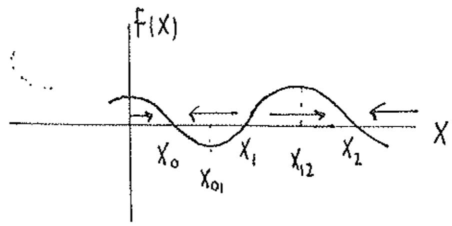
FLOWS ON THE LINE

CONSIDER $x' = f(x)$ AND ASSUME THERE ARE 3 EQ. POINTS AS SHOWN



x_0, x_2 stable

x_1 UNSTABLE

NOW LET ξ BE $f(\xi) = 0$. WE LINEARIZE BY LETTING

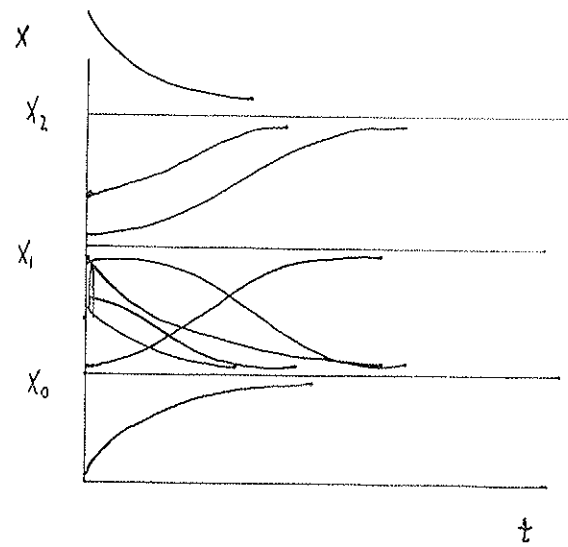
$$x = \xi + \varepsilon e^{\sigma t}$$

THEN $$\varepsilon \sigma e^{\sigma t} = f(\xi + \varepsilon e^{\sigma t}) = f(\xi) + \varepsilon f'(\xi) e^{\sigma t}.$$

THUS $\sigma = f'(\xi)$. $\longrightarrow$ STABLE IF $f'(\xi) < 0$ UNSTABLE OTHERWISE

NOW $x'' = f'(x)\, x'$.

	x'	f'	x''
$(0, x_0)$	>0	<0	<0
(x_0, x_{01})	<0	<0	>0
(x_{01}, x_1)	<0	>0	<0
(x_1, x_{12})	>0	>0	>0
(x_{12}, x_2)	>0	<0	<0
(x_2, ∞)	<0	<0	>0

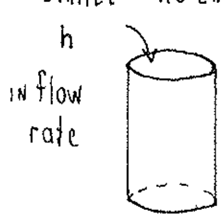


EXAMPLE $x' = \pm x^3$ LINEARIZATION FAILS.

DEFINITION x_0 IS A LINEARLY STABLE FIXED POINT IF FOR ALL INITIAL CONDITIONS SUFFICIENTLY CLOSE TO x_0 THAT $x(t)$ REMAINS CLOSE TO (OR TENDS TO) x_0 AS t INCREASES.

EXAMPLE 1 (TORICELLI'S LAW)

FLUID FLOWS UNDER GRAVITY WHEN A TANK IS DRAINED THROUGH A SMALL HOLE AT THE BOTTOM



A: CROSS-sectional area of cylinder

$$\frac{dV}{dt} = h - a s$$

s = speed of outflow

a : cross-sectional area of hole

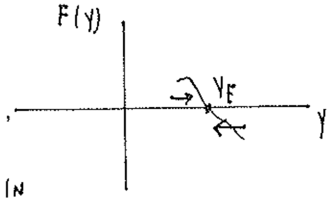
NOW TORICELLI'S LAW: $s = \alpha\sqrt{2gy}$ $\quad 0 < \alpha \le 1$

THUS
$$\frac{dV}{dt} = h - a\alpha\sqrt{2gy}$$

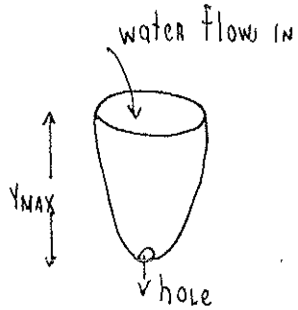
BUT NOW $V(y) = \int_0^y A(s)\,ds$. THUS $\dfrac{dV}{dt} = V'(y)\dfrac{dy}{dt} = A(y)\dfrac{dy}{dt}$

SO
$$\begin{cases} A(y)\dfrac{dy}{dt} = h - a\alpha\sqrt{2gy} \\ y(0) = y_0 \end{cases}$$

EQUILIBRIUM POINT IS $\left(\dfrac{h}{a\alpha}\right)^2 = 2gy$ $\qquad y_E = \dfrac{1}{2g}\left(\dfrac{h}{a\alpha}\right)^2$.



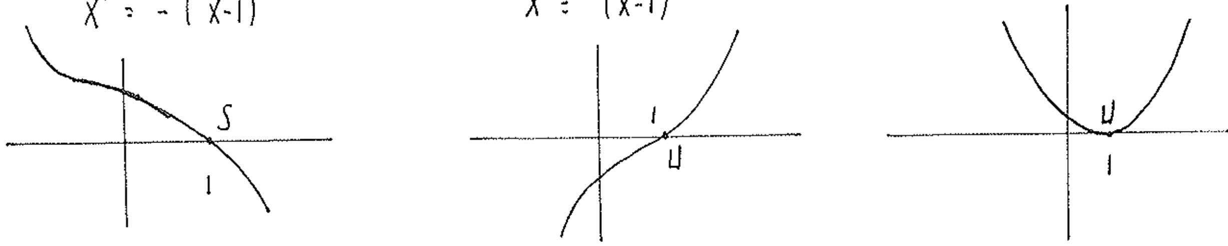
$$F(y) = \frac{h - a\alpha\sqrt{2gy}}{A(y)}$$



IF $y_E > y_{MAX}$ THEN TANK WILL FILL UP AND OVERFLOW BEFORE EQUILIBRIUM IS REACHED.

EXAMPLE 2 (LINEARIZATION FAILS)

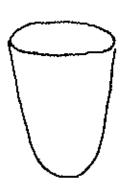
$x' = -(x-1)^3$ $\qquad x' = (x-1)^3$ $\qquad x' = (x-1)^2$



EXAMPLE 3 (TIME FOR ALL WATER OUT).

SUPPOSE $h = 0$ NO INFLOW SO THAT $A(y)\dfrac{dy}{dt} = -K\sqrt{y}$ $\qquad K = a\alpha\sqrt{2g}$.

ASSUME $y(0) = y_0$. FIND T SUCH THAT $y(T) = 0$.



NOW $\displaystyle\int_0^{y_0} y^{-1/2} A(y)\,dy = -K\int_T^0 dt = KT$.

THUS $\displaystyle T = \frac{1}{K}\int_0^{y_0} y^{-1/2} A(y)\,dy$.

NOW LET'S ASSUME THAT $V(y_m) = V_0$ BUT $A(y) = c\,y^p$.

THEN $$V(y_m) = \int_0^{y_m} c\,y^p\,dy = \frac{c}{p+1}\,y_m^{\,p+1} = V_0 \qquad C = \frac{V_0(p+1)}{y_m^{\,p+1}}$$

THEN $$A(y) = \frac{V_0(p+1)}{y_m}\left(\frac{y}{y_m}\right)^p.$$

THE TIME TO ESCAPE IS
$$T = \frac{1}{K}\int_0^{y_0} \frac{V_0(p+1)}{y_m^{\,p+1}}\,y^{\,p-1/2}\,dy = \frac{1}{K}\,\frac{V_0(p+1)}{y_m^{\,p+1}}\,\frac{1}{p+1/2}\,y_m^{\,p+1/2}.$$

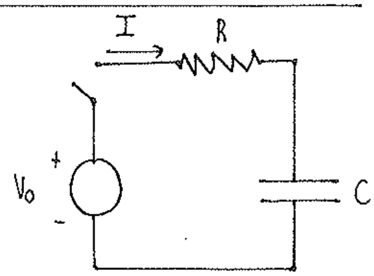
THUS $$T = \frac{1}{K}\,V_0\left(\frac{p+1}{p+\frac{1}{2}}\right)\frac{1}{\sqrt{y_m}}$$

FOR A STRAIGHT CYLINDER $p=0$ AND $\dfrac{V_0}{y_m} = \pi r_0^2$.

THIS GIVES $$T = \frac{1}{K}\left(\frac{V_0}{y_m}\right)\left(\frac{1}{1/2}\right)\sqrt{y_m} = \frac{2\pi r_0^2}{K}\sqrt{y_m}.$$

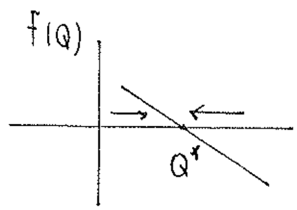
BUT $V_0 = \pi r_0^2\,y_m$ $\qquad \sqrt{y_m} = \dfrac{\sqrt{V_0}}{\sqrt{\pi}\,r_0}$

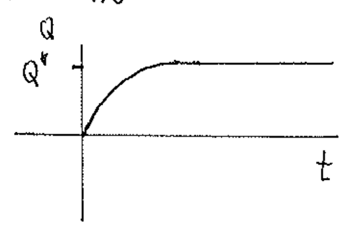
EXAMPLE (NONLINEAR CIRCUITS)

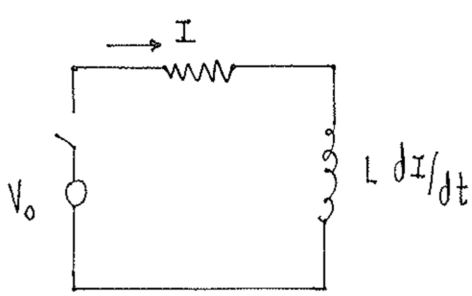


$$-V_0 + R\dot{Q} + Q/C = 0 \qquad Q\text{: charge on capacitor}$$

SO $$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}$$

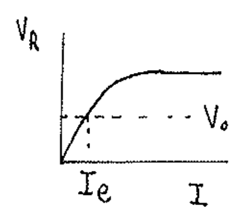


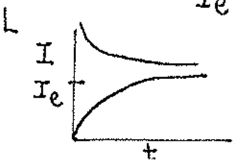




$$L\,dI/dt + V_R(I) = V_0$$

NOW $$\frac{dI}{dt} = \frac{V_0 - V_R(I)}{L}$$



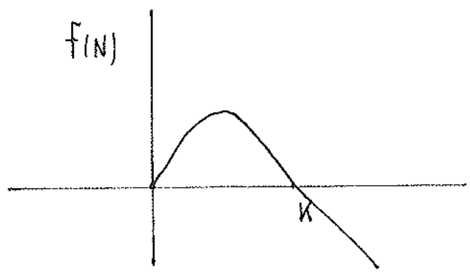


EXAMPLE (LOGISTIC GROWTH)

$$\frac{dN}{dt} = r\left(1 - N/K\right)N$$

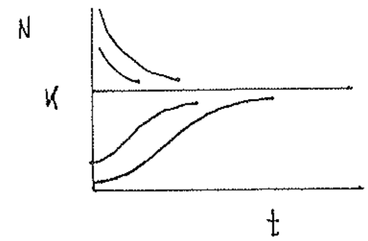
r = rate of growth for small N

K: carrying capacity.



$N = 0$ UNSTABLE

$N = K$ stable



EXISTENCE AND UNIQUENESS

THEOREM CONSIDER $\dot{x} = f(x)$

$x(0) = x_0$

SUPPOSE THAT f, f' ARE CONTINUOUS ON AN OPEN INTERVAL R CONTAINING x_0. THEN THE IVP HAS A SOLUTION $x(t)$ ON SOME INTERVAL $(-\tau, \tau)$ ABOUT $t=0$ AND THE SOLUTION IS UNIQUE.

EXAMPLE (BLOW-UP) $\dot{x} = 1 + x^2$

$x(0) = x_0$.

THE SOLUTION IS $x = \tan(t + c)$ $c = \tan^{-1} x_0$. (principal value)

NOTICE THE SOLUTION IS DEFINED ON

$-\pi/2 < t + c < \pi/2$ $\quad -\pi/2 - \tan^{-1}x_0 < t < \pi/2 - \tan^{-1} x_0$

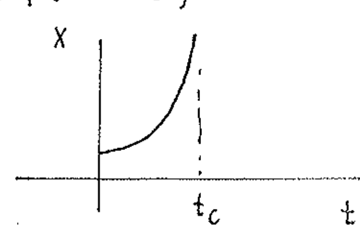
EXAMPLE (BLOW-UP) THE SOLUTION TO

$x' = e^x$

$x(0) = x_0$

IS $e^{-x}dx = dt \rightarrow -e^{-x} + e^{-x_0} = t$ $\quad x = -\ln(e^{-x_0} - t)$

HENCE THE SOLUTION BLOWS UP AT $t_c = e^{-x_0}$.



EXAMPLE (GENERIC BLOW-UP)

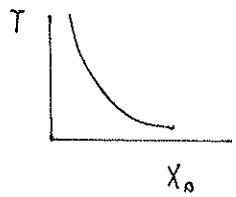
CONSIDER
$$\frac{dx}{dt} = a x^p \qquad p > 1 \qquad a > 0$$
$$X(0) = X_0$$

THE SOLUTION BLOW UP AT A FINITE TIME T depending ON X_0.

SOLVE TO get
$$\frac{1}{1-p}\left(X^{1-p} - X_0^{1-p} \right) = a t$$

THEN $X \to \infty$ AT $t = T$ WHERE $aT = \frac{1}{p-1} X_0^{1-p}$ OR $T = \frac{1}{a(p-1)} X_0^{1-p}$

THUS,



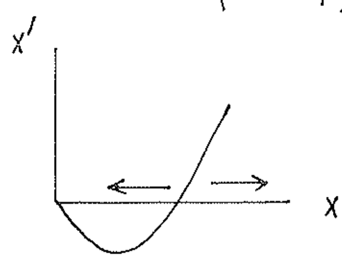
$T \uparrow$ AS $X_0 \downarrow$.

NOTE THAT FOR $X' = 1 + X^2$
$$X(0) = X_0 > 0$$

THEN FOR X LARGE, $X' \approx X^2$ WHICH BLOWS UP IN FINITE t.

EXAMPLE (THRESHOLD)

$$X' = -X\left(1 - X/_T\right)$$

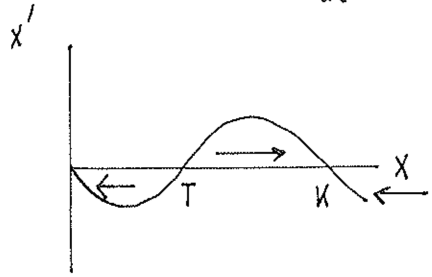


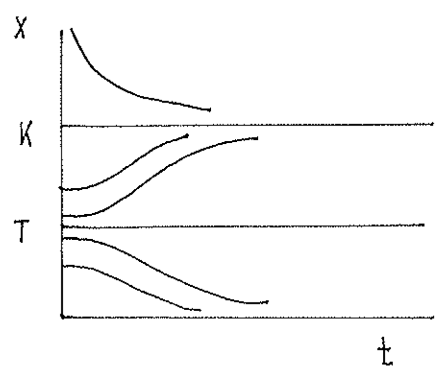
$X = T$ UNSTABLE POINT.

$X(t) \to 0$ AS $t \to \infty$ IF $0 < X(0) <$

NOTE THAT FOR X LARGE WE HAVE
$$dX/_{dt} \approx X^2/_T \quad \text{WHICH HAS FINITE TIME BLOW-UP.}$$

THIS IS NOT ACCEPTABLE FOR A BIOLOGICAL MODEL SO WE INSTEAD CONSIDER
$$\frac{dX}{dt} = -X\left(1 - X/_T\right)\left(1 - X/_K\right) \qquad K > T$$





EXAMPLE (NON-UNIQUENESS)

$$* \quad \begin{cases} x' = -c\,x^{1/2} & c > 0 \\ x(0) = 0 \end{cases}$$

$f(x) = x^{1/2}$ IS NOT DIFFERENTIABLE AT $x = 0$ SO THAT THE UNIQUENESS THEOREM DOES NOT APPLY. $\longrightarrow$ NO GUARANTEE OF UNIQUE SOLUTION.

$$x^{-1/2}\,dx = -c\,dt \qquad x(0) = 0$$

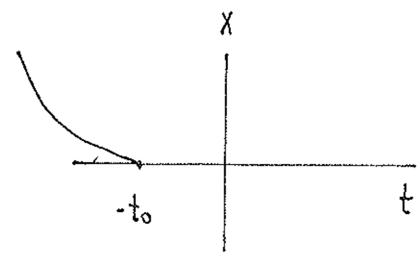
WE SOLVE TO GET $\quad 2x^{1/2} = -c(t - t_0)$

BUT $\quad x = \dfrac{c^2}{4}(t - t_0)^2$

THUS $\quad x = 0$ AND $x = \dfrac{c^2}{4}t^2$ ARE BOTH SOLUTIONS.

ALTERNATIVELY, WE CAN HAVE A C^1 SOLUTION

$$x(t) = \begin{cases} 0 & -t_0 \leq t \leq 0 \\ \dfrac{c^2}{4}(t + t_0)^2 & t \leq -t_0 \end{cases}$$



NOTE: FOR THE LEAKY BUCKET PROBLEM, * ASKS THE QUESTION CAN WE DETERMINE THE TIME WHEN THE LAST DROP OF WATER LEAKED OUT OF THE BUCKET GIVEN THAT WE OBSERVE NO WATER IN THE BUCKET AT TIME $t = 0$?

ON PHYSICAL GROUNDS THIS QUESTION CANNOT BE ANSWERED. MATHEMATICA IT PRODUCES A NON-UNIQUE SOLUTION.

POTENTIALS $\quad$ CONSIDER $\quad \dfrac{dx}{dt} = f(x)$

AND DEFINE $V(x)$ BY $\quad V'(x) = -f(x) \qquad V(x)$ IS THE POTENTIAL.

THUS $\quad \dfrac{dx}{dt} = -V'(x).$

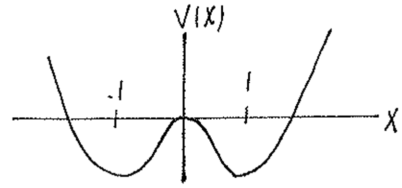
NOW WE CALCULATE $\quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = -[V'(x)]^2. \qquad \dfrac{dV}{dt} \leq 0.$

THUS, MINIMA OF $V(x)$ ARE STABLE EQUILIBRIA.

FOR EXAMPLE IF

$$f(x) = -(-x + x^3) \qquad V'(x) = x^3 - x$$

THEN $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$



MINIMA OF TWO WELLS IS STABLE.

<u>CRITICAL POINTS</u>

$$V'(x^*) = 0$$

$V''(x^*) > 0$ MINIMA (stable)

$V''(x^*) < 0$ MAXIMA (unstable)

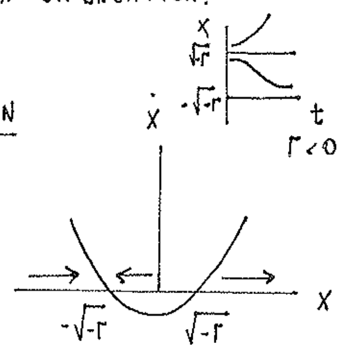
$V''(x^*) = 0 \rightarrow$ NO INFORMATION

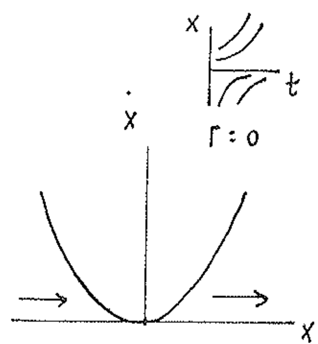
BIFURCATIONS

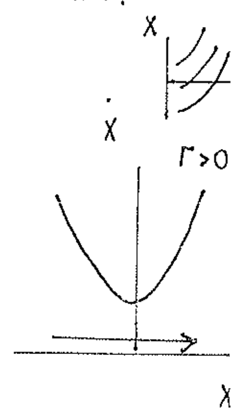
QUALITATIVE STRUCTURE OF THE FLOW CAN CHANGE AS PARAMETERS ARE VARIED. THIS PHENOMENA IS CALLED A BIFURCATION.

<u>SADDLE NODE BIFURCATION</u>
(FOLD BIFURCATION)

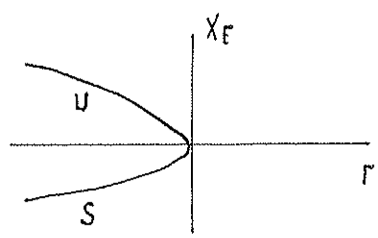
$$\dot{x} = r + x^2$$







THE EQUILIBRIUM POINTS ARE $x_E = \pm\sqrt{-r}$



THUS AS r IS VARIED, TWO EQUILIBRIUM POINTS GO TO ZERO EQ. POINTS AS r CROSSES PAST A CRITICAL VALUE $r = r_c$.

<u>NORMAL FORM</u> $\dot{x} = f(x, r)$ UNDER WHAT CONDITION DOES A SADDLE NODE BIFURCATION OCCUR? SUPPOSE THAT $\exists$ A POINT x_E, r_c SUCH THAT

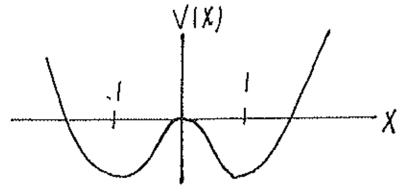
$$f(x_E, r_c) = 0, \quad \partial f/_{\partial x}(x_E, r_c) = 0 \quad \text{WITH}$$

$$f_{xx}(x_E, r_c) \neq 0 \quad \text{AND} \quad f_r(x_E, r_c) \neq 0.$$

FOR EXAMPLE IF

$f(x) = -(-x + x^3)$ $\qquad V'(x) = x^3 - x$

THEN $\quad V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$



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$V''(x^*) > 0$ MINIMA (stable)

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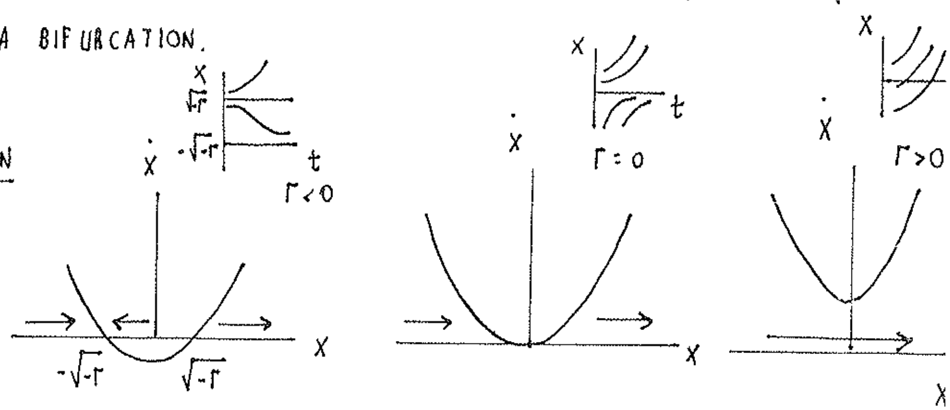
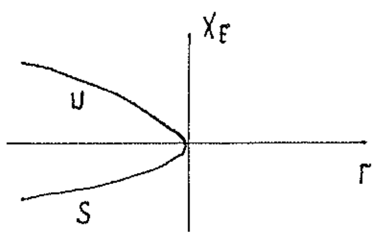


THE EQUILIBRIUM POINTS ARE $\quad X_E = \pm\sqrt{-r}$



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$$f(X_E, r_c) = 0, \quad \partial f/\partial x\,(X_E, r_c) = 0 \quad \text{WITH}$$

$$f_{xx}(X_E, r_c) \neq 0 \quad \text{AND} \quad f_r(X_E, r_c) \neq 0.$$

WE THEN LINEARIZE NEAR X_E AND r_c. LET

$$X = X_E + y, \quad r = r_c + \delta$$

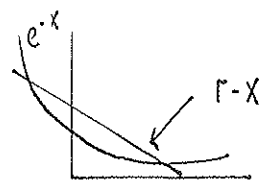
SUBSTITUTE TO GET $\dot{y} = f(X_E + y, r_c + \delta) = f(X_E, r_c) + f_X(X_E, r_c)y + f_{XX}(X_E, r_c)\, y^2/2 + f_r(X_E, r_c)\delta + \ldots$

THUS THE LOCAL ODE IS

$$\dot{y} = \delta f_r^0 + \frac{y^2}{2} f_{XX}^0 + \ldots \quad \text{WHERE} \quad f_{XX}^0 \equiv f_{XX}(X_E, r_c), \quad f_r^0 = f_r(X_E, r_c)$$

<u>EXAMPLE</u> SUPPOSE THAT $f(X, r) = r - X - e^{-X}$

NOW $r - X = e^{-X}$



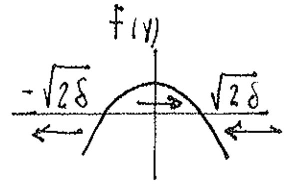
AS $r \uparrow$ TWO ROOTS

$r \downarrow$ NO ROOTS.

NOW THE BIFURCATION POINT IS WHERE $f = f_X = 0$ SO THAT

$$\left.\begin{array}{l} r - X = e^{-X} \\ -1 = -e^{-X} \end{array}\right\} \Longrightarrow \text{THIS GIVES } r = 1 \text{ AND } X = 0.$$

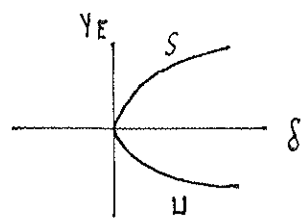
NOW $f_r^0 = 1, \quad f_{XX}^0 = -e^{-X}\big|_{X=0} = -1.$

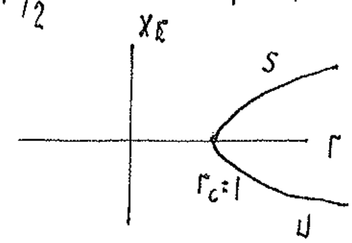


THUS NEAR $X_E = 0$ AND $r_c = 1$ WE GET

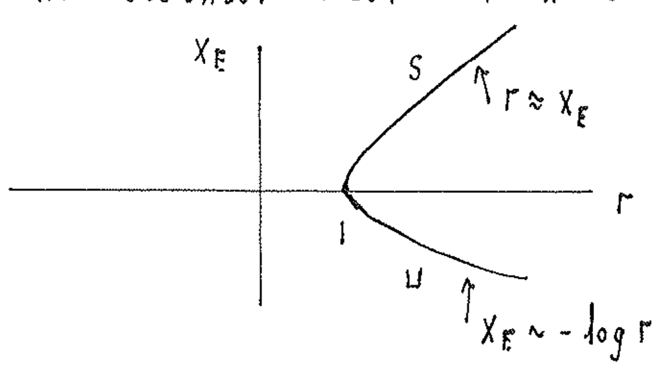
$$X = y, \quad r = 1 + \delta \quad \text{WHERE} \quad y' = \delta - y^2/2 + \ldots \quad y \ll 1$$

THUS <u>LOCALLY</u> $y_E = \pm\sqrt{2\delta}$





NOW IF WE GLOBALLY PLOT $r - X = e^{-X}$ WE GET



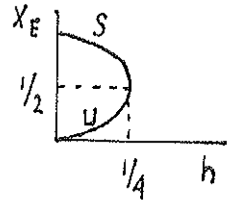
IS THE GLOBAL BIFURCATION DIAGRAM.

<u>EXAMPLE</u> LOGISTIC GROWTH WITH HARVESTING

$$\frac{dx}{dt} = x(1-x) - h$$

THE EQUILIBRIUM POINTS SATISFY $x^2 - x + h = 0$ $\quad x_E = \dfrac{1 \pm \sqrt{1-4h}}{2}$

THUS WE CAN PLOT



NOW THE LOCAL ANALYSIS PROCEEDS AS FOLLOWS:

$$f(x,h) = x(1-x) - h \longrightarrow h = 1/4$$

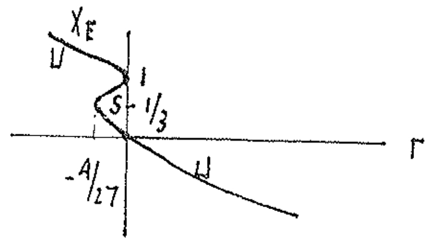
$$f_x(x,h) = 1 - 2x \longrightarrow x = 1/2$$

NOW $f_h = -1$, $f_{xx} = -2$ THE LOCAL EQUATION IS

$$x = 1/2 + y, \quad h = 1/4 + \delta \quad \text{so that} \quad y' = -\delta - y^2$$

SO THAT $y = \pm\sqrt{-\delta}$. THUS $\quad x = 1/2 \pm \sqrt{1/4 - h}$ locally.

<u>EXAMPLE</u> $\quad \dfrac{dx}{dt} = x(1-x)^2 + r$



NOW THE BIFURCATION POINTS SATISFY

$f(x,r) = f_x(x,r) = 0$ SO THAT

$$x(1-x)^2 + r = 0$$

$$(1-x)^2 - 2x(1-x) = 0$$

NOW THIS GIVES $\quad (1-x)[1-3x] = 0$ SO THAT $x = 1$, $x = 1/3$.

IF $x = 1 \rightarrow r = 0$

$\quad x = 1/3 \rightarrow r = -4/27$

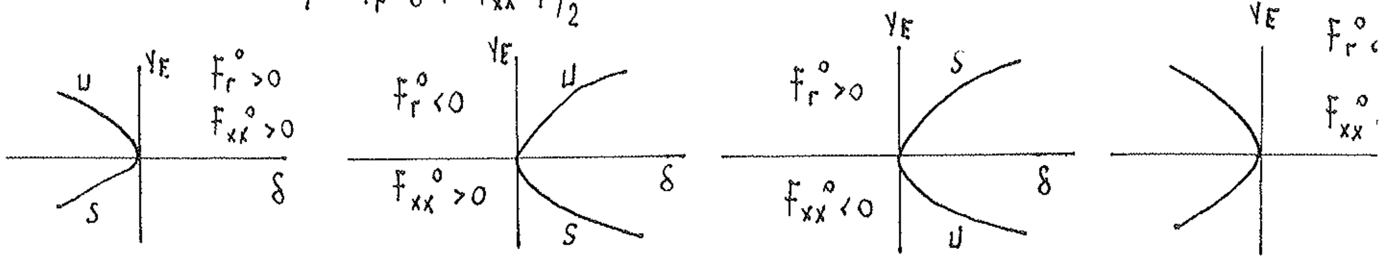
NOW WE CALCULATE $f_r = 1$, $f_{xx} = 6x - 4$.

THUS NEAR $x = 1, r = 0 \qquad \dot{y} = \delta + y^2 \qquad x = 1 + y, \quad r = \delta$

$\qquad x = 1/3, r = -4/27 \qquad \dot{y} = \delta - y^2 \qquad x = \frac{1}{3} + y \qquad r = -\frac{4}{27} + \delta$

THE NORMAL FORM EQUATION IS

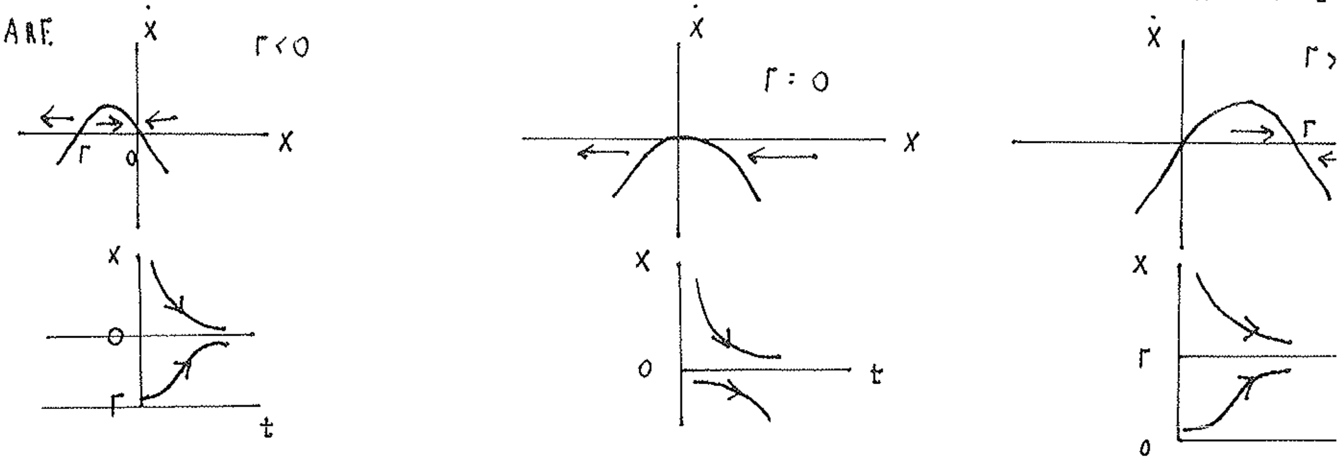
$$\dot{y} = F_r^\circ \delta + F_{xx}^\circ y^2/2$$



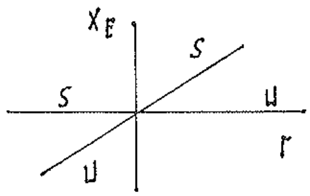
TRANSCRITICAL BIFURCATIONS

NOW CONSIDER $\quad x' = rx - x^2$

THEN TWO CRITICAL POINTS EXCHANGE STABILITIES AS r CROSSES THROUGH A CRITICAL VALUE. NOTICE $r = 0$ AND $x = r$ ARE EQUILIBRIUM POINTS. THE PLOTS ARE



THEN WE HAVE THE FOLLOWING TRANSCRITICAL BIFURCATION DIAGRAM



EXCHANGE OF STABILITY AS r INCREASES PAST $r =$

<u>EXAMPLE</u> CONSIDER $\quad \dot{x} = r \log x + x - 1 = f(x, r)$

NOTICE THAT $x = 1$ IS AN EQUILIBRIUM FOR ANY r.

NOW SET $\quad f(x, r) = 0 \longrightarrow r \log x + x - 1 = 0$

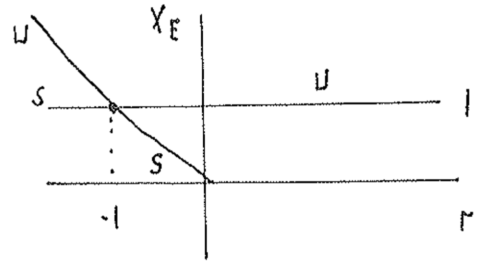
$\qquad\qquad f_x(x, r) = 0 \qquad \dfrac{r}{x} + 1 = 0 \quad$ SO $\quad x = -r.$

THEN $\quad -x \log x + x - 1 = 0 \longrightarrow x(1 - \log x) = 1.$ THE ONLY SOLUTION TO THIS IS $x = 1$ WHICH IMPLIES $r_c = -1$. THUS WE GET THE FOLLOWING PLOTS

$$x - 1 = -r \log x \qquad r = \frac{1 - x}{\log x} \longrightarrow \frac{-1}{1/x} \longrightarrow -1 \text{ AS } x \to$$

NOW NOTICE THAT $r < 0$.

$$r = \frac{1-X}{\log X}$$



$r \to 0$ AS $X \to 0$

$r \to -\infty$ AS $X \to +\infty$

NOW $\dot{X} = f(X, r)$ $\quad f(X, r) = r \log X + X - 1$.

WE GET THE NORMAL FORM AT $r = -1$, $X = 1$

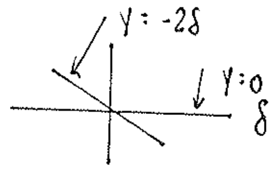
$$f_r^0 = \log X = 0 \quad \text{AT} \quad X = 1$$

NOW WE LET $r = -1 + \delta$, $X = 1 + y$ SO THAT

$$\dot{y} = (-1+\delta)\log(1+y) + y = (-1+\delta)(y - y^2/2 + \dots) + y$$

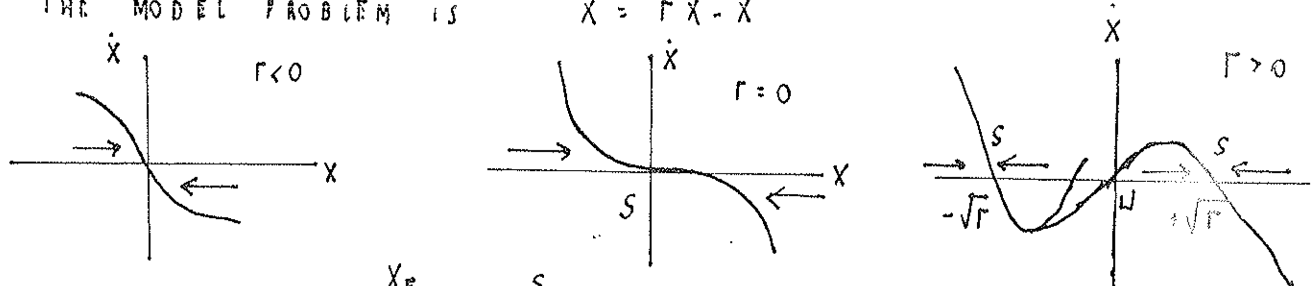
THUS $\dot{y} = \delta y + y^2/2 + O(\delta y^2)$

THUS $y = 0$ AND $y = -2\delta$.

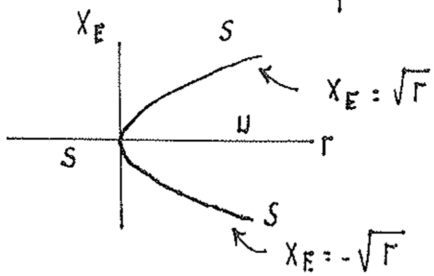


PITCHFORK BIFURCATIONS (common when we have a symmetry)

THE MODEL PROBLEM IS $\dot{X} = rX - X^3$

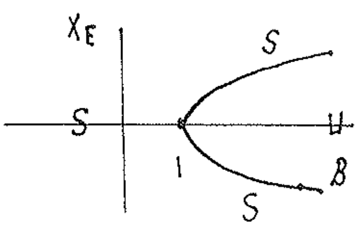


THUS WE OBTAIN



supercritical pitchfork bifurcation.

EXAMPLE CONSIDER $\dot{X} = -X + B \tanh X$. NOW THE EQUILIBRIA ARE



$$B = \frac{X}{\tanh X} \longrightarrow \begin{array}{l} B \sim X \quad \text{AS} \quad X \to +\infty \\ B \sim -X \quad \text{AS} \quad X \to -\infty \end{array}$$

WITH $B \to 1$ AS $X \to 0$.

FOR X SMALL WE GET $\dot{X} = (B-1)X - BX^3/3 + \dots$ NORMAL FORM.

REMARK CONSIDER $\dot{x} = f(x)$. DEFINE A POTENTIAL $V(x)$ BY

$$V'(x) = -f(x).$$

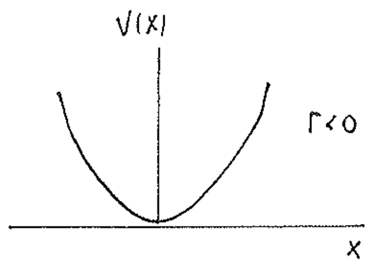
THEN $\dot{x} = -V'(x)$

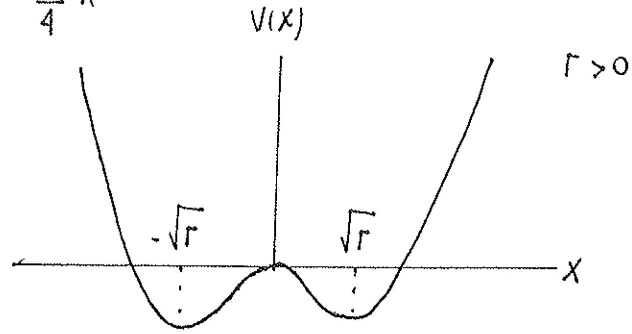
SO MINIMA OF V ARE STABLE $V'' > 0$

MAXIMA OF V ARE UNSTABLE $V'' < 0$.

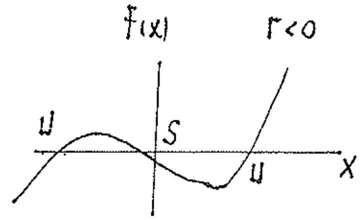
NOW IF $f(x) = rx - x^3$ WE HAVE

$$V(x) = -\frac{1}{2} r x^2 + \frac{1}{4} x^4$$



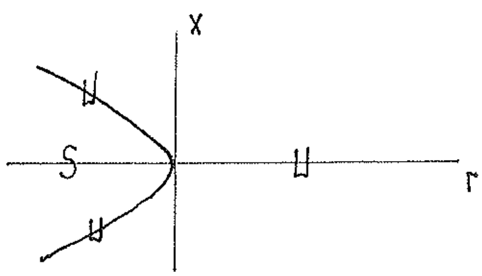


EXAMPLE SUBCRITICAL PITCHFORK



NOW CONSIDER $\dot{x} = rx + x^3 = f(x)$

EQUILIBRIA AT $x = 0$, $x = \pm\sqrt{-r}$



FOR ANY $x_0 \neq 0$ WE HAVE THAT THE SOLUTION TO

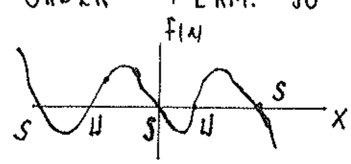
$$\dot{x} = rx + x^3 \qquad r < 0$$
$$x(0) = x_0$$

SATISFIES $x(t) \to \infty$ AT SOME FINITE T IF $x_0 > 0$

HENCE WE MUST INCLUDE A HIGHER ORDER TERM. SO CONSIDER

$-1/4 < r < 0$

$$\dot{x} = rx + x^3 - x^5 = f(x)$$



$$x(0) = x_0$$

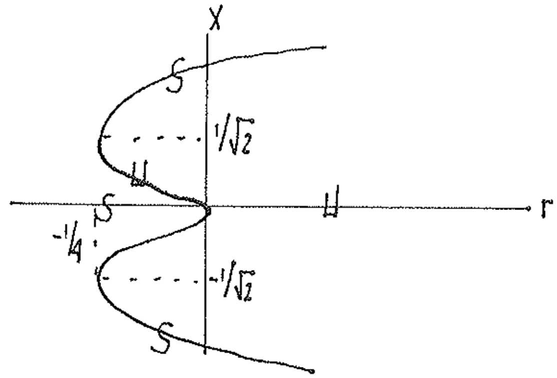
THE EQUILIBRIA ARE $x = 0$ AND $r = x^4 - x^2$. $\quad dr/dx = 0$ WHEN $4x^3 = 2x$

SO THAT $x = 0$ AND $x = \pm 1/\sqrt{2}$

$$r_{xx} = 12x^2 - 2 = \begin{cases} -2 & \text{AT } x = 0 \to \text{MAX} \\ 4 & \text{AT } x = \pm 1/\sqrt{2} \ \text{MIN.} \end{cases}$$

NOW AT $x = \pm 1/\sqrt{2}$ WE GET $r = 1/4 - 1/2 = -1/4$.

HENCE $(x_E, r_E) = (\pm 1/\sqrt{2}, -1/4)$ ARE BIFURCATION POINTS.

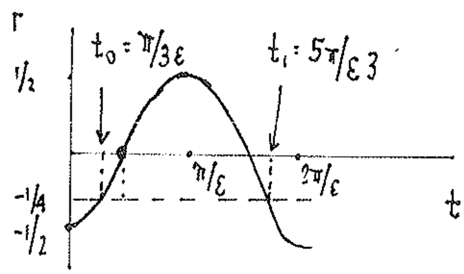


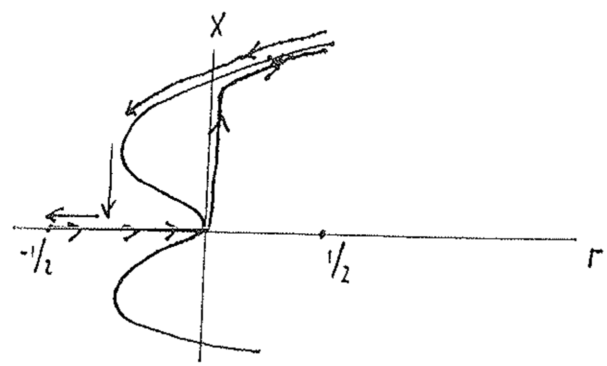
THIS EXAMPLE SHOWS THE POSSIBILITY OF HYSTERESIS. HYSTERESIS IS A LACK OF REVERSIBILITY IN THE DYNAMICS. FOR INSTANCE CONSIDER

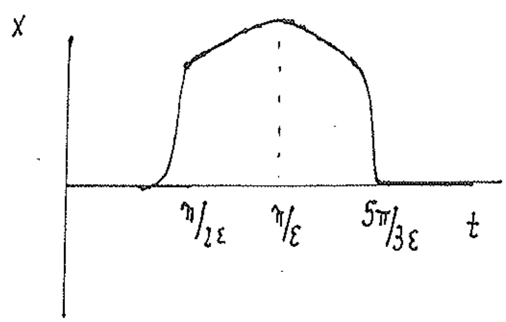
$$\dot{x} = r x + x^3 - x^5$$

$$x(0) = 0.05$$

$$r = -\frac{1}{2}\cos(\varepsilon t) \qquad \varepsilon = .05$$

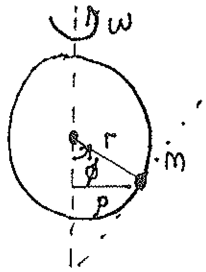






IT WOULD BE INTERESTING TO COMPUTE THIS SOLUTION.

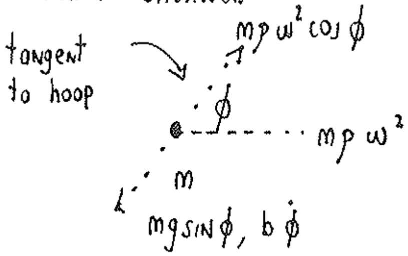
EXAMPLE (OVERDAMPED BEAD ON A ROTATING HOOP).



THERE IS A BEAD OF MASS m ON A WIRE HOOP. A CENTRIFUGAL FORCE, GRAVITY AND FRICTION ACT ON THE MASS.

$p = r \sin\phi$ $\qquad$ $r\ddot{\phi}$ tangential acceleration.

FORCE BALANCE



HENCE $\quad m r\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mp\omega^2\cos\phi$

NOW $\quad p = r\sin\phi$

THUS $$m r\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + m r\omega^2\cos\phi\sin\phi.$$

LET $\quad t = T\tau \qquad \dfrac{d\phi}{dt} = \dfrac{d\phi}{d\tau}\dfrac{1}{T}$

$$\frac{mr}{T^2}\phi'' = -\frac{b}{T}\phi' - mg\sin\phi + mr\omega^2\cos\phi\sin\phi$$

$$\frac{r}{gT^2}\phi'' = -\frac{b}{mgT}\phi' - \sin\phi + \frac{r\omega^2}{g}\cos\phi\sin\phi.$$

NOW CHOOSE $\quad T = b/mg \qquad$ DEFINE $\quad \gamma = \dfrac{r\omega^2}{g}, \qquad \varepsilon = \dfrac{r}{gT^2} = \dfrac{r}{g}\left(\dfrac{mg}{b}\right)^2$

suppose that the damping is large so that

$$\varepsilon = \frac{r}{g}\left(\frac{mg}{b}\right)^2 \ll 1 \qquad [b] = Kg\cdot L/S^2\cdot S$$

$$[\varepsilon] = \frac{L}{(L/sec^2)}\left(\frac{Kg\cdot L/sec^2}{Kg\cdot L/sec}\right)^2 = \frac{sec^2}{sec^2} =$$

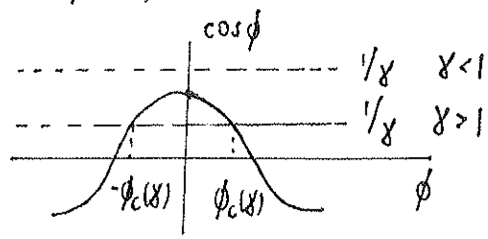
THEN $\quad \varepsilon\phi'' = -\phi' - \sin\phi + \gamma\cos\phi\sin\phi \quad$ REDUCES TO

$$\phi' = -\sin\phi + \gamma\cos\phi\sin\phi \qquad -\pi \le \phi \le \pi.$$

NOW THE EQUILIBRIUM POINTS ARE

$$\sin\phi = 0 \longrightarrow \phi = 0, \pm\pi$$

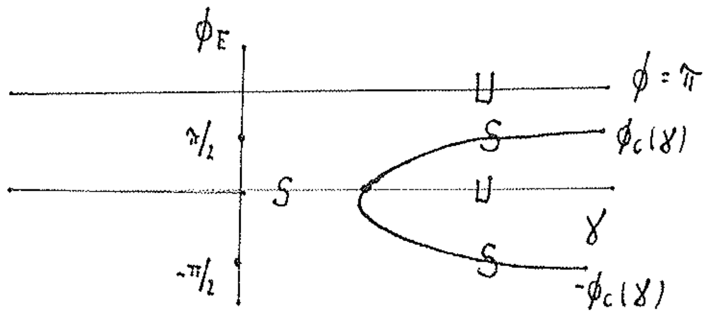
$$\cos\phi = 1/\gamma$$

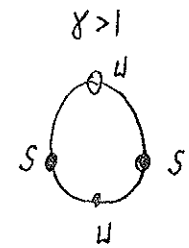


THUS FOR $\gamma > 1$ WE HAVE,

$$\phi = \begin{cases} \phi_c(\gamma) \\ -\phi_c(\gamma) \end{cases} \qquad \phi_c = \cos^{-1}(1/\gamma)$$

AS $\gamma \longrightarrow 1$, $\phi_c(\gamma) \longrightarrow 0$; AS $\gamma \longrightarrow \infty$, $\phi_c(\gamma) \sim \pi/2$.
(low rotation) (high rotation)





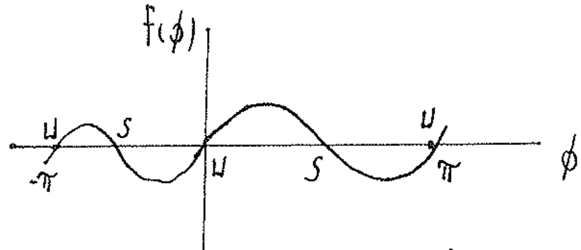
NOW FOR $\phi \ll 1$, $\phi' \simeq (\gamma - 1)\phi$. TO GET HIGHER ORDER

$$\phi' \sim -(\phi - \phi^3/6) + \gamma(1 - \phi^2/2)(\phi - \phi^3/3)$$

THIS GIVES $\phi' \sim (\gamma - 1)\phi + \phi^3\left(\frac{1}{6} - \frac{2\gamma}{3}\right)$.

NOW ON THE INTERVAL $-\pi < \phi < \pi$ IF WE PLOT

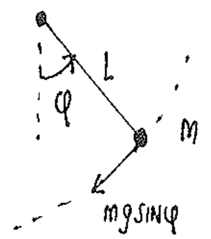
$$f(\phi) = \sin\phi\,(\gamma\cos\phi - 1) \qquad \text{WITH} \qquad \gamma > 1$$



SO AS $\omega \uparrow$ THE EQUILIBRIUM AT $\phi = 0$ LOSES STABILITY AND THE STABLE EQUILIBRIUM BEAD POSITION GOES UP THE SIDES.

EXAMPLE (OVERDAMPED PENDULUM)

CONSIDER A PENDULUM DRIVEN BY A CONSTANT TORQUE F.
FORCES ACTING ARE GRAVITY AND FRICTION



SO $$m L \ddot{\varphi} + b\dot{\varphi} + mg\sin\varphi = F > 0$$

$$\dot{\varphi} = \frac{d\varphi}{dt}.$$

NOW NON-DIMENSIONALIZE. LET $t = \tau T$. THEN $d/dt = d/d\tau \; 1/T$.

HENCE $$\frac{mL}{T^2}\varphi'' + \frac{b}{T}\varphi' + mg\sin\varphi = F.$$

SO $$\frac{L}{gT^2}\varphi'' + \frac{b}{mgT}\varphi' + \sin\varphi = \Gamma \qquad \Gamma = F/mg > 0$$

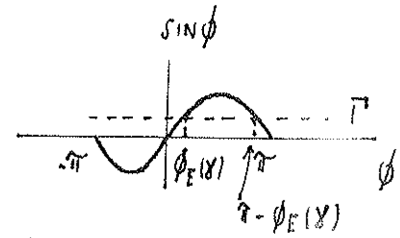
CHOSE $T = b/mg$. THEN

SUPPOSE $$\varepsilon = \frac{L}{gT^2} = \frac{L}{g}\left(\frac{mg}{b}\right)^2$$

SO THAT $$\varepsilon\varphi'' + \varphi' + \sin\varphi = \Gamma.$$

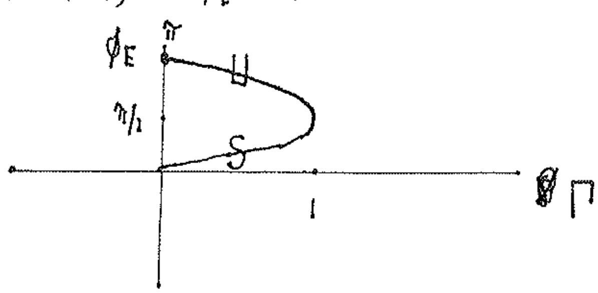
IF $\varepsilon \ll 1$ THEN CAN APPROXIMATE BY

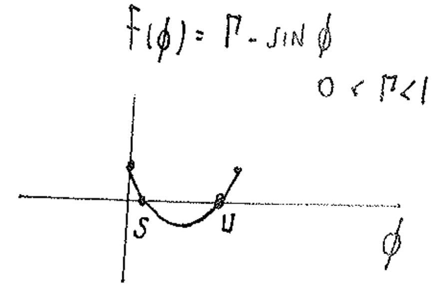
$$\varphi' = \Gamma - \sin\varphi. \qquad -\pi < \varphi < \pi.$$



IF $0 < \Gamma < 1 \longrightarrow$ TWO EQUILIBRIA. IF $\Gamma > 1$ NO EQUILIBRIA.

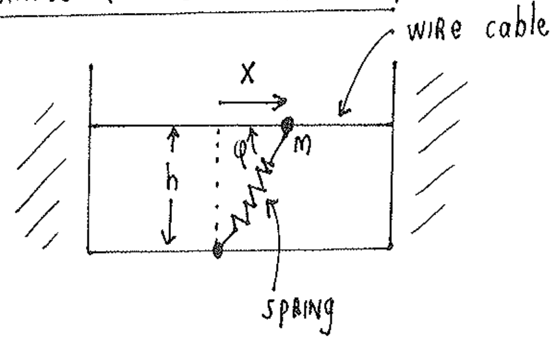
AS $\Gamma \to 0$, $\phi_E(\gamma) \to 0$ AND $\pi - \phi_E(\gamma) \to \pi$.





SADDLE-NODE OR FOLD BIFURCATION.

EXAMPLE (BEAD ON A WIRE)



THE EQUATION IS $\quad m\ddot{x} + b\dot{x} + K\left[L_{ext}\right]\cos\varphi = 0.$

THE ELONGATION OF THE SPRING IS

$$L_{ext} = -L_0 + \sqrt{h^2+x^2} \qquad L_0 : \text{Relaxed length of spring.}$$

THUS

$$m\ddot{x} + b\dot{x} + K\left[\sqrt{h^2+x^2} - L_0\right]\frac{x}{\sqrt{x^2+h^2}} = 0 \qquad \cos\varphi = \frac{x}{\sqrt{x^2+h^2}}.$$

THUS,

$$m\ddot{x} + b\dot{x} + Kx\left[1 - \frac{L_0}{\sqrt{x^2+h^2}}\right] = 0. \qquad \dot{x} = dx/dt$$

NOW LET $\quad x = hu, \quad t = \tau T.$ THEN,

$$\frac{mh}{T^2}\ddot{u} + \frac{bh}{T}\dot{u} + Kh\,u\left[1 - \frac{L_0}{h\sqrt{u^2+1}}\right] = 0$$

THEN,

$$\frac{m}{KT^2}\ddot{u} + \frac{b}{KT}\dot{u} + u\left[1 - \frac{\alpha}{\sqrt{u^2+1}}\right] = 0 \qquad \alpha = L_0/h.$$

CHOOSE $\quad T = b/K \quad$ DEFINE $\quad \varepsilon = \frac{m}{KT^2} = \frac{m}{K}\left(\frac{K}{b}\right)^2 = \frac{mK}{b^2}.$

$$\varepsilon\ddot{u} + \dot{u} + u\left[1 - \frac{\alpha}{\sqrt{u^2+1}}\right] = 0 \qquad \varepsilon = \frac{mK}{b^2}$$

ASSUME $\varepsilon \ll 1$.

IF $\alpha = L_0/h > 1$ $\quad$ SPRING IS UNDER COMPRESSION (EXPECT $u=0$ IS UNSTABLE)

IF $\alpha = L_0/h < 1$ $\quad$ SPRING IS UNDER TENSION. IT IS BEING PULLED (EXPECT $u=0$ IS STABLE)

THE EQUATION FOR $\varepsilon \ll 1$ IS

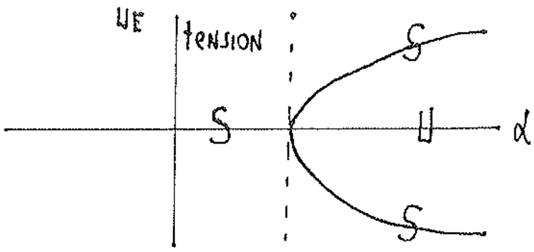
$$\dot{u} = -u\left[1 - \frac{\alpha}{\sqrt{u^2+1}}\right]$$

THE EQUILIBRIA ARE AT $u = 0$ AND $\sqrt{u^2+1} = \alpha$ FOR $\alpha > 1$.

THIS YIELDS,

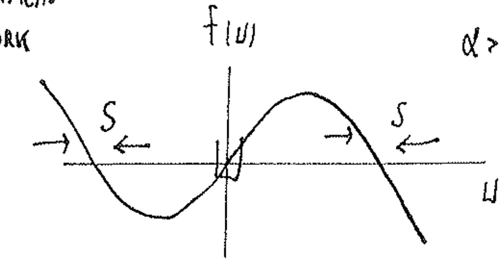
$$u = \pm\left(\alpha^2 - 1\right)^{1/2} \quad \text{FOR} \quad \alpha > 1.$$

THE PICTURE IS



FOR $u \ll 1$, $(1+u^2)^{-1/2} \sim 1 - u^2/2$

SO $\ddot{u} \sim -(1-\alpha)u - \alpha u^3/2$.

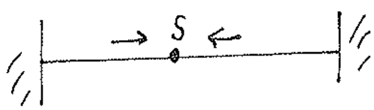


LET $f(u) = -u\left(1 - \frac{\alpha}{\sqrt{u^2+1}}\right)$

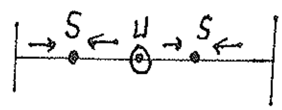
$f(u) \sim -u$ AS $u \to \infty$

$f(u) \sim (\alpha - 1)u$ AS $u \to 0$

<u>TENSION</u>



<u>COMPRESSION</u>

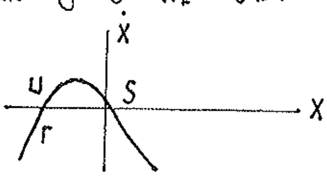


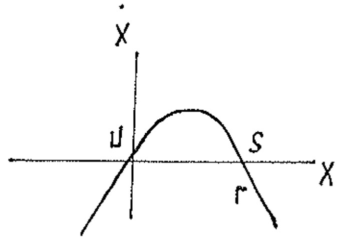
IMPERFECT BIFURCATIONS

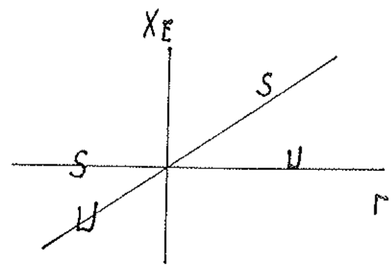
<u>EXAMPLE</u> (TRANSCRITICAL BIFURCATION)

$$\dot{x} = rx - x^2 + \delta$$

WHEN $\delta = 0$ WE GET







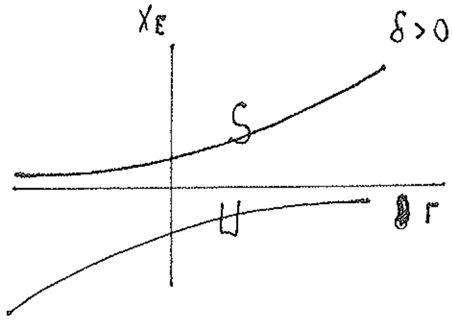
NOW SUPPOSE $\delta \neq 0$ THEN EQUILIBRIA SATISFY

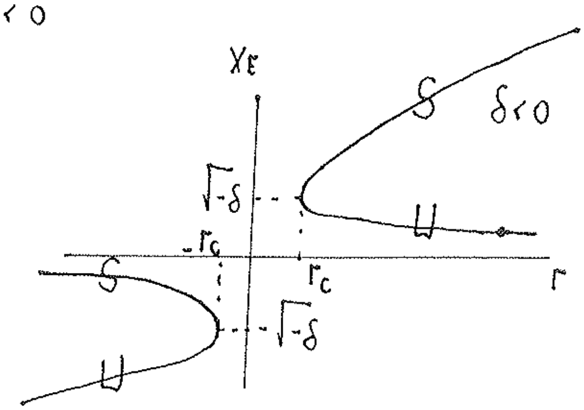
$$r = x - \delta/x$$

$x \to 0^+ \Rightarrow \begin{cases} r \to -\infty & \text{when } \delta > 0 \\ r \to +\infty & \text{when } \delta < 0. \end{cases}$

$$dr/dx = 1 + \delta/x^2 = 0 \quad \text{when} \quad \underset{\delta < 0}{x = \pm\sqrt{-\delta}}$$

$$x \longrightarrow 0^- \implies \begin{cases} r \longrightarrow +\infty & \text{when } \delta > 0 \\ r \longrightarrow -\infty & \text{when } \delta < 0 \end{cases}$$

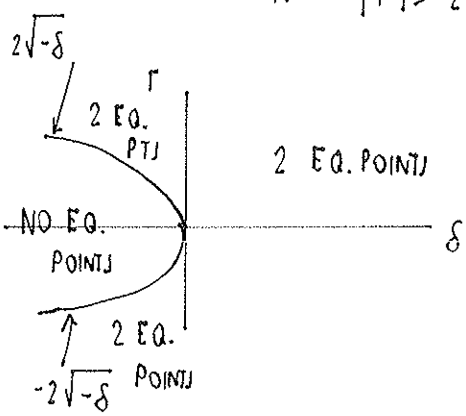




NOW $\quad r_c = \sqrt{-\delta} - \dfrac{\delta}{\sqrt{-\delta}} = \sqrt{-\delta} + \sqrt{-\delta} = 2\sqrt{-\delta}.$

THUS IF $\quad -2\sqrt{-\delta} < r < 2\sqrt{-\delta} \longrightarrow$ NO EQ. POINTS

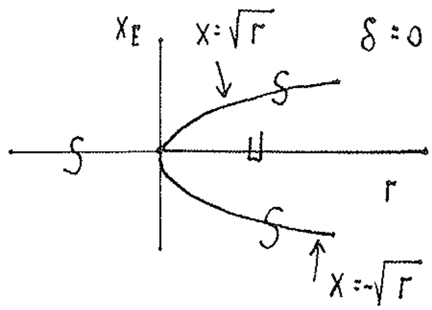
IF $\quad |r| > 2\sqrt{-\delta} \longrightarrow$ 2 EQUILIBRIUM POINTS



EXAMPLE (PERTURBED PITCHFORK)

$$\frac{dx}{dt} = + r x - x^3 - \delta$$

THE EQUILIBRIUM POINTS ARE $\quad r = x^2 + \delta / x$.



$$\frac{dr}{dx} = 2x - \delta / x^2$$

$2x^3 = \delta$

$x = (\delta/2)^{1/3} \quad \delta >$

$r_c = 3(\delta/2)^{2/3}$

$x = -(-\delta/2)^{1/3} \quad \delta <$

$r_c = 3(-\delta/2)^{2/3}$

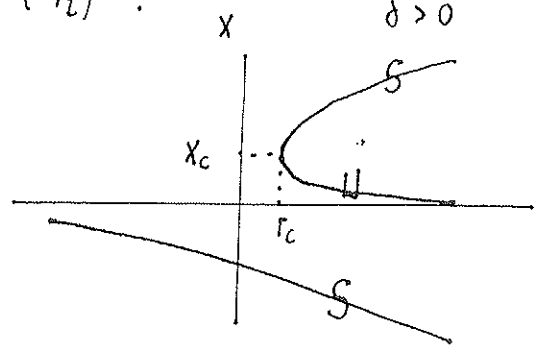
NOW $\delta > 0$ IF $X \to 0^+ \Rightarrow r \to \infty$

$X \to \infty \Rightarrow r \to \infty$

turning point at $X = (\delta/2)^{1/3}$, $r = 3(\delta/2)^{2/3}$.

AS $X \to 0^- \Rightarrow r \to -\infty$

$X \to -\infty \Rightarrow r \to +\infty$

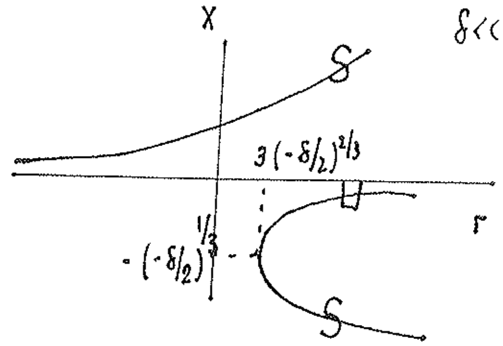


NOW IF $\delta < 0$ $X \to 0^+ \Rightarrow r \to -\infty$

$X \to \infty \Rightarrow r \to +\infty$

$X \to 0^- \Rightarrow r \to \infty$

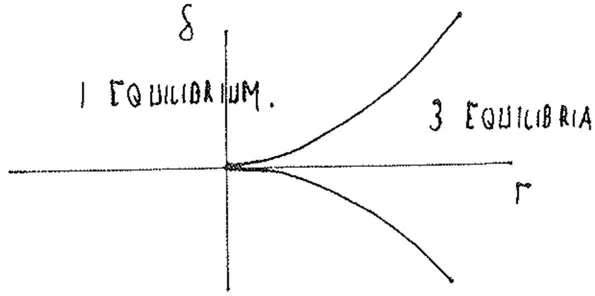
$X \to -\infty \Rightarrow r \to \infty$



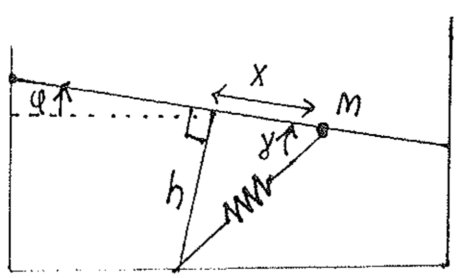
NOTICE: WE HAVE 3 EQUILIBRIA IF

$$r > 3\left(\frac{|\delta|}{2}\right)^{2/3} \qquad |\delta| < 2\left(\frac{r}{3}\right)^{3/2}$$

AND ONE EQUILIBRIA OTHERWISE



EXAMPLE (IMPERFECTION SENSITIVITY) BEAD ON A WIRE THAT IS SLIGHTLY TILTED AT ANGLE φ TO HORIZONTAL.



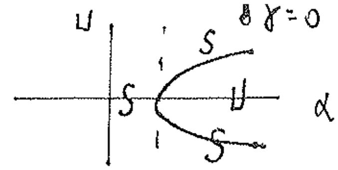
$$m\ddot{x} + b\dot{x} - mg\sin\varphi + K\,L_{ext}\cos\gamma = 0$$

$$L_{ext} = \sqrt{h^2 + x^2} - L_0 \qquad L_0 \text{ Resting length}$$

$$\cos\gamma = \frac{x}{\sqrt{x^2 + h^2}}$$

THE MODEL IS

$$m\ddot{x} + b\dot{x} - mg\sin\varphi + Kx\left[1 - \frac{L_0}{\sqrt{h^2+x^2}}\right] = 0.$$



LET $x = hu$, $t = \tau T$. $\quad u' = du/d\tau \quad \alpha = L_0/h$

$$\frac{mh}{T^2}u'' + \frac{bh}{T}u' + Khu\left[1 - \frac{\alpha}{\sqrt{1+u^2}}\right] - mg\sin\varphi = 0.$$

THEN $$\frac{m}{KT^2}u'' + \frac{b}{KT}u' + u\left[1 - \frac{\alpha}{\sqrt{1+u^2}}\right] - \gamma = 0 \qquad \gamma = \frac{mg\sin\varphi}{Kh} > 0$$

CHOOSE $T = b/K$ LET $\varepsilon = \dfrac{m}{KT^2} = \dfrac{m}{K}\dfrac{K^2}{b^2} = mK/b^2$.

$$\varepsilon u'' + u' = \gamma - u\left[1 - \frac{\alpha}{\sqrt{1+u^2}}\right] \qquad \alpha = L_0/h$$

$\alpha > 1 \rightarrow$ COMPRES.

$\alpha < 1 \rightarrow$ TENSIO

NOW IF $\varepsilon \ll 1$,

$$u' = \gamma - u\left[1 - \frac{\alpha}{\sqrt{1+u^2}}\right]$$

IF $\gamma > \gamma_c$ (tilt is too big) we expect that uphill equilibrium may suddenly disappear.

WE SUPPOSE $\gamma > 0$. THE EQUILIBRIAM ARE

$$\frac{\gamma}{u} = 1 - \frac{\alpha}{\sqrt{1+u^2}} \qquad \text{OR} \qquad \alpha = \sqrt{1+u^2}\left(1 - \gamma/u\right)$$

SINCE $\gamma > 0$
$\alpha \rightarrow -\infty$ AS $u \rightarrow 0^+$
$\alpha \rightarrow +\infty$ AS $u \rightarrow 0^-$
$\alpha \rightarrow \infty$ AS $u \rightarrow \pm\infty$.

NOW
$$\alpha = (1+u^2)^{1/2}\left(1 - \gamma/u\right)$$

$$\frac{d\alpha}{du} = u(1+u^2)^{-1/2}\left(1 - \gamma/u\right) + \frac{\gamma}{u^2}(1+u^2)^{1/2}$$

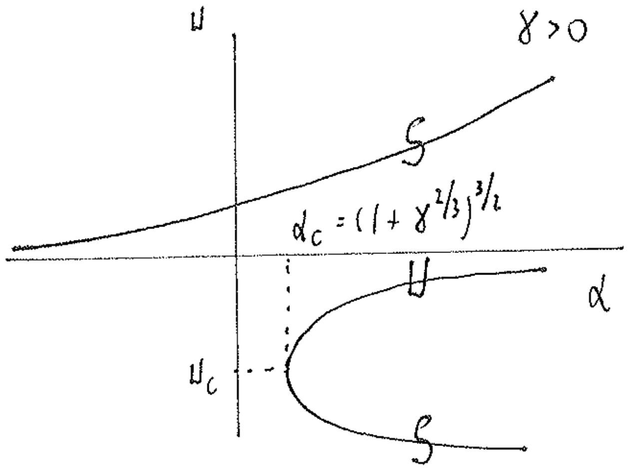
THEN $$\frac{d\alpha}{du} = (1+u^2)^{-1/2}\left(u + \gamma/u^2\right)$$

IF $u > 0 \quad d\alpha/du > 0$

$u < 0 \quad d\alpha/du = 0 \quad u = -$

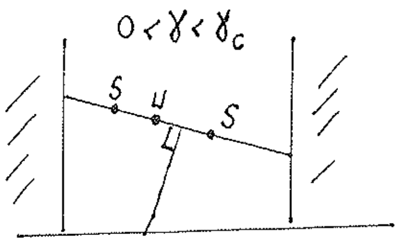
THUS AT THIS VALUE

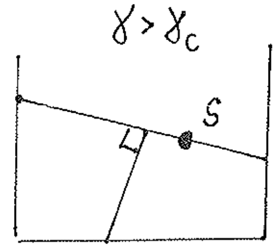
$$\alpha = \left(1+\gamma^{2/3}\right)^{1/2}\left(1+\gamma^{2/3}\right) = \left(1+\gamma^{2/3}\right)^{3/2}.$$



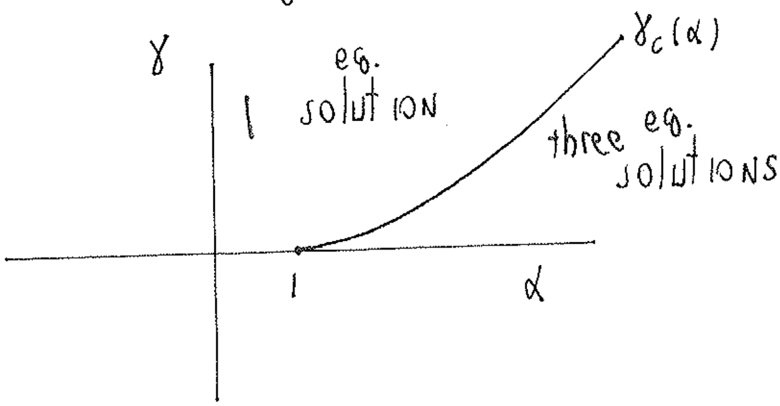
$$U_c = -\gamma^{1/3}.$$

THUS LET $\alpha = L_0/h$ BE FIXED. ASSUME THAT $\alpha > 1$. THEN IF THE TILT IS STRONG ENOUGH SO THAT $\gamma_0 > \left(\alpha_0^{2/3}-1\right)^{3/2}$ WE GET ONLY THE EQUILIBRIUM WITH $U > 0$.



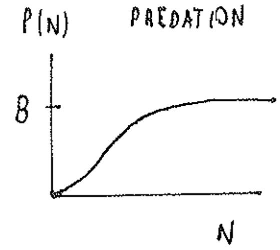


$$\gamma_c = \left(\alpha^{2/3}-1\right)^{3/2}$$ which exists IF $\alpha > 1$.



EXAMPLE (INSECT OUTBREAK)

$$\frac{dN}{dt} = RN\left(1 - N/K\right) - P(N)$$



$$P(N) = \frac{BN^2}{A^2 + N^2}.$$

r grow rate for small x

K foliage left on trees (carrying capacity)

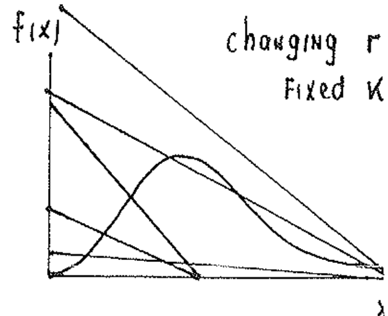
NOW LET $N = Ax$, $t = T\tau$

$$\frac{A}{T} x' = RAx\left(1 - \frac{A}{K}x\right) - B\frac{x^2}{1+x^2}$$

$$\frac{A}{BT} x' = x\frac{RA}{B}\left(1 - x/K\right) - \frac{x^2}{1+x^2}$$

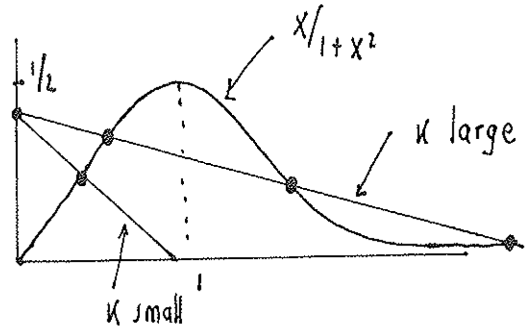
LET $T = A/B$, $r = RA/B$ AND $K = K/A$.

THEN $$\frac{dx}{d\tau} = rx\left(1 - x/K\right) - x^2/_{1+x^2} = f(x)$$



THUS $\left\{ \begin{array}{l} r\left(1 - x/K\right) = x/_{1+x^2} \\ x = 0 \end{array} \right.$ EQUILIBRIA

changing K



plot $r(1 - x/K)$

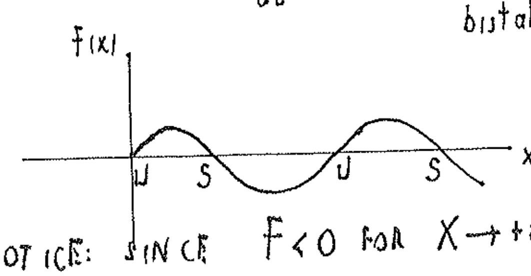
AND $x/_{1+x^2}$

- K SMALL $\rightarrow$ 1 EQ. POINT
- K large $\rightarrow$ three equilibrium points

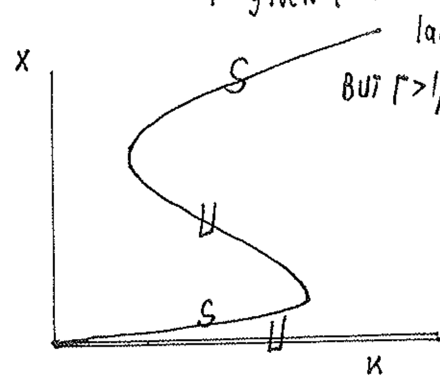
} certain Range of r.

NEAR $x = 0$ $\frac{dx}{dt} \approx rx$ $x = 0$ is UNSTABLE.

bistability Range

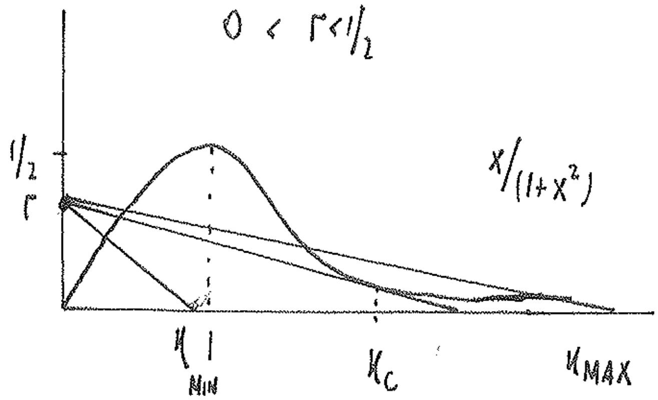


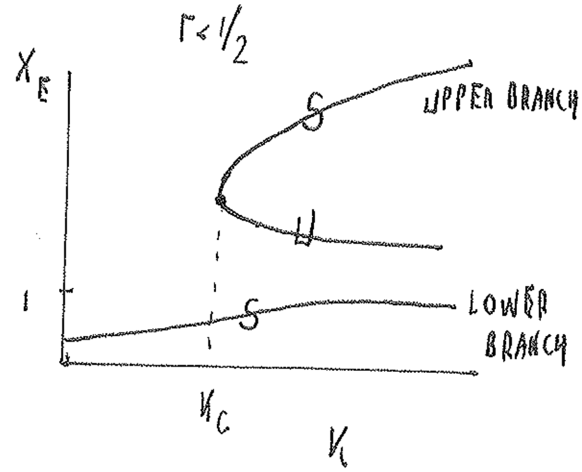
r given (NOT too large) BUT $r > 1/2$



NOTICE: SINCE $F < 0$ FOR $x \to +\infty$ THE "LARGEST" EQUILIBRIA IS ALWAYS STABLE.

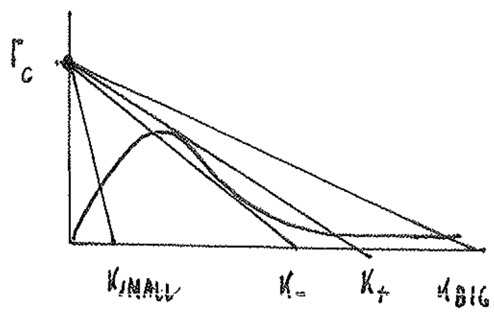
REMARK WE HAVE UPON MORE REFINED PLOTTING



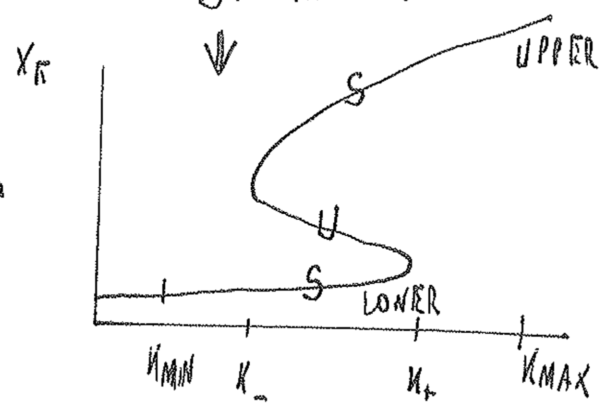


IF $\Gamma < 1/2$, $\exists$ SADDLE NODE BIFURCATION WHEN $K = K_C$!

• NOW SUPPOSE $1/2 < \Gamma < \Gamma_C$ FOR SOME Γ_C, THEN



BISTABILITY

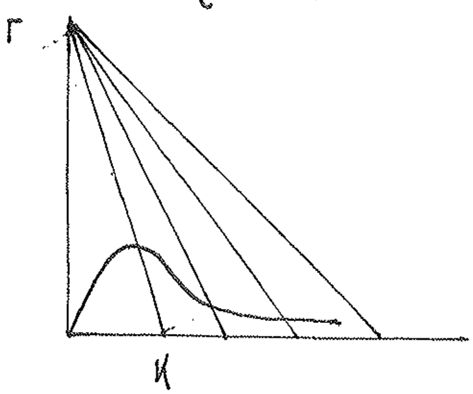


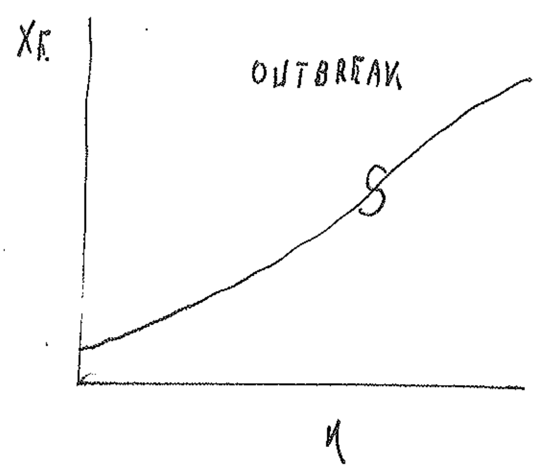
THUS $\exists$ TWO SADDLE-NODE BIF AT K_- AND K_+

AND WE HAVE BISTABILITY IF $K_- < K < K_+$

IN THIS REGIME WE EXPECT HYSTERESIS AS K IS SWEEPED FROM K_{MIN} TO K_{MAX}

• NOW SUPPOSE $\Gamma > \Gamma_C$ THEN





HENCE $\exists$ ONE SOLUTION $\forall$ K.

QUALITATIVELY CORRECT STATEMENTS

- IF κ SMALL THEN ONLY ONE NON-TRIVIAL SOLUTION $\forall\, r>0$.
- IF κ FIXED, THEN AS $r \to 0^+$ $\exists$ ONLY ONE NONTRIVIAL SOLUTION.
- IF κ FIXED, THEN ONLY ONE NON-TRIVIAL SOLUTION AS $r \to \infty$.
    - LOWER SOLUTION BRANCH : "SAFE OR REFUGE" LEVEL OF INSECTS.
    - UPPER SOLUTION BRANCH : OUTBREAK OF INSECTS.

NOW IF r IS FIXED, WE CAN SOLVE FOR κ AS $\kappa = \dfrac{r x (1+x^2)}{r(1+x^2) - x}$.

THE PLOTS ARE ON PREVIOUS PAGE.

Q. NOW WHERE IN (r, κ) PARAMETER PLANE IS THERE BISTABILITY?

TO OBTAIN THIS, WE WRITE $\dfrac{dx}{d\tau} = x\,g(x) \equiv f(x)$ $\qquad g(x) = r\left(1 - x/\kappa\right) - \dfrac{x}{1+x^2}$.

A SADDLE-NODE BIFURCATION OCCURS AT $x \neq 0$ WHEN

$$f(x) = 0 \qquad f'(x) = x g'(x) + g(x) = 0.$$

THUS FOR $x \neq 0$ WE GET

$$g(x) = 0 \longrightarrow r\left(1 - x/\kappa\right) = \frac{x}{1+x^2} \qquad (1)$$

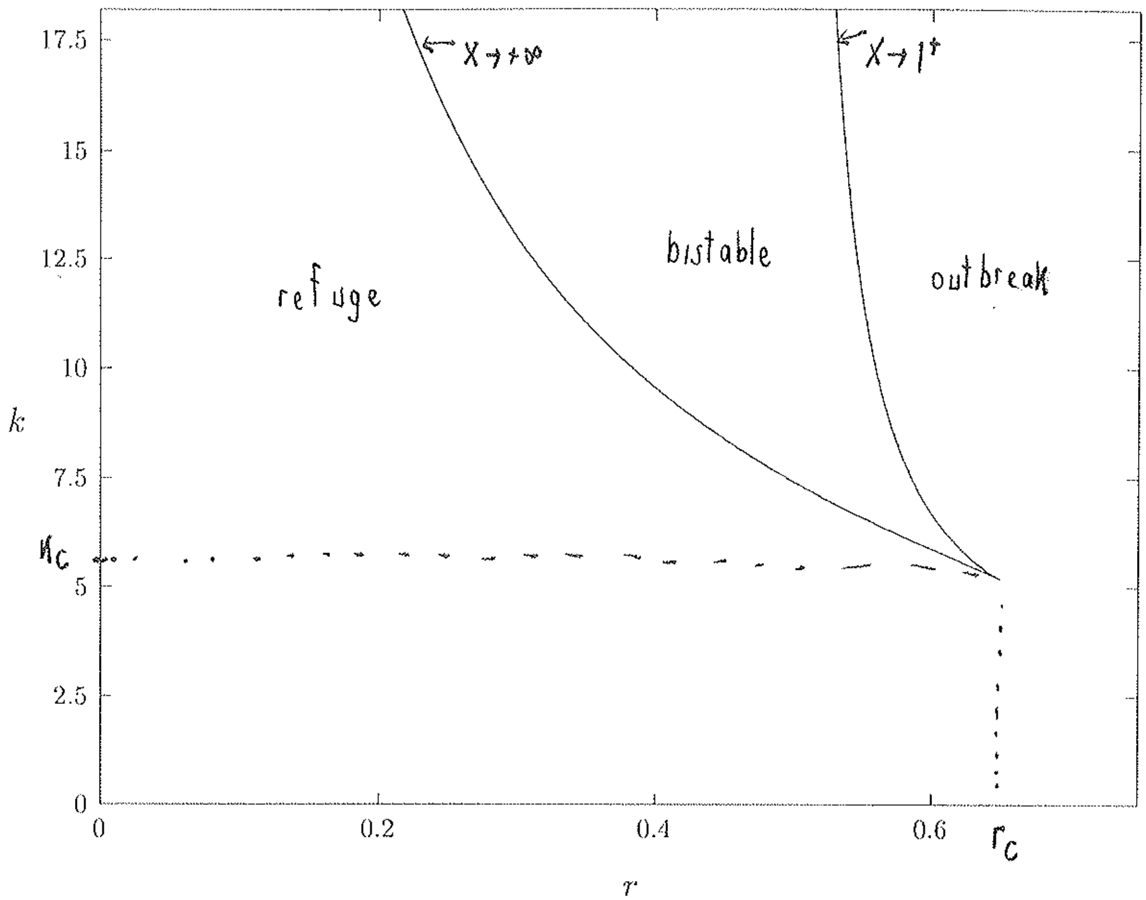
$$g'(x) = 0 \longrightarrow -r/\kappa = \frac{d}{dx}\left(\frac{x}{(1+x^2)}\right) = +\frac{(1-x^2)}{(1+x^2)^2}. \qquad (2)$$

WE SOLVE (1) AND (2) IN THE FORM $r = r(x)$, $\kappa = \kappa(x)$ WHICH GIVES A PARAMETRICALLY DEFINED CURVE IN THE (r, κ) PLANE (PARAMETRIZED BY x) WHERE SADDLE-NODE BIFURCATIONS OCCUR. TO OBTAIN THIS CURVE COMBINE (1) AND (2) AS

$$r - \frac{r}{\kappa}x = \frac{x}{(1+x^2)} \implies r + \frac{x(1-x^2)}{(1+x^2)^2} = \frac{x}{1+x^2} \implies r = \frac{x}{(1+x^2)} - \frac{x(1-x^2)}{(1+x^2)^2}$$

THUS $r = \dfrac{1}{(1+x^2)^2}\left[x(1+x^2) - x(1-x^2)\right] = \dfrac{2x^3}{(1+x^2)^2} \implies r = \dfrac{2x^3}{(1+x^2)^2}$ THEN $\kappa = \dfrac{-r(1+x^2)^2}{(1-x^2)} = \dfrac{2x^3}{x^2-1}$.

INSECT MODEL   PLOT OF BISTABILITY REGION IN (r, K) space.



THUS IN (r, K) PLANE THE PARAMETRIC CURVE IS

$$r = \frac{2x^3}{(1+x^2)^2}, \quad K = \frac{2x^3}{x^2-1} \quad \text{FOR} \quad 1 < x < \infty.$$

NOTE: ALSO CAN WRITE

$$K = 2x + \frac{2x}{x^2-1}$$

THIS CURVE IS PLOTTED ABOVE

- AS $x \to 1^+$ THEN $K \to +\infty$ AND $r \to 1/2$
- AS $x \to +\infty$, $K \to +\infty$ AND $r \approx 2/x \to 0$

NOTICE THAT K IS MINIMIZED WHEN $K_x = 0$. SO $K_x = 2 + \frac{2}{x^2-1} - \frac{2x(2x)}{(x^2-1)^2}$

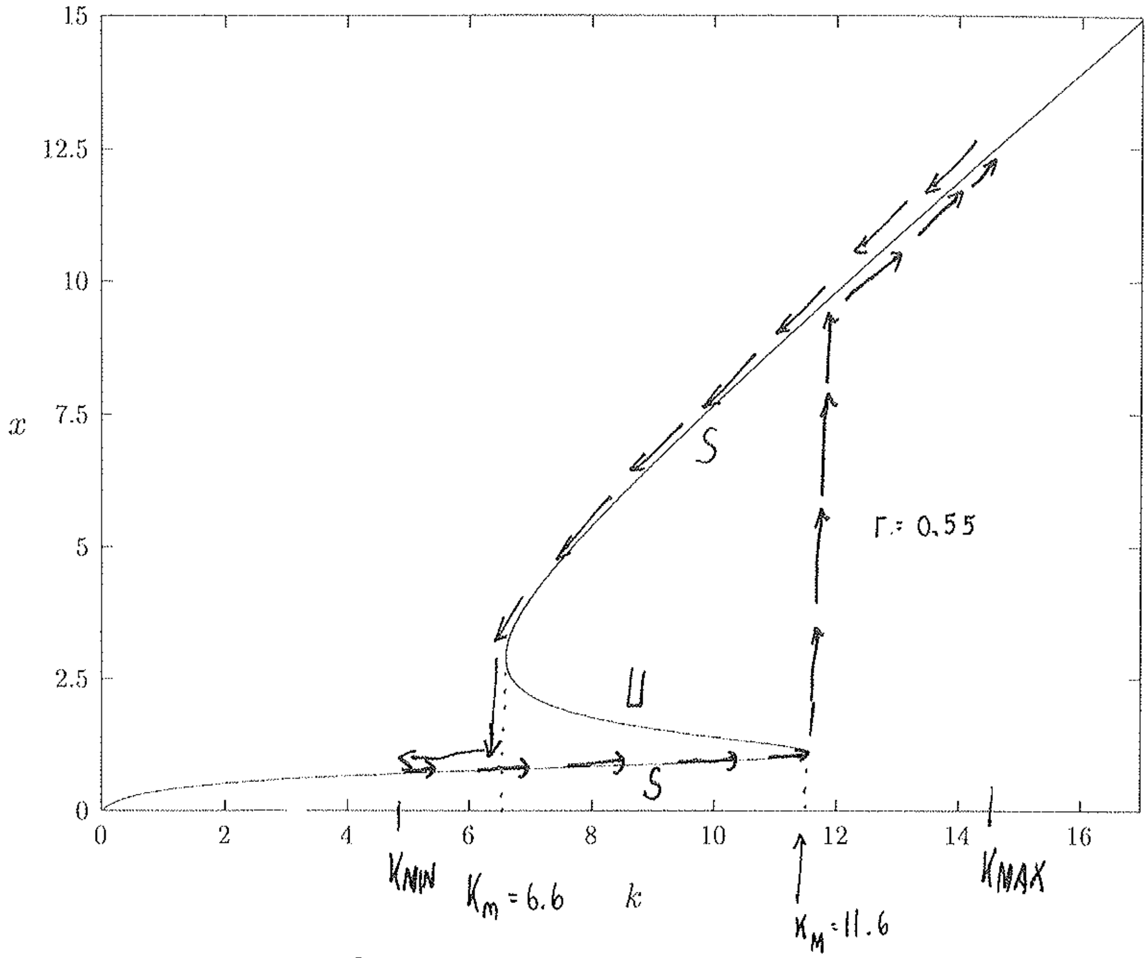
SO $K_x = 2 - \frac{2(1+x^2)}{(1-x^2)^2}$. THUS $K_x = 0$ AT $1 = \frac{(1+x^2)}{(1-x^2)^2} \rightarrow (1-x^2)^2 = (1+x^2)$

SO $x^4 - 2x^2 + 1 = x^2 + 1$.

THUS $x^4 = 3x^2$ OR $x^2 = 3 \rightarrow x = \sqrt{3}$. AT $x = \sqrt{3}$, WE HAVE

$$r = r_c = \frac{2(\sqrt{3})^3}{((\sqrt{3})^2+1)^2} = \frac{3\sqrt{3}}{8} \doteq .649, \quad K = K_c = \frac{2(\sqrt{3})^3}{(\sqrt{3})^2-1} = 3\sqrt{3} \doteq 5.196.$$

(SEE FIG ABOVE).



TAKE $\Gamma = 0.55$. SINCE $\frac{1}{2} < \Gamma < \Gamma_C \doteq .649$ WE HAVE SECOND FIGURE ON PAGE 24.

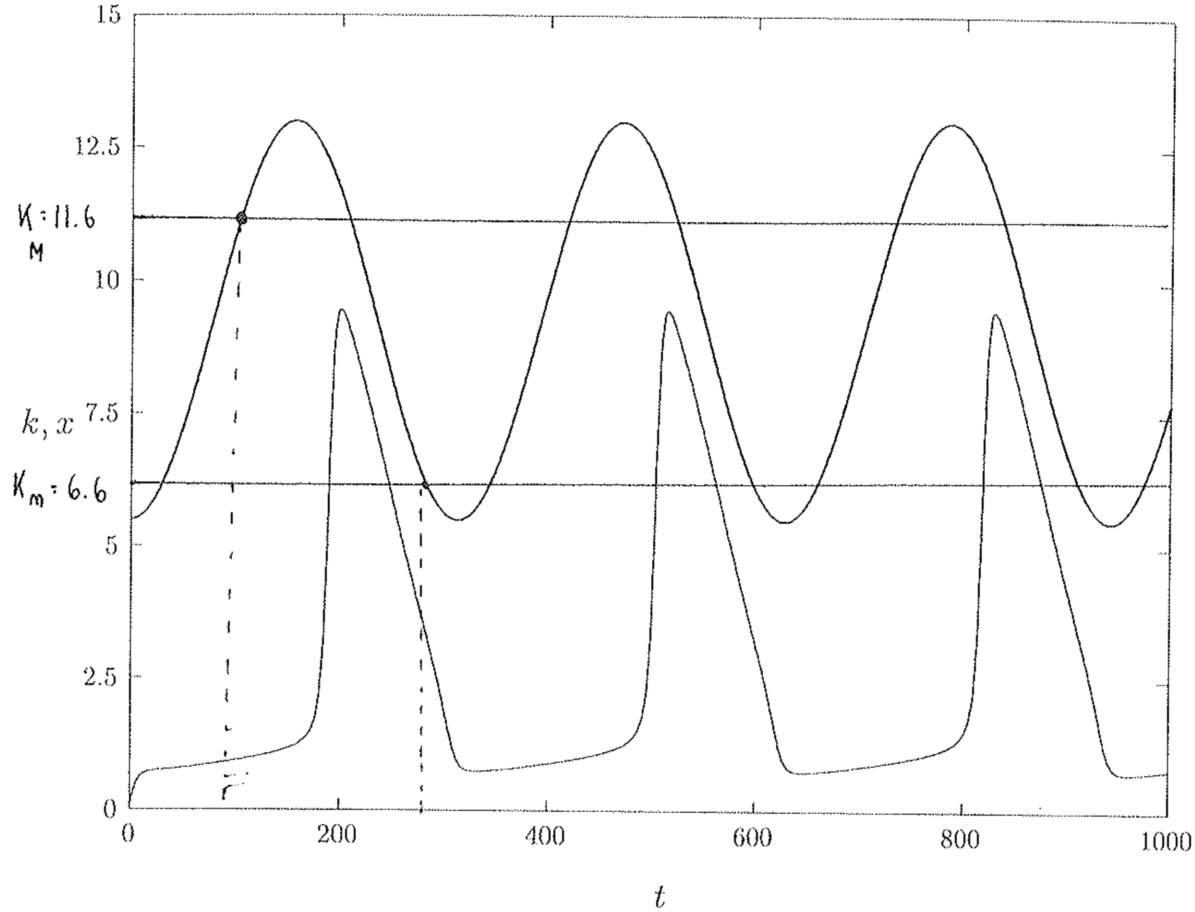
THIS suggests that to get hysteresis we let

$$K = \left(\frac{K_{MIN} + K_{MAX}}{2} \right) + \left(\frac{K_{MAX} - K_{MIN}}{2} \right) \sin\left(\varepsilon t - \pi/2 \right)$$

$\varepsilon = .02$

$K_{MIN} = 5.5 \qquad K_{MAX} = 13.0$

K SWEEPING SLOWLY IN TIME COULD MODEL THE CHANGE IN AMOUNT OF FOILAGE OVER A YEAR. WE PREDICT A SUDDEN LARGE OUTBREAK OF INSECTS AS WE SWEEP ABOVE $K_M = 11.6$



$$K = \frac{(K_{MIN} + K_{MAX})}{2} + \frac{(K_{MAX} - K_{MIN})}{2} \sin\left(\varepsilon t - \pi/2\right)$$

$\varepsilon = .02$

Notice the delayed bifurcation.