+^*)| = |1 + \sqrt{1-4r}| > 1 \longrightarrow X_+^*$ IS UNSTABLE FOR ALL $r \leq 1/4$
- $\lambda = |F'(X_+^*)| = |1 - \sqrt{1-4r}|$

NOW $-1 \leq 1 - \sqrt{1-4r} \leq 1$ WHEN $\sqrt{1-4r} \leq 2$

SO $1 - 4r \leq 4 \longrightarrow r \geq -3/4$.

WE CONCLUDE THAT FOR

$-3/4 < r < 1/4 \longrightarrow X_-^*$ IS STABLE. IF $r < -3/4 \rightarrow X_-^*$ UNSTABLE

WHEN $r = -3/4$ THEN $X_-^* = -1/2$ AND $\lambda = -1$. THIS IS THE FLIP BIFURCATION WHERE A TWO-CYCLE IS FORMED.

NOW WE ANALYZE 2-CYCLES. WE LOOK FOR FIXED POINTS OF THE MAP

$$X = F^2(X) = F[F(X)] = [F(X)]^2 + r = [X^2 + r]^2 + r.$$

THIS GIVES

$$X^4 + 2rX^2 + r^2 - X + r = 0.$$

NOW $X^2 - X + r = 0$ MUST BE A FACTOR SINCE THESE ARE FIXED POINTS.

POLYNOMIAL DIVISION

$$
\begin{array}{r l}
 & x^2 + x + (r+1) \\
x^2 - x + r \;\big) & x^4 + 2 r x^2 - x + r^2 + r \\
 & -(x^4 - x^3 + r x^2) \\
\hline
 & x^3 + r x^2 - x + r^2 + r \\
 & -(x^3 - x^2 + r x) \\
\hline
 & (r+1) x^2 - (r+1) x + (r+1) r
\end{array}
$$

THUS THE PERIOD 2 POINTS ARE ROOTS OF

$$(x^2 - x + r)(x^2 + x + r + 1) = 0.$$

THUS SET $x^2 + x + r + 1 = 0$. THE ROOTS ARE LABELLED BY p, q

SO $p = \dfrac{-1 \pm \sqrt{1 - 4(r+1)}}{2} \;\rightarrow\; p = \dfrac{-1 - \sqrt{-3 - 4r}}{2}$

$$q = \frac{-1 + \sqrt{-3 - 4r}}{2}.$$

NOTICE THAT THESE ROOTS EXIST WHEN $r \leq -3/4$. THEY ARE BORN AT FLIP BIFURCATION POINT $r = -3/4$.

NOTICE $\quad pq = r + 1 \quad$ SINCE $\quad x^2 + x + r + 1 = (x - p)(x - q)$.

$$p + q = -1$$

NOW WE CALCULATE

$$f(p) = p^2 + r = \frac{1}{4}\left(-1 - \sqrt{-3-4r}\right)^2 + r = \frac{1}{4}\left(1 + 2\sqrt{3-4r} + (-3-4r)\right) + r$$

$$= -\frac{1}{2} + \frac{1}{2}\sqrt{-3-4r} = q. \qquad \text{SO } f(p) = q$$

AND $f(f(p)) = f(q) = p$.

SO NOW THE 2-CYCLE IS COMPOSED OF $\{p, q\}$.

NOW THE STABILITY OF THE 2-CYCLE IS DETERMINED BY

$$\lambda = \frac{d}{dx} f[f(x)]\Big|_{x=p} = f'(f(p))\, f'(p) = f'(q)\, f'(p)$$

$$= 2q\, 2p = 4qp = 4(r+1).$$

BUT $|\lambda| < 1$ IF $4(r+1) > -1 \rightarrow r > -5/4$. <u>NOTE</u> $\lambda = -1$ when $r = -5/4$.

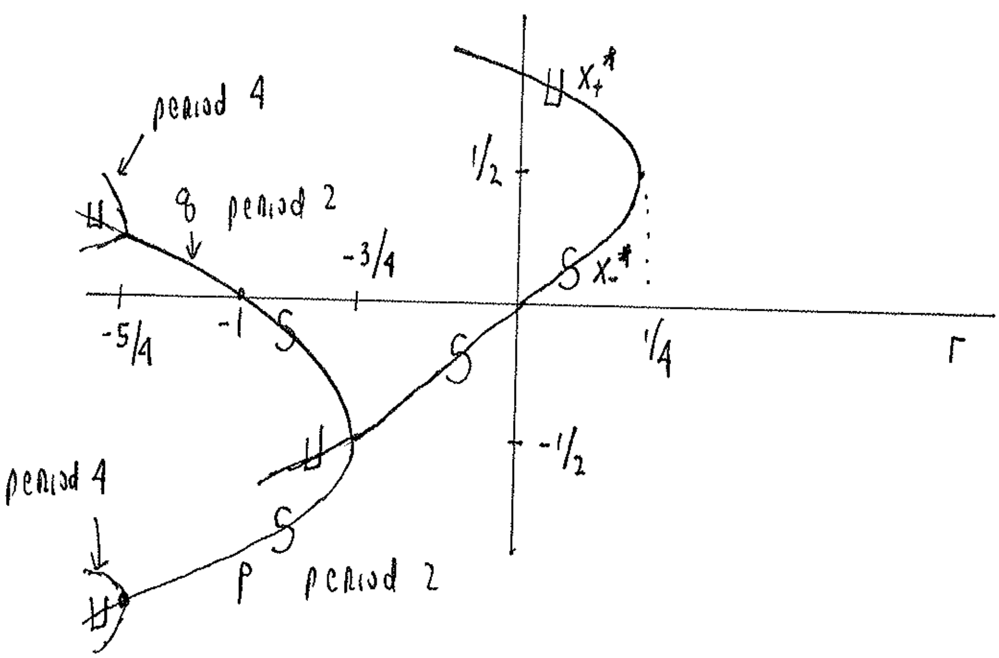
SO ON $-5/4 < r < -3/4$ 2-cycle is stable

$r < -5/4$ 2-cycle unstable

$r = -5/4$ flip bifurcation of 2-cycle (creation of period 4 cycle).

NOTICE ALSO THAT WHEN $r = -1$ THEN $\lambda = 0$, SO AT THIS VALUE OF r THE 2-CYCLE IS SUPERSTABLE.

WE PLOT A BIFURCATION DIAGRAM AS



EXAMPLE CONSIDER THE CUBIC MAP

$$x_{n+1} = f(x_n) = r x_n - x_n^3 \quad \text{WITH} \quad -\infty < r < \infty.$$

(i) FIND ALL THE FIXED POINTS AND DETERMINE FOR WHICH r THEY EXIST.

(ii) DETERMINE THE STABILITY OF THE FIXED POINTS FOUND IN (i) AS A FUNCTION OF r.

(iii) FIND ALL PERIOD-2 CYCLES. [HINT: IF $f(p) = q$ AND $f(q) = p$ SHOW THAT p, q ARE ROOTS OF $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - r x^2 + 1) = 0$.

(iv) DETERMINE THE STABILITY OF THE PERIOD-2 CYCLES AS A FUNCTION OF r.

(v) PLOT A PARTIAL BIFURCATION DIAGRAM FOR THE REGION

$$-2 \leq r \leq 4$$

(SOLUTION TO BE WORKED OUT BY STUDENTS)

MAPS OF HIGHER DIMENSION

IF $\underline{x}_{n+1} = \underline{F}(\underline{x}_n)$

WITH $\underline{x}_n = (x_{1n}, \ldots, x_{Mn})^T, \quad \underline{F} = (F_1, \ldots, F_M)^T$

THEN DEFINE $\underline{x}^*$ TO BE A FIXED POINT

$$\underline{x}^* = \underline{F}(\underline{x}^*)$$

TO DETERMINE THE STABILITY OF $\underline{x}^*$ CALCULATE THE JACOBIAN J

$$J = \left. \begin{pmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_M \\ \vdots & & \\ \partial F_M/\partial x_1 & \cdots & \partial F_M/\partial x_M \end{pmatrix} \right|_{\underline{x} = \underline{x}^*}$$

THE LINEARIZATION IS $\underline{x}_n = \underline{x}^* + \underline{y}_n$, WHERE $\underline{y}_n$ SATISFIES THE LINEAR RECURRENCE $\underline{y}_{n+1} = J\,\underline{y}_n$

ASSUME THAT J IS DIAGONALIZABLE IN THAT

$$J = S \Lambda S^{-1} \qquad \Lambda = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_M \end{pmatrix}$$

μ_j eigenvalue of J.

THEN,

$$\underline{y}_{n+1} = S \Lambda (S^{-1} \underline{y}_n)$$

MULTIPLY BY S^{-1}

$$S^{-1} \underline{y}_{n+1} = \Lambda (S^{-1} \underline{y}_n)$$

DEFINE $\underline{\xi}_n = S^{-1} \underline{y}_n$. THEN $\underline{\xi}_{n+1} = \Lambda \underline{\xi}_n$.

WE CONCLUDE THAT x^* IS STABLE IF $\|\xi_n\| \to \infty$ AS $n \to \infty$.
THIS REQUIRES THAT $|\mu_i| < 1$ FOR $i = 1, \ldots, M$.